Canad. Math. Bull. Vol. 15 (4), 1972.

ON THE LOCAL CONNECTEDNESS OF $\beta X - X$

BY
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Let X be any completely regular Hausdorff topological space, and let βX denote its Stone-Čech compactification. This note is devoted to proving the following result:

5. THEOREM. *Let X be realcompact and noncompact. Then $\beta X - X$ is not connected im kleinen at any point.*

The question of when βX itself is locally connected was settled in [2], in which the authors prove that βX is locally connected if and only if the (completely regular Hausdorff) space X is locally connected and pseudocompact.

All topological spaces considered in this paper are assumed to be completely regular and Hausdorff. The notation and terminology used in the Gillman-Jerison text [1] will be used here without further comment. If p is a point of a space X, then X is said to be *locally connected at p* if every neighborhood of p contains a connected open neighborhood of p. The space X is said to be *connected im kleinen at p* if every neighborhood of p contains a connected neighborhood of p. The latter property is strictly weaker than the former, but a space is connected im kleinen at every point if and only if it is locally connected at every point (see [3, Ch. 3]; the definition of connectedness im kleinen given therein is different from the one given here, but is easily seen to be equivalent to it).

ACKNOWLEDGMENT. The author wishes to thank the referee for suggesting a much shorter proof of Lemma 3.

1. **Notation.** Let A be a closed subset of a space X. The set $(\mathrm{cl}_{\beta X} A) - X$ will be denoted by the symbol A^* (in particular $X^* = \beta X - X$). The symbol αR will denote the one-point compactification of the space R of real numbers. The cardinality of a set S will be denoted by $|S|$.

The following result is an immediate consequence of 8.4 of [1].

2. LEMMA. *Let X be a realcompact, noncompact space and let $p \in X^*$. Then there exists $f \in C(X)$ such that $f^*(p) = \omega$, where f^* denotes the Stone extension to βX of the map $f: X \to \alpha R$, and $\{\omega\} = \alpha R - R$.*

Received by the editors February 8, 1971 and, in revised form, March 12, 1971.

3. LEMMA. *Let X be any space. The family* $\{\mathrm{int}_{X^*}A^*: A$ *closed in* $X\}$ *is a base for the open subsets of* X.

Proof. Let V be open in βX and let $p \in V-X$. As βX is regular, there exists an open subset W of βX such that $p \in W \subseteq \mathrm{cl}_{\beta X} W \subseteq V$. It follows quickly from 0.12 of [1] that $W \subseteq \mathrm{cl}_{\beta X}(X \cap \mathrm{cl}_{\beta X} W)$; hence

$$p \in W \cap X^* \subseteq \mathrm{cl}_{\beta X}(X \cap \mathrm{cl}_{\beta X} W) - X \subseteq V - X.$$

Thus

$$p \in \mathrm{int}_{X^*}[X \cap \mathrm{cl}_{\beta X} W]^* \subseteq V - X,$$

and the lemma follows.

The next result is Lemma 2.4 of [4].

4. LEMMA. *Let X be realcompact, let A and B be closed subsets of X, and let* $A^* \subseteq B^*$. *Then* $\mathrm{cl}_X(A-B)$ *is compact if* $A = \mathrm{cl}_X(\mathrm{int}_X A)$.

We can now prove Theorem 5, stated at the beginning of the paper.

Proof of Theorem 5. Let $p \in X^*$. By Lemma 2 there exists $f \in C(X)$ such that $f^*(p) = \omega$; without loss of generality assume that $f \geq 0$. Obviously f is unbounded on X, so there exists a sequence $(q_n)_{n \in N} \subseteq f[X]$ such that $q_{n+1} \geq q_n + 1$ for each $n \in N$ (N denotes the set of nonnegative integers). Let $(I_n)_{n \in N}$ and $(J_n)_{n \in N}$ be two sequences of closed subintervals of $[0, \infty)$ such that:

(a) If $i \neq j$ then $I_i \cap I_j = J_i \cap J_j = \varnothing$.

(b) $q_{2n+1} \in \mathrm{int}_R I_n - \bigcup_{k \in N} J_k$ and $q_{2n} \in \mathrm{int}_R J_n - \bigcup_{k \in N} I_k$ for each $n \in N$.

(c) $[0, \infty) - \bigcup_{n \in N} I_n$ and $[0, \infty) - \bigcup_{n \in N} J_n$ are completely separated in R.

It is obvious that $(I_n)_{n \in N}$ and $(J_n)_{n \in N}$ can be chosen as described, and that if $p_n \in I_n$ for each $n \in N$, then $\lim_{n \to \infty} p_n = \infty$. Put $Z_1 = f^{\leftarrow}[\bigcup_{n \in N} I_n]$ and $Z_2 = f^{\leftarrow}[\bigcup_{n \in N} J_n]$. Evidently Z_1 and Z_2 are in $Z(X)$. We claim that $\mathrm{int}_{X^*} Z_1^* \cup \mathrm{int}_{X^*} Z_2^* = X^*$. To verify this we note that by (c) above there exist disjoint zero-sets S_1 and S_2 of $[0, \infty)$ containing $[0, \infty) - \bigcup_{n \in N} I_n$ and $[0, \infty) - \bigcup_{n \in N} J_n$ respectively. Hence $f^{\leftarrow}[S_2]$ and $f^{\leftarrow}[S_2]$ are disjoint zero-sets of X containing $X - Z_1$ and $X - Z_2$ respectively. Thus $\mathrm{cl}_{\beta X}(X - Z_1) \cap \mathrm{cl}_{\beta X}(X - Z_2) = \varnothing$ (see [1, 6.5]). Now

$$\begin{aligned}\beta X - [\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_1 \cup \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_2] &= (\beta X - \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_1) \cap (\beta X - \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_2) \\ &= \mathrm{cl}_{\beta X}[\beta X - \mathrm{cl}_{\beta X} Z_1] \cap \mathrm{cl}_{\beta X}[\beta X - \mathrm{cl}_{\beta X} Z_2].\end{aligned}$$

Since $(\beta X - \mathrm{cl}_{\beta X} Z_1) \cap X = X - Z_1$, it follows from 0.12 of [1] that

$$\mathrm{cl}_{\beta X}(\beta X - \mathrm{cl}_{\beta X} Z_1) = \mathrm{cl}_{\beta X}(X - Z_1).$$

Hence by the preceding remarks it follows that $\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_1 \cup \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_2 = \beta X$. As $\mathrm{int}_{X^*} Z^*_i \supseteq \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z_i - X$ $(i = 1, 2)$, our claim immediately follows. Hence without loss of generality we may assume that $p \in \mathrm{int}_{X^*} Z^*$.

https://doi.org/10.4153/CMB-1972-102-5 Published online by Cambridge University Press

We now claim that if W is an X^*-neighborhood of p contained in $\text{int}_{X^*} Z_1^*$, then W is not connected. This will show that X^* is not connected im kleinen at p. If W is such a neighborhood, by the regularity of X^* and Lemma 3 we can find a closed subset G of X such that $G = \text{cl}_X(\text{int}_X G)$ and

$$p \in \text{int}_{X^*} G^* \subseteq G^* \subseteq W \subseteq \text{int}_{X^*} Z_1^* \subseteq Z_1^*.$$

By Lemma 4 $\text{cl}_X(G - Z_1)$ is compact, and so

$$\begin{aligned} G^* &= (G \cap Z_1)^* \cup [\text{cl}_X(G-Z_1)]^* \\ &= (G \cap Z_1)^*. \end{aligned}$$

Let $M = \{n \in N : G \cap f^{\leftarrow}[I_n] \neq \varnothing\}$. We claim that $|M| = \aleph_0$. For since $p \in G^* = (G \cap Z_1)^* \subset \text{cl}_{\beta X}(\bigcup_{n \in M} f^{\leftarrow}[I_n])$, it follows that

$$\begin{aligned} f^*(p) &\in f^*\left[\text{cl}_{\beta X}\left(\bigcup_{n\in M} f^{\leftarrow}[I_n]\right)\right] \\ &= \text{cl}_{\alpha R}\left(f^*\left[\bigcup_{n \in M} f^{\leftarrow}[I_n]\right]\right) \\ &= \text{cl}_{\alpha R}\left[\bigcup_{n\in M} I_n\right] \end{aligned}$$

If M is finite, then $\text{cl}_{\alpha R}[\bigcup_{n \in M} I_n] = \bigcup_{n\in M} I_n \subseteq R$, which contradicts the fact that $f^*(p) = \omega$. Hence M is infinite.

Thus we can find subsets N_1 and N_2 of N such that:

(a) $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \varnothing$

(b) $|N_1| = |N_2| = \aleph_0$

(c) $|\{n \in N_i : G \cap f^{\leftarrow}[I_n] \neq \varnothing\}| = \aleph_0$, $i = 1, 2$.

Put $E = \bigcup_{n \in N_1} f^{\leftarrow}[I_n]$ and $F = \bigcup_{n\in N_2} f^{\leftarrow}[I_n]$. Then $E^* \cup F^* = (E \cup F)^* = Z_1^*$, and $E^* \cap F^* = \varnothing$ as E and F are disjoint zero-sets of X (see [1, 6.5]). For each $n \in N_1$, choose $p_n \in G \cap f^{\leftarrow}[I_n]$. Put $P = (p_n)_{n \in N}$. Then $\text{cl}_X P$ is not compact, so

$$\varnothing \neq (cl_X P)^* \subseteq G^* \cap E^*.$$

A similar argument shows that $G^* \cap F^* \neq \varnothing$. Thus $\{W \cap E^*, W \cap F^*\}$ is a disconnection of W into two disjoint closed nonempty subsets whose union is W (since $E^* \cup F^* = Z_1^*$). As W was an arbitrary neighborhood of p contained in $\text{int}_{X^*} Z_1^*$, it follows that X^* is not connected im kleinen at p. As p was arbitrary, the theorem follows.

6. Remark. It follows from Problem 9K of [1] that if X is any completely regular Hausdorff space, then there exists a pseudocompact space Y such that $\beta Y - Y$ is homeomorphic to X. This suggests that a characterization of those pseudocompact spaces Y for which $\beta Y - Y$ is locally connected may be somewhat involved.

https://doi.org/10.4153/CMB-1972-102-5 Published online by Cambridge University Press

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