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ON A LATTICE CHARACTERISATION OF FINITE SOLUBLE *PST*-GROUPS

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Abstract

Let $\mathfrak{F}$ be a class of finite groups and G a finite group. Let $\mathcal{L}_{\mathfrak{F}}(G)$ be the set of all subgroups A of G with $A^G/A_G \in \mathfrak{F}$. A chief factor H/K of G is $\mathfrak{F}$-*central* in G if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. We study the structure of G under the hypothesis that every chief factor of G between A_G and A^G is $\mathfrak{F}$-central in G for every subgroup $A \in \mathcal{L}_{\mathfrak{F}}(G)$. As an application, we prove that a finite soluble group G is a *PST*-group if and only if $A^G/A_G \leq Z_\infty(G/A_G)$ for every subgroup $A \in \mathcal{L}_{\mathfrak{N}}(G)$, where $\mathfrak{N}$ is the class of all nilpotent groups.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathcal{L}(G)$ denotes the lattice of all subgroups of G and $\mathcal{L}_n(G)$ is the lattice of all normal subgroups of G. We use A^G to denote the normal closure of the subgroup A in G and set $A_G = \bigcap_{x \in G} A^x$. If $L \leq T$ are normal subgroups of G, then we say that T/L is a *normal section* of G. Finally, $\mathfrak{F}$ is a class of groups containing all identity groups and $\mathfrak{N}$ denotes the class of all nilpotent groups.

Wielandt [12] proved that the set $\mathcal{L}_{sn}(G)$, of all subnormal subgroups of a finite group G, forms a sublattice of the lattice $\mathcal{L}(G)$. Later, Kegel [7] proposed a generalisation of the lattice $\mathcal{L}_{sn}(G)$ based on the theory of group classes. The papers [7, 12] motivated many studies to find and apply sublattices of the lattices $\mathcal{L}(G)$ and $\mathcal{L}_{sn}(G)$ (see, for example, [1, 6, 11], [4, Chapter 6] and the recent paper [10]).

In this paper, we discuss a new approach that allows us to locate two new classes of sublattices in the lattice $\mathcal{L}(G)$ and we give some applications of these sublattices in the theory of generalised T-groups.

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Let Δ be any set of normal sections of G. We say that Δ is *G-closed* provided that, for any two G-isomorphic normal sections H/K and T/L where $T/L \in \Delta$, we have $H/K \in \Delta$. If $L \leq T$ are normal subgroups of G, then we write $T/L \leq Z_\Delta(G)$ (or simply $T \leq Z_\Delta(G)$ if $L = 1$) provided either $L = T$ or $L < T$ and $H/K \in \Delta$ for every chief factor H/K of G between L and T.

Now let $\mathcal{L}_\Delta(G)$ be the set of all subgroups A of G such that $A^G/A_G \leq Z_\Delta(G)$, and let $\mathcal{L}_{\mathfrak{F}}(G)$ be the set of all subgroups A of G such that $A^G/A_G \in \mathfrak{F}$. Then $\mathcal{L}_n(G) \subseteq \mathcal{L}_\Delta(G) \cap \mathcal{L}_{\mathfrak{F}}(G)$.

Before continuing, we recall some notation and concepts of the theory of group classes. The symbol $G^{\mathfrak{F}}$ denotes the *$\mathfrak{F}$-residual* of G, that is, the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$, and $G_{\mathfrak{F}}$ denotes the *$\mathfrak{F}$-radical* of G, that is, the product of all normal subgroups N of G with $N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be *normally hereditary* if $H \in \mathfrak{F}$ whenever $H \trianglelefteq G \in \mathfrak{F}$, *saturated* if $G \in \mathfrak{F}$ whenever $G^{\mathfrak{F}} \leq \Phi(G)$, a *formation* if every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G and a *Fitting class* if every normal subgroup of $G_{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G.

Our first observation is the following theorem.

Theorem 1.1.

(i) *If Δ is a G-closed set of chief factors of G, then $\mathcal{L}_\Delta(G)$ is a sublattice of the lattice $\mathcal{L}(G)$.*

(ii) *If $\mathfrak{F}$ is a normally hereditary formation, then the set $\mathcal{L}_{\mathfrak{F}}(G)$ is a lattice (a meet-sublattice of $\mathcal{L}(G)$* [8, page 7]*).*

(iii) *If $\mathfrak{F}$ is a Fitting formation, then $\mathcal{L}_{\mathfrak{F}}(G)$ is a sublattice of the lattice $\mathcal{L}(G)$.*

A subgroup M of G is called *modular* in G if M is a modular element (in the sense of Kurosh (see [8, page 43])) of the lattice $\mathcal{L}(G)$. From [8, Theorem 5.2.3], for every modular subgroup A of G, all chief factors of G between A_G and A^G are cyclic. Consequently, despite the fact that in the general case the intersection of two modular subgroups of G may be nonmodular, the following result holds.

Corollary 1.2. *If A and B are modular subgroups of G, then every chief factor of G between $(A \cap B)_G$ and $(A \cap B)^G$ is cyclic.*

A subgroup A of G is said to be *quasinormal* (respectively, *S-quasinormal* or *S-permutable* [3]) in G if A permutes with all subgroups (respectively, with all Sylow subgroups) H of G, that is, $AH = HA$. For every quasinormal subgroup A of G, we have $A^G/A_G \leq Z_\infty(G/A_G)$ [3, Corollary 1.5.6]. In general, the intersection of quasinormal subgroups of G may be nonquasinormal. Nevertheless, the following fact holds.

Corollary 1.3. *If A and B are quasinormal subgroups of G, then*

$$(A \cap B)^G/(A \cap B)_G \leq Z_\infty(G/(A \cap B)_G).$$

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A chief factor H/K of G is said to be $\mathfrak{F}$-*central* in G if $(H/K) \rtimes (G/C_G(H/K))$ belongs to $\mathfrak{F}$ [9]. This leads to our next result.

Theorem 1.4. *Let $\mathfrak{F}$ be a normally hereditary saturated formation containing all nilpotent groups and Δ the set of all $\mathfrak{F}$-central chief factors of G.*

(i) *If the $\mathfrak{F}$-residual $D = G^{\mathfrak{F}}$ of G is soluble and $\mathcal{L}_{\mathfrak{F}}(G) = \mathcal{L}_{\Delta}(G)$, then D is an abelian Hall subgroup of odd order of G, every element of G induces a power automorphism in $D/\Phi(D)$ and every chief factor of G below D is cyclic.*
(ii) *Let G be soluble and let Δ be the set of all central chief factors H/K of G, that is, $H/K \leq Z(G/K)$. If $\mathcal{L}_{\mathfrak{N}}(G) = \mathcal{L}_{\Delta}(G)$, then every element of G induces a power automorphism in $G^{\mathfrak{N}}$.*

Now we consider some applications of Theorem 1.4 in the theory of generalised T-groups. Firstly recall that G is said to be a T-*group* (respectively, a *PT-group* or a *PST-group*) if every subnormal subgroup of G is normal (respectively, permutable or S-permutable) in G. Theorem 1.4 allows us to give a new characterisation of soluble *PST*-groups.

Theorem 1.5. *Suppose that G is soluble. Then G is a PST-group if and only if $\mathcal{L}_{\mathfrak{N}}(G) = \mathcal{L}_{\Delta}(G)$, where Δ is the set of all central chief factors of G.*

Since clearly $\mathcal{L}_{\mathfrak{N}}(G) \subseteq \mathcal{L}_{sn}(G)$ and, in the general case, the lattices $\mathcal{L}_{\mathfrak{N}}(G)$ and $\mathcal{L}_{sn}(G)$ do not coincide, Theorem 1.5 allows us to strengthen the following known result.

Corollary 1.6 (Ballester-Bolinches and Esteban-Romero [2]). *If G is soluble and $A/A_G \leq Z_{\infty}(G/A_G)$ for every subnormal subgroup A of G, then G is a PST-group.*

From Theorem 1.4, we also derive the following well-known result.

Corollary 1.7 (Zacher (see [3, Theorem 2.1.11])). *If G is a soluble PT-group, then G has an abelian normal Hall subgroup D of odd order such that G/D is nilpotent and every element of G induces a power automorphism in D.*

Finally, Theorem 1.5 and [3, Corollary 2.1.12] yield the following result.

Corollary 1.8. *Suppose that G is soluble. Then G is a PT-group if and only if $\mathcal{L}_{\mathfrak{N}}(G) = \mathcal{L}_{\Delta}(G)$, where Δ is the set of all central chief factors H/K of G, and every two subgroups A and B of any Sylow subgroup of G are permutable, that is, $AB = BA$.*

2. Proof of Theorem 1.1

Direct verification gives the following two lemmas.

Lemma 2.1. *Let N, M and $K < H \leq G$ be normal subgroups of G, where H/K is a chief factor of G.*

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(1) *If $N \leq K$, then*

$$(H/K) \rtimes (G/C_G(H/K)) \simeq ((H/N)/(K/N)) \rtimes ((G/N)/C_{G/N}((H/N)/(K/N))).$$

(2) *If T/L is a chief factor of G and H/K and T/L are G-isomorphic, then $C_G(H/K) = C_G(T/L)$ and*

$$(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L)).$$

Lemma 2.2. *Let Δ be a G-closed set of chief factors of G. Let $K \leq H$, $K \leq V$, $W \leq V$ and $N \leq H$ be normal subgroups of G, where $H/K \leq Z_\Delta(G)$.*

(1) *$KN/K \leq Z_\Delta(G)$ if and only if $N/(K \cap N) \leq Z_\Delta(G)$.*
(2) *If $H/N \leq Z_\Delta(G)$, then $H/(K \cap N) \leq Z_\Delta(G)$.*
(3) *If $V/K \leq Z_\Delta(G)$, then $HV/K \leq Z_\Delta(G)$.*

Proof of Theorem 1.1. Let A and B be subgroups of G such that $A, B \in \mathcal{L}_\Delta(G)$ (respectively, $A, B \in \mathcal{L}_{\mathfrak{F}}(G)$).

Claim 1: $A \cap B \in \mathcal{L}_\Delta(G)$ (respectively, $A \cap B \in \mathcal{L}_{\mathfrak{F}}(G)$).

First note that $(A \cap B)_G = A_G \cap B_G$. On the other hand, from the G-isomorphism

$$(A^G \cap B^G)/(A_G \cap B^G) = (A^G \cap B^G)/(A_G \cap B^G \cap A^G) \simeq A_G(B^G \cap A^G)/A_G \leq A^G/A_G,$$

we see that $(A^G \cap B^G)/(A_G \cap B^G) \leq Z_\Delta(G)$ (respectively, $(A^G \cap B^G)/(A_G \cap B^G) \in \mathfrak{F}$ since $\mathfrak{F}$ is normally hereditary). Similarly, $(B^G \cap A^G)/(B_G \cap A^G) \leq Z_\Delta(G)$ (respectively, $(B^G \cap A^G)/(B_G \cap A^G) \in \mathfrak{F}$). Then

$$(A^G \cap B^G)/((A_G \cap B^G) \cap (B_G \cap A^G)) = (A^G \cap B^G)/(A_G \cap B_G) \leq Z_\Delta(G)$$

by Lemma 2.2(2) (respectively, $(A^G \cap B^G)/(A_G \cap B_G) \in \mathfrak{F}$ since $\mathfrak{F}$ is a formation). But $(A \cap B)^G \leq A^G \cap B^G$, so

$$(A \cap B)^G/(A_G \cap B_G) = (A \cap B)^G/(A \cap B)_G \leq Z_\Delta(G)$$

(respectively, $(A \cap B)^G/(A \cap B)_G \in \mathfrak{F}$). Therefore, $A \cap B \in \mathcal{L}_\Delta(G)$ (respectively, $A \cap B \in \mathcal{L}_{\mathfrak{F}}(G)$).

Claim 2: Statement (ii) holds for G.

The set $\mathcal{L}_{\mathfrak{F}}(G)$ is partially ordered with respect to set inclusion and G is the greatest element of $\mathcal{L}_{\mathfrak{F}}(G)$. Moreover, Claim 1 implies that for any set $\{A_1, \dots, A_n\} \subseteq \mathcal{L}_{\mathfrak{F}}(G)$, we have $A_1 \cap \cdots \cap A_n \in \mathcal{L}_{\mathfrak{F}}(G)$. Therefore, the set $\mathcal{L}_{\mathfrak{F}}(G)$ is a lattice (a meet-sublattice of $\mathcal{L}(G)$ [8, page 7]).

Claim 3: Statements (i) and (iii) hold for G.

In view of Claim 1, we only need to show that $\langle A, B \rangle \in \mathcal{L}_\Delta(G)$ (respectively, $\langle A, B \rangle \in \mathcal{L}_{\mathfrak{F}}(G)$). From the G-isomorphisms

$$\begin{aligned} A^G(A_GB_G)/A_GB_G &\simeq A^G/(A^G \cap A_GB_G) = A^G/A_G(A^G \cap B_G) \\ &\simeq (A^G/A_G)/(A_G(A^G \cap B_G)/A_G), \end{aligned}$$

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we see that $A^G(A_GB_G)/A_GB_G \leq Z_\Delta(G)$ (respectively, $A^G(A_GB_G)/A_GB_G \in \mathfrak{F}$ since $\mathfrak{F}$ is closed under taking homomorphic images). Similarly, $B^G(A_GB_G)/A_GB_G \leq Z_\Delta(G)$ (respectively, $B^G(A_GB_G)/A_GB_G \in \mathfrak{F}$). Moreover,

$$A^GB^G/A_GB_G = (A^G(A_GB_G)/A_GB_G)(B^G(A_GB_G)/A_GB_G)$$

and so $A^GB^G/A_GB_G \leq Z_\Delta(G)$ by Lemma 2.2(3) (respectively, $A^GB^G/A_GB_G \in \mathfrak{F}$ since $\mathfrak{F}$ is a Fitting formation).

Next, we note that $\langle A, B\rangle^G = A^GB^G$ and $A_GB_G \leq \langle A, B\rangle_G$. It follows that $\langle A, B\rangle^G/\langle A, B\rangle_G \leq Z_\Delta(G)$ (respectively, $\langle A, B\rangle^G/\langle A, B\rangle_G \in \mathfrak{F}$ since $\mathfrak{F}$ is closed under taking homomorphic images). Hence, $\langle A, B\rangle \in \mathcal{L}_\Delta(G)$ (respectively, $\langle A, B\rangle \in \mathcal{L}_\mathfrak{F}(G)$). The theorem is proved. □

3. Proofs of Theorems 1.4 and 1.5

Remark 3.1. If $G \in \mathfrak{F}$, where $\mathfrak{F}$ is a formation, then every chief factor of G is $\mathfrak{F}$-central in G by a well-known result of Barnes and Kegel (see [5, Chapter IV, Lemma 1.5]). On the other hand, if $\mathfrak{F}$ is a saturated formation and every chief factor of G is $\mathfrak{F}$-central in G, then $G \in \mathfrak{F}$ by [9, Theorem 17.14].

Proof of Theorem 1.4. (i) Assume that the assertion is false and let G be a counterexample of minimal order. Let $D = G^\mathfrak{F}$ be the $\mathfrak{F}$-residual of G and let R be a minimal normal subgroup of G.

Claim 1: Statement (i) holds for G/R.

Let Δ^* be the set of all $\mathfrak{F}$-central chief factors of G/R. By [4, Proposition 2.2.8], $(G/R)^\mathfrak{F} = RG^\mathfrak{F}/R = RD/R \simeq D/(D \cap R)$ is soluble. Now let $A/R \in \mathcal{L}_\mathfrak{F}(G/R)$. From the G-isomorphism

$$A^G/A_G \simeq (A^G/R)/(A_G/R) = (A/R)^{G/R}/(A/R)_{G/R},$$

we see that $A^G/A_G \in \mathfrak{F}$, so $A \in \mathcal{L}_\mathfrak{F}(G)$ and, by hypothesis, $A \in \mathcal{L}_\Delta(G)$, that is, $A^G/A_G \leq Z_\Delta(G)$. By Lemma 2.1(1), it follows that

$$(A/R)^{G/R}/(A/R)_{G/R} \leq Z_{\Delta^*}(G/R).$$

Hence, $A/R \in \mathcal{L}_{\Delta^*}(G/R)$. Therefore, the hypothesis holds for G/R, so we have established Claim 1 by the choice of G.

Claim 2: D is nilpotent.

Assume that this is false. Claim 1 implies that $(G/R)^\mathfrak{F} = RD/R \simeq D/(R \cap D)$ is nilpotent. Therefore, if $R \nleq D$, then $D \simeq D/(R \cap D) = D/1$ is nilpotent. Consequently, every minimal normal subgroup N of G is contained in D and D/N is nilpotent. Hence, R is abelian. If $N \neq R$, then $D \simeq D/1 = D/((R \cap D) \cap (N \cap D))$ is nilpotent. Therefore, R is the unique minimal normal subgroup of G and $R \nleq \Phi(G)$ by [5, Chapter A, Lemma 13.2]. Hence, $R = C_G(R)$ by [5, Chapter A, Theorem 15.6]. If $|R|$ is a prime,

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then $G/R = G/C_G(R)$ is cyclic and so $R = D$ is nilpotent. Thus, $|R|$ is not a prime. Let V be a maximal subgroup of R. Then $V_G = 1$ and $V^G = R \in \mathcal{L}_{\mathfrak{F}}(G)$ since $\mathfrak{F}$ contains all nilpotent groups. Therefore, $V \in \mathcal{L}_{\Delta}(G)$. Hence, $V^G/V_G = R/1$ is $\mathfrak{F}$-central in G and so $G/R = G/C_G(R) = G/D$, which implies that $D = R$ is nilpotent, which is a contradiction. This proves Claim 2.

Claim 3: Every subgroup V of D containing $\Phi(D)$ is normal in G.

Let $V/\Phi(D)$ be a maximal subgroup of $D/\Phi(D)$. Suppose that $V/\Phi(D)$ is not normal in $G/\Phi(D)$. Then $V^G = D$ and $V \in \mathcal{L}_{\mathfrak{F}}(G) = \mathcal{L}_{\Delta}(G)$ by Claim 2. Hence, $D/V_G \leq Z_{\Delta}(G)$ and so $G/V_G \in \mathfrak{F}$ by Remark 3.1. But then $D \leq V_G < D$. This contradiction shows that $V/\Phi(D)$ is normal in $G/\Phi(D)$. Since $D/\Phi(D)$ is the direct product of elementary abelian Sylow subgroups of $D/\Phi(D)$, every subgroup of $D/\Phi(D)$ can be written as the intersection of some maximal subgroups of $D/\Phi(D)$. Hence, we have Claim 3.

Claim 4: Every chief factor of G below D is cyclic.

This follows from Claim 3 and [5, Chapter IV, Theorem 6.7].

Claim 5: D is a Hall subgroup of G.

Suppose that this assertion is false and let P be a Sylow p-subgroup of D such that $1 < P < G_p$ for some prime p and some Sylow p-subgroup G_p of G. Then p divides $|G : D|$.

(a) *$D = P$ is a minimal normal subgroup of G.*

Let N be a minimal normal subgroup of G contained in D. Then N is a q-group for some prime q and NP/N is a Sylow p-subgroup of D/N. Moreover, $D/N = (G/N)^{\mathfrak{F}}$ is a Hall subgroup of G/N by Claim 1 and p divides $|(G/N) : (D/N)| = |G : D|$. Hence, $N = P$ is a Sylow p-subgroup of D. Since D is nilpotent by Claim 2, a p-complement V of D is characteristic in D and so it is normal in G. Therefore, $V = 1$ and $D = N = P$.

(b) *If $R \neq D$, then $G_p = D \times R$. Hence, $O_{p'}(G) = 1$ and $R/1$ is $\mathfrak{F}$-central in G.*

Indeed, $DR/R \simeq D$ is a Sylow subgroup of G/R by Claim 1 and (a) and hence $G_pR/R = DR/R$, which implies that $G_p = D(G_p \cap R)$. But then $G_p = D \times R$ since $D < G_p$ by (a). Thus, $O_{p'}(G) = 1$. Finally, from the G-isomorphism $DR/D \simeq R$, it follows that $R/1$ is $\mathfrak{F}$-central in G.

(c) *$D = R \nleq \Phi(G)$ is the unique minimal normal subgroup of G.*

Suppose that $R \neq D$. Then $G_p = D \times R$ is an elementary abelian p-group by (a) and (b). Hence, $R = \langle a_1 \rangle \times \cdots \times \langle a_t \rangle$ for some elements $a_1, \ldots, a_t$ of order p. On the other hand, by Claim 3, $D = \langle a \rangle$, where $|a| = p$. Now let $Z = \langle aa_1 \cdots a_t \rangle$. Then $|Z| = p$ and $ZR = DR = G_p$ since $Z \cap D = 1 = Z \cap R$ and $|G_p : R| = p$. If $Z = Z_G$ is normal in G, then from the G-isomorphism $DZ/D \simeq Z$ it follows that $Z/1$ is $\mathfrak{F}$-central in G. Hence, $G_p = ZR \leq Z_{\Delta}(G)$ by Lemma 2.2(3) since $R/1$ is $\mathfrak{F}$-central in G by (b). In the case when $Z_G = 1$, by hypothesis $Z < Z^G \leq Z_{\Delta}(G)$ and again $G_p = ZR \leq Z_{\Delta}(G)$. But then $G \in \mathfrak{F}$ by Remark 3.1. This contradiction establishes (c).

(d) *G is supersoluble, so G_p is normal in G.*

Since $\mathfrak{F}$ is a saturated formation, $D \nleq \Phi(G)$ and so $D = C_G(D)$ by (c) and [5, Chapter A, Theorem 15.6]. On the other hand, $|D| = p$ by Claim 4 and (a), so $G/D = G/C_G(D)$ is cyclic. Hence, G is supersoluble and so for some prime q dividing $|G|$ a Sylow q-subgroup Q of G is normal in G. Now (b) implies that, in fact, $Q = G_p$. Hence, we have (d).

The final contradiction for Claim 5.

Since $\Phi(G_p)$ is characteristic in G_p, (d) implies that $\Phi(G_p)$ is normal in G and so $\Phi(G_p) \leq \Phi(G) = 1$. Hence, G_p is an elementary abelian p-group and it follows that $G_p = N_1 \times \cdots \times N_n$ for some minimal normal subgroups $N_1, \ldots, N_n$ of G by Maschke's theorem. But then $G_p = D$ by (c). This contradiction completes the proof of Claim 5.

Claim 6: Every subgroup H of D is normal in D.

If $H_G \neq 1$, then H/H_G is normal in $D/H_G = G^{\mathfrak{F}}/H_G$ by Claim 1 and so H is normal in D. Now suppose that $H_G = 1$. Then $H^G \leq Z_\Delta(G)$ by hypothesis and hence $G/C_G(H^G) \in \mathfrak{F}$ by [9, Theorem 17.14] and [5, Chapter IV, Theorem 6.10]. It follows that $D \leq C_G(H^G)$, which implies that H is normal in D.

Claim 7: $|D|$ is odd.

Suppose that 2 divides $|D|$. Then G has a chief factor D/K with $|D/K| = 2$ by Claims 2 and 4. But then $D/K \leq Z(G/K)$ and so $G/K \in \mathfrak{F}$ by Remark 3.1, which implies that $D \leq K < D$. This contradiction proves Claim 7.

Claim 8: The group D is abelian.

In view of Claims 6 and 7, D is a Dedekind group of odd order, giving Claim 8.

Conclusion of the proof of Theorem 1.4.

From Claims 3–8, it follows that Statement (i) holds for G, contrary to the choice of G. This final contradiction completes the proof of (i).

(ii) We have to show that if H is any subgroup of $D = G^{\mathfrak{N}}$, then $x \in N_G(H)$ for each $x \in G$. It is enough to consider the case when H is a p-group for some prime p. Moreover, in view of Statement (i), we can assume that x is a p'-element of G.

If $H_G \neq 1$, then $H/H_G \leq D/H_G = (G/H_G)^{\mathfrak{N}}$ and so the hypothesis holds for $(G/H_G, H/H_G)$ (see the proof of Claim 1). Thus, H/H_G is normal in G/H_G by induction, which implies that H is normal in G. If $H_G = 1$, then $H^G \leq Z_\infty(G) \cap O_p(G)$ since H is subnormal in G. But then $[H, x] = 1$. The theorem is proved. □

Proof of Theorem 1.5. First observe that if $\mathcal{L}_{\mathfrak{N}}(G) = \mathcal{L}_\Delta(G)$, then G is a *PST*-group by Theorem 1.4 and [3, Theorem 2.1.8].

Now assume that G is a soluble *PST*-group and let $A \in \mathcal{L}_{\mathfrak{N}}(G)$, that is, A^G/A_G is nilpotent. Then A is subnormal in G and so $A/A_G \leq Z_\infty(G/A_G)$ by [2, Corollary 2] (see also [3, Theorem 2.4.4]), which implies that $A^G/A_G \leq Z_\infty(G/A_G)$. Hence, $A \in \mathcal{L}_\Delta(G)$, so $\mathcal{L}_{\mathfrak{N}}(G) \subseteq \mathcal{L}_\Delta(G)$. The inverse inclusion follows from the fact that if $A \in \mathcal{L}_\Delta(G)$, then $A^G/A_G \leq Z_\infty(G/A_G) \leq F(G/A_G)$. The theorem is proved. □

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