SOME CLASSES OF INDECOMPOSABLE VARIETIES OF GROUPS

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(Received 14 August 1967)

1. Introduction

A variety of groups is an equationally defined class of groups: equivalently, it is a class of groups closed under the operations of taking cartesian products, subgroups, and quotient groups. If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, then $\mathfrak{U}\mathfrak{V}$ is the class of all groups G with a normal subgroup N in $\mathfrak{U}$ such that G/N is in $\mathfrak{V}$; $\mathfrak{U}\mathfrak{V}$ is a variety, called the product of $\mathfrak{U}$ and $\mathfrak{V}$. We denote by $\mathfrak{E}$ the variety generated by the unit group, and by $\mathfrak{O}$ the variety of all groups. We say that a variety $\mathfrak{V}$ is indecomposable if $\mathfrak{V} \neq \mathfrak{E}$, $\mathfrak{V} \neq \mathfrak{O}$, and $\mathfrak{V}$ cannot be written as a product $\mathfrak{X}\mathfrak{Y}$, with both $\mathfrak{X} \neq \mathfrak{E}$ and $\mathfrak{Y} \neq \mathfrak{E}$. One of the basic results in the theory of varieties of groups is that the set of varieties, excluding $\mathfrak{O}$, and with multiplication of varieties as above, is a free semigroup, freely generated by the indecomposable varieties. Thus one would like to be able to decide whether a given variety is indecomposable or not. In connection with this question, Hanna Neumann raises the following problem (as part of Problem 7 in her book [7]):

PROBLEM 1. *If* $\mathfrak{U} \nleqq \mathfrak{V}$, *and* $\mathfrak{V} \nleqq \mathfrak{U}$, *prove that* $[\mathfrak{U}, \mathfrak{V}]$ *is indecomposable unless both* $\mathfrak{U}$ *and* $\mathfrak{V}$ *have a common non-trivial right hand factor.*

(If G is an arbitrary group, and $\mathfrak{U}$ any variety, denote by $U(G)$ the intersection of all normal subgroups of G whose quotient group is in $\mathfrak{U}$: clearly $G/U(G) \in \mathfrak{U}$, and $U(G)$ is the smallest normal subgroup of G with this property. Then $[\mathfrak{U}, \mathfrak{V}]$ is the variety of all groups G for which $U(G)$ and $V(G)$ centralize each other.)

In this paper, we solve Problem 1 for a class of varieties which includes many of the well known varieties. To state our theorem, we need some notation. Following Philip Hall, we denote by $\mathscr{F}$ the class of all finite groups, and by $\mathscr{N}$ the class of all nilpotent groups: then $\mathscr{F}\mathscr{N}$ denotes the class of all groups which have a finite normal subgroup whose quotient group is nilpotent. The main result of this paper is then

THEOREM 1. *Suppose that* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are varieties each of which can be*

[1] The author is a Fulbright-Hays scholar.

https://doi.org/10.1017/S1446788700007308 Published online by Cambridge University Press

generated by a group in $\mathscr{F}\mathscr{N}$. Then if $\mathfrak{U} \neq \mathfrak{V}$, $[\mathfrak{U}, \mathfrak{V}]$ is indecomposable unless both $\mathfrak{U}$ and $\mathfrak{V}$ have a common non-trivial right hand factor.

Theorem 1 has a couple of special cases that are worth noting. Firstly, if both $\mathfrak{U}$ and $\mathfrak{V}$ are nilpotent, then they are indecomposable ([7] Theorem 24.34), and hence we have

COROLLARY 1. *If both $\mathfrak{U}$ and $\mathfrak{V}$ are nilpotent, and $\mathfrak{U} \neq \mathfrak{V}$, then $[\mathfrak{U}, \mathfrak{V}]$ is indecomposable.*

Another special case, which is a partial result on the way to Theorem 1, is

THEOREM 2. *If either $\mathfrak{U}$ or $\mathfrak{V}$ cannot be generated by a finite group (but each can be generated by a group in $\mathscr{F}\mathscr{N}$), then $[\mathfrak{U}, \mathfrak{V}]$ is indecomposable.*

2. Notation and preliminaries

The main tool used in the proof of Theorem 1 is the (standard) wreath product of groups, and we will assume familiarity with the construction and basic properties of this wreath product: for a detailed description, see [7] Section 2.2. We will also adopt the notation used there.

Other notation is in general standard. We denote the fact that H is a subgroup of G by $H \leqq G$: if H is a proper subgroup, by $H < G$. When H is normal in G, we put $H \trianglelefteq G$. As usual, $[x, y] = x^{-1}y^{-1}xy$: if $H \leqq G$, $K \leqq G$, then $[H, K]$ is the subgroup of G generated by all $[h, k]$, $h \in H$, $k \in K$. If S is a subset of G, the centraliser of S in G is denoted by $C_G(S)$. If S is a subset of G, the subgroup of G generated by S is denoted by $\langle S \rangle$: if $S = \{x_1, \cdots, x_n\}$, $\langle S \rangle = \langle x_1, \cdots, x_n \rangle$. The centre of a group G is denoted by $\zeta(G)$.

If $\mathfrak{U}$, $\mathfrak{V}$ are varieties, then $\mathfrak{U} \leqq \mathfrak{V}$ means that $\mathfrak{U}$ is a subvariety of $\mathfrak{V}$: $\mathfrak{U} < \mathfrak{V}$ means that $\mathfrak{U}$ is a proper subvariety of $\mathfrak{V}$. The union $\mathfrak{U} \cup \mathfrak{V}$ of two varieties is the variety generated by the union of the classes $\mathfrak{U}$ and $\mathfrak{V}$: the intersection $\mathfrak{U} \cap \mathfrak{V}$ is just the class of all groups G such that $G \in \mathfrak{U}$ and $G \in \mathfrak{V}$. If $\mathfrak{C}$ is a class of groups, then var $\mathfrak{C}$ denotes the variety generated by $\mathfrak{C}$: if $\mathfrak{C}$ consists of a single group G, var G = var $\mathfrak{C}$. If G is finite, we call var G a Cross variety. If $\mathfrak{U}$ is locally finite, there is a smallest integer e such that the exponent of any group in $\mathfrak{U}$ divides e: we call e the exponent of $\mathfrak{U}$. $\mathfrak{A}_n$ will denote the variety of all abelian groups of exponent dividing n: $\mathfrak{A}$ denotes the variety of all abelian groups. If p is a prime, $\mathfrak{Q}_p$ denotes the variety of all groups which are central extensions of elementary abelian p-groups by elementary abelian p-groups and are of exponent dividing p^2.

If G is a finite group, and $1 \leqq N \trianglelefteq H \leqq G$, we say that H/N is a factor of G: if either $1 \neq N$ or $H \neq G$, we say that H/N is a proper factor. If G is not in the variety generated by its proper factors, we say that G

https://doi.org/10.1017/S1446788700007308 Published online by Cambridge University Press

is critical. If G is critical, then G has a unique minimal normal subgroup, called the monolith of G, and denoted by σG.

Next, some observations about varieties which can be generated by a group in $\mathscr{F}\mathscr{N}$. In [4], Philip Hall showed that $\mathscr{F}\mathscr{N}$ was also the class of all groups in which some finite term of the upper central series was of finite index: it is this characterisation of the class $\mathscr{F}\mathscr{N}$ that seems more useful from the viewpoint of varieties. Groups in $\mathscr{F}\mathscr{N}$ are closely related to both finite and nilpotent groups, and in fact the varieties they generate enjoy many of the pleasant properties of nilpotent and Cross varieties.

LEMMA 2.1. *Suppose that* $\mathfrak{U}$ *and* $\mathfrak{V}$ *are varieties which can be generated by a group in* $\mathscr{F}\mathscr{N}$. *Then we have*

(a) $\mathfrak{U} \cup \mathfrak{V}$ *can also be generated by a group in* $\mathscr{F}\mathscr{N}$.
(b) $\mathfrak{U}$ *and all its subvarieties are generated by finitely generated groups.*
(c) *Every finitely generated group in* $\mathfrak{U}$ *is in* $\mathscr{F}\mathscr{N}$.
(d) $\mathfrak{U}$ *is generated by its finite groups.*
(e) *There is a bound on the class of nilpotent groups in* $\mathfrak{U}$.
(f) *There is a bound on the minimal number of generators of chief factors of finite groups in* $\mathfrak{U}$.

These results are either easy to prove or are contained in [2].

We often need the following fact that, though well known, does not seem to be readily available in the literature. Denote by $C(p, q)$ the critical group with an elementary abelian normal p-subgroup, with quotient group of order q, p, q distinct primes: we have var $C(p, q) = \mathfrak{A}_p\mathfrak{A}_q$.

LEMMA 2.2. *If* $\mathfrak{U}$ *is a variety which contains non-abelian finite groups, then it contains either a non-abelian group of order* p^3 *for some prime* p, *or a non-nilpotent group* $C(p, q)$, p, q *distinct primes. If* $\mathfrak{U}$ *contains non-nilpotent finite groups, it contains a* $C(p, q)$ *for distinct primes* p, q.

PROOF. We give a sketch of the proof. There are two cases to consider.

If every non-abelian finite group in $\mathfrak{U}$ is nilpotent, $\mathfrak{U}$ contains finite non-abelian groups which are nilpotent of class two. Let G be a non-abelian nilpotent finite group in U such that every proper factor of G is abelian. Firstly, G will be a p-group for some prime p, and if $x, y \in G$ are such that $[x, y] \neq 1$, then we must have $\langle x, y\rangle = G$. Also, every proper homomorphic image of G is abelian, and so, by Theorem 5 of M. F. Newman [9], G is either non-abelian of order p^3 and exponent p, in which case we are finished, or G is isomorphic to a group of the form

$$\{a, b, z : a^p = b^p = z,\ z^{p^{n-1}} = [a, b],\ [a, b]^p = 1\},\ n \geqq 1.$$

In this case, var G is defined by the laws $x^{p^{n+1}} = 1$, $[x, y]^p = 1$, $[x, y, t] = 1$,

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and any non-abelian group of order p^3 satisfies these laws, and so is contained in var G and hence in $\mathfrak{U}$.

Now, suppose that $\mathfrak{U}$ contains a non-nilpotent finite group. Let G be a non-nilpotent finite group of minimal order in $\mathfrak{U}$. Then every subgroup of G is nilpotent, and so G is soluble (L. Redei [11]). Let N be a minimal normal subgroup of $G : N$ is an elementary abelian p-group for some prime p. Also, N cannot be in the centre of G, for G/N is nilpotent. It follows that there is an element $x \in G$ of prime order q such that $x \notin C_G(N)$. But then $\langle N, x\rangle = N\langle x\rangle$ is non-nilpotent, and so $G = N\langle x\rangle$. It now follows that $G \cong C(p, q)$ ([10] p. 364).

Another trivial but important fact is

LEMMA 2.3. *For any varieties* $\mathfrak{U}$, $\mathfrak{V}$, *we have*

$$[\mathfrak{U}, \mathfrak{V}] \leqq [\mathfrak{U} \cup \mathfrak{V}, \mathfrak{U} \cup \mathfrak{V}] = \mathfrak{A}(\mathfrak{U} \cup \mathfrak{V}).$$

Thus, for $G \in [\mathfrak{U}, \mathfrak{V}]$, *there is an abelian normal subgroup* N *of* G *such that* $G/N \in \mathfrak{U} \cup \mathfrak{V}$.

LEMMA 2.4. *If* G *is a non-abelian finite group,* H *a finite group, and if* A *is a maximal abelian normal subgroup of* $G \operatorname{wr} H$, *then there is a maximal abelian normal subgroup* A_0 *of* G *such that if* $B(= G^H)$ *is the base group of* G *wr* H,

$$A = \{f \in B : f(h) \in A_0, \text{ for all } h \in H\}.$$

PROOF. We claim firstly that $A \leqq B$. For suppose that $fh \in A$, $f \in B$, $h \in H$, and $h \neq 1$. Let $A_1 = A \cap B$, and for $k \in H$ the epimorphism $\pi_k : B \to G$ be defined by

$$f\pi_k = f(k).$$

Then $A_1\pi_k$ is abelian: in particular $A_1\pi_1$ is abelian. Since G is non-abelian, there is an $x \in G$, such that $x \notin A_1\pi_1$. Define $g \in B$ by $g(1) = x^{-1}$, $g(k) = 1$, $k \neq 1$. Now since both A and B are normal, $[g, fh] \in A \cap B = A_1$. Since $h \neq 1$, $(f^{-1}gf)^h(1) = f^{-1}gf(h^{-1}) = 1$, from the definition of g, and so $[g, fh](1) = x$: i.e. $x \in A_1\pi_1$, a contradiction. Thus $A \leqq B$.

Now, it is easy to check that $A\pi_h = A\pi_1$ for all $h \in H$. Thus if A_0 is a maximal abelian normal subgroup of G containing $A\pi_1$,

$$A \leqq \{f \in B : f(h) \in A_0, h \in H\}.$$

But clearly

$$\{f \in B : f(h) \in A_0, h \in H\}$$

is an abelian normal subgroup of G *wr* H, and hence the result follows.

Then we have as a consequence of Lemma 2.4 and Theorems 22.11 and 22.12 of [7] the following lemma.

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LEMMA 2.5. *If G is a non-abelian finite group, H a finite group, A a maximal abelian normal subgroup of $W = G \operatorname{wr} H$, B the base group of W, then W/A contains a factor isomorphic to $C_p \operatorname{wr} H$, where C_p is a cyclic group of order p, for all primes p dividing $|B/A|$.*

Finally in this section we prove:

LEMMA 2.6. *If $\mathfrak{U}$ and $\mathfrak{B}$ are varieties, such that $\mathfrak{U} \neq \mathfrak{B}$, and both $\mathfrak{U}$ and $\mathfrak{B}$ are generated by their finite groups, then*

$$\mathfrak{A}(\mathfrak{U} \cap \mathfrak{B}) \neq [\mathfrak{U}, \mathfrak{B}].$$

PROOF. First, we show that it is sufficient to prove the lemma for $\mathfrak{U} < \mathfrak{B}$, and $\mathfrak{B}$ generated by its finite groups. For suppose $\mathfrak{U} \nless \mathfrak{B}$, $\mathfrak{B} \nless \mathfrak{U}$, and $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{A}(\mathfrak{U} \cap \mathfrak{B})$. Then

$$\begin{aligned} \mathfrak{A}(\mathfrak{U} \cap \mathfrak{B}) &= [\mathfrak{U} \cap \mathfrak{B}, \mathfrak{U} \cap \mathfrak{B}] \\ &\leqq [\mathfrak{U} \cap \mathfrak{B}, \mathfrak{B}] \\ &\leqq [\mathfrak{U}, \mathfrak{B}] \\ &= A(\mathfrak{U} \cap \mathfrak{B}) \end{aligned}$$

and so $[\mathfrak{U} \cap \mathfrak{B}, \mathfrak{B}] = \mathfrak{A}((\mathfrak{U} \cap \mathfrak{B}) \cap \mathfrak{B})$.

Now, if $\mathfrak{U} < \mathfrak{B}$, and $\mathfrak{B}$ is generated by its finite groups, then there is a finite group G of minimal order such that $G \in \mathfrak{B}$, $G \notin \mathfrak{U}$. Then we have that G is critical and $G/\sigma G \in \mathfrak{U}$. If σG is non-abelian, then clearly $G \notin [\mathfrak{U}, \mathfrak{U}]$. Hence σG is abelian: and so — as a minimal normal subgroup of G — elementary abelian of exponent p for some prime p: hence $G \in \mathfrak{A}_p\mathfrak{U}$. Let $\mathfrak{B}_1$ be the variety generated by G. Then for some positive integer n, if H is the relatively free group of rank n of $\mathfrak{B}_1$, $U(H)$ is a non-cyclic elementary abelian p-group. Let F be the absolutely free group of rank n,

$$U/V_1(F) = U(F/V_1(F)).$$

Then $F/V_1(F)$ is a finite group ([8] Theorem 14.2), and so by Schreier's theorem U is an absolutely free group of finite rank. To complete the proof of Lemma 2.6, it is now sufficient to prove:

LEMMA 2.7. *Let F be an absolutely free group of finite rank, N a normal subgroup of F such that F/N is a non-cyclic elementary abelian p-group for some prime p. Then $F/[F, N]$ is non-abelian.*

PROOF.[2] With p as in the statement of the lemma, we have $N > Q_p(F)$. Put $H = F/Q_p(F)$, $M = N/Q_p(F)$: then it is sufficient to prove that $H/[H, M]$ is non-abelian. Observe that since F is of finite rank, n say, H is finite. Now $H/\Phi(H) = M/\Phi(H) \times L/\Phi(H)$, where $\Phi(H)$ is the Frattini

[2] This proof was suggested to me by Professor G. Baumslag.

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subgroup of H. Let $x_1\Phi(H), \cdots, x_k\Phi(H)$ be a minimal set of generators for $L/\Phi(H)$, and $y_1\Phi(H), \cdots, y_{n-k}\Phi(H)$ be a minimal set of generators for $M/\Phi(H)$. Since H is a finite relatively free group of rank n, and H is generated by $\{x_1, \cdots, x_k, y_1, \cdots, y_{n-k}\} = S$, it is freely generated by S. Thus the commutators $[x_i, x_j]$, $i \neq j$, $1 \leqq i, j \leqq k$, $[x_i, y_j]$, $1 \leqq i \leqq k$, $1 \leqq j \leqq n-k$, $[y_i, y_j]$, $i \leqq j$, $1 \leqq i, j \leqq n-k$, are all independent. But $[H, M]$ is generated by $[x_i, y_j]$, $1 \leqq i \leqq k$, $1 \leqq j \leqq n-k$, $[y_i, y_j]$, $i \neq j$, $1 \leqq i, j \leqq n-k$, and since H/M was non-cyclic, $k \geqq 2$. Hence $[x_1, x_2] \notin [H, M]$, and Lemma 2.7 is proved.

3. The proof of Theorem 1

Suppose that $\mathfrak{U}$ and $\mathfrak{B}$ are varieties which can be generated by a group in $\mathscr{F}\mathscr{N}$ and $\mathfrak{U} \neq \mathfrak{B}$. Then, using Lemmas 2.1 and 2.3, finitely generated groups in $[\mathfrak{U}, \mathfrak{B}]$ are abelian-by-nilpotent-by-finite, and so, as a consequence of Theorem 1 of P. Hall [5], are residually finite. Hence $[\mathfrak{U}, \mathfrak{B}]$ is generated by its finite groups.

The proof is broken up into several steps, which we number consecutively. The first step gives some necessary conditions which varieties $\mathfrak{X}$, $\mathfrak{Y}$ must satisfy if $[\mathfrak{U}, \mathfrak{B}]$ is to equal $\mathfrak{X}\mathfrak{Y}$.

3.1 *Suppose that* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are varieties: then* $[\mathfrak{U}, \mathfrak{B}] \neq \mathfrak{X}\mathfrak{Y}$ *if any of the following conditions hold:*

(a) $\mathfrak{X}$ *is abelian,*

(b) $\mathfrak{X}$ *contains a non-abelian group of order* p^3 *for some prime* p, *and* $\mathfrak{A}_p \leqq \mathfrak{Y}$,

(c) $\mathfrak{A}_p\mathfrak{A}_q \leqq \mathfrak{X}$, $\mathfrak{A}_q \leqq \mathfrak{Y}$, *for distinct primes* p, q,

(d) $\mathfrak{X}$ *contains a non-abelian finite simple group* G, *and for some prime* p *dividing* $|G|$, $\mathfrak{A}_p \leqq \mathfrak{Y}$.

PROOF. (a) Suppose that $\mathfrak{X}$ is abelian and $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$. Then $\mathfrak{Y} \leqq \mathfrak{U} \cap \mathfrak{B}$ ([7] Theorem 24.31), and so

$$\begin{aligned}[\mathfrak{U}, \mathfrak{B}] &= \mathfrak{X}\mathfrak{Y} \\ &\leqq \mathfrak{A}(\mathfrak{U} \cap \mathfrak{B}) \\ &\leqq [\mathfrak{U}, \mathfrak{B}],\end{aligned}$$

giving $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{A}(\mathfrak{U} \cap \mathfrak{B})$, contradicting Lemma 2.6.

(b) Suppose $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$, and that $G \in \mathfrak{X}$ is a non-abelian group of order p^3, and C_p is a cyclic group of order p. Then the set

$$\{G \text{ wr } C_p^n : n = 1, 2, \cdots\} \subseteq \mathfrak{X}\mathfrak{Y}.$$

But then, applying Lemma 2.5,

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$$\{C_p \operatorname{wr} C_p^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{U} \cup \mathfrak{B}.$$

But $C_p \operatorname{wr} C_p^n$ is nilpotent of class $n(p-1)+1$ ([6] Theorem 5.1), and so $\mathfrak{U} \cup \mathfrak{B}$ contains nilpotent groups of arbitrarily large class, contradicting Lemma 2.1 (e).

(c) Again, suppose $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$, and let $C(p, q) \in \mathfrak{X}$, $C_q \in \mathfrak{Y}$, for distinct primes p, q. Then the set $\{C(p, q) \operatorname{wr} C_q^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{X}\mathfrak{Y}$. Now $C(p, q)$ has a unique maximal abelian normal subgroup, which has index q: hence we may conclude from Lemma 2.5 that

$$\{C_q \operatorname{wr} C_q^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{U} \cup \mathfrak{B},$$

again giving a contradiction.

(d) If $G \in \mathfrak{X}$ is a non-abelian finite simple group, p a prime dividing $|G|$ such that $C_p \in \mathfrak{Y}$, then $\{G \operatorname{wr} C_p^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{X}\mathfrak{Y}$. But $G \operatorname{wr} C_p^n$ has no non-unit abelian normal subgroups, and so, using Lemma 2.5 again,

$$\{C_p \operatorname{wr} C_p^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{U} \cup \mathfrak{B}$$

if $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$: again giving a contradiction.

Now suppose that $[\mathfrak{U}, \mathfrak{B}]$ is decomposable: that is $[\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$ for some $\mathfrak{X}$, $\mathfrak{Y}$. Then we have that $\mathfrak{X}$ is non-abelian, and $\mathfrak{Y} \leqq \mathfrak{U} \cap \mathfrak{B}$. If $\mathfrak{Y}$ is not locally finite, then $\mathfrak{A} \leqq \mathfrak{Y}$, and we see from Lemma 2.2 that $\mathfrak{X}$ and $\mathfrak{Y}$ must satisfy either (b) or (c) of Lemma 3.1, giving a contradiction. Thus $\mathfrak{Y}$ is locally finite: let the exponent of $\mathfrak{Y}$ be e.

3.2 *Suppose that G is a finite group in $\mathfrak{X}$. Then G has an abelian normal subgroup N such that $(e, |G/N|) = 1$, and for some integer k, $|N|$ divides e^k.*

Proof. If H is any subgroup of G for which there is an integer k such that $|H|$ divides e^k, then H is abelian: for otherwise, we may conclude from Lemma 2.2 that $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy either (b) or (c) of 3.1, a contradiction.

We now use induction on the length of a chief series of G. If the length is one, the result is trivial. Suppose now G has a chief series of length n, and the result is true for groups in $\mathfrak{X}$ with a chief series of length $n-1$. Let M be a minimal normal subgroup of G: then G/M has a chief series of length $n-1$. Hence G/M has an abelian normal subgroup N/M satisfying the requirements of the lemma. Now, either $|M| = p^l$ for some prime p dividing e, or $|M|$ is prime to e. In the first case, N is abelian, from the first paragraph of the proof, and we are finished. For the second case, M is complemented in N, by L say, using the Schur-Zassenhaus theorem: let $C = C_L(M)$. If $C \neq L$, then N is not nilpotent, and so by Lemma 2.2 and the assumption on L contains a factor isomorphic to $C(p, q)$, for p, q distinct primes, with q dividing e. But then $\mathfrak{A}_p\mathfrak{A}_q \leqq \mathfrak{X}$, $\mathfrak{A}_q \leqq \mathfrak{Y}$, a contradiction. Hence $C = L$,

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and so $N = M \times L$, and now L has the properties required of 'N' in the statement 3.2.

We now prove:

3.3 $\mathfrak{Y}$ *is abelian.*

Proof. Since $\mathfrak{X}$ is non-abelian it contains a group G isomorphic to either a non-abelian group of order p^3 or a $C(q, p)$, p, q distinct primes: it is an immediate consequence of 3.2 that p does not divide e. If $\mathfrak{Y}$ is non-abelian it contains a non-abelian group H. Then the set

$$\{G \operatorname{wr} H^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{X}\mathfrak{Y}.$$

But then, using Lemma 2.5, we have that

$$\{C_p \operatorname{wr} H^n : n = 1, 2, \cdots\} \subseteqq \mathfrak{U} \cup \mathfrak{V}.$$

Now the base group of $C_p \operatorname{wr} H^n$ may be regarded as a vector space over $GF(p)$, the field of p elements, on which H^n acts as a group of linear transformations. Since p is prime to $|H|$, the base group is completely reducible, and it contains irreducible components of degree at least 2^n. These irreducible components may then be thought of as chief factors of $C_p \operatorname{wr} H^n$, and so the set $\{C_p \operatorname{wr} H^n : n = 1, 2, \cdots\}$ contains groups with chief factors having an arbitrarily large minimal number of generators, contradicting Lemma 2.1 (f). Hence $\mathfrak{Y}$ is abelian.

Now $\mathfrak{U}$ and $\mathfrak{V}$ can be generated by finitely generated groups, G and H say. Now we can choose G and H such that $\mathfrak{Y} = \operatorname{var} G/Y(G) = \operatorname{var} H/Y(H)$. Since $\mathfrak{Y}$ is abelian of finite exponent, $G/Y(G)$ and $H/Y(H)$ are finite. We have further:

3.4 *With G and H as above, $Y(G)$ and $Y(H)$ are finite, and*

$$(|G/Y(G)|, |Y(G)|) = (|H/Y(H)|, |Y(H)|) = 1.$$

Proof. By the symmetry of the situation, it is enough to prove 3.4 for G. Note that if $Y(G)$ is of finite exponent, it is finite. Hence if $Y(G)$ is not finite, then $\mathfrak{A} \leqq \mathfrak{U}$. Let B be a free nilpotent group of class two and rank 2, generated by x and y: then $B \in [\mathfrak{U}, \mathfrak{V}]$. Also $B_1 = \langle x^e, y^e \rangle \leqq Y(B)$, and by [1] Theorem 1, B_1 is also a free nilpotent group of class 2 and rank 2. If now p is any prime dividing e, B_1 has non-abelian factors of p-power order. But $B_1 \in \mathfrak{X}$, and so $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy condition (b) of 3.1, a contradiction. Thus $Y(G)$ is finite.

Put $Y(G) = A$: by 3.2, there is an abelian normal subgroup N of A such that $(|A/N|, e) = 1$, and $|N|$ divides e^k for some integer k. Suppose that $N \neq 1$. There are two cases to consider

(i) $C_A(N) \neq A$. Then there is an element x of prime order such that

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$x \notin C_A(N)$. Then from Lemma 2.2, $C(p, q) \in \text{var}(N\langle x\rangle) \leqq \mathfrak{X} \cap \mathfrak{U}$ for primes p, q: from the choice of $N\langle x\rangle$, it follows that p divides e, and q does not divide e. Now (i) divides into two subcases. Firstly, suppose that $V(C(p, q)) < C(p, q)$. Then, if $G \in \mathfrak{Q}_p\mathfrak{A}_q$, $U(G) \leqq \zeta(V(G))$, and so $\mathfrak{Q}_p A_q \leqq [\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y}$. If $|\sigma C(p, q)| = p^t$, let F be the free group of $\mathfrak{Q}_p$ of rank $2t$ freely generated by $x_1, \cdots, x_t, y_1, \cdots, y_t$: let

$$H_1 = \langle x_1\zeta(F), \cdots, x_t\zeta(F)\rangle,$$
$$H_2 = \langle y_1\zeta(F), \cdots, y_t\zeta(F)\rangle.$$

On each of H_1, H_2, define the action of C_q, the cyclic group of order q, by the action of $C(p, q)/\sigma C(p, q)$ on $\sigma C(p, q)$, and extend this action to F. Let $G = FC_q$: from its definition it follows that G has no quotient groups of p-power order, and so we may conclude that $G \in \mathfrak{X}$. But G has a non-abelian Sylow p-subgroup, and so $\mathfrak{X}$, $\mathfrak{Y}$ satisfy (b) of 3.1, a contradiction.

Thus, suppose $V(C(p, q)) = C(p, q)$: it follows that q does not divide the exponent of $\mathfrak{B}$. Further $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_p \leqq \mathfrak{X}\mathfrak{Y}$, and $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_p$ cannot be generated by a finite group ([7] Theorem 24.62). Since G is finite, $\mathfrak{U}$ can be generated by a finite group, and so there is a critical group $F \in \mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_p$ such that $1 < M < K \lhd F$, with M, the unique minimal normal subgroup of F, and F/K elementary abelian p-groups, and K/M an elementary abelian q-group. Further, F has the property that $[M, K] \neq 1$, and $U(F) \geqq M$, $U(F) \geqq K$. Hence $[U(F), V(F)] \neq 1$, a contradiction.

(ii) $C_A(N) = A$. Then $A = N \times N^*$, where $N^* \cong A/N$, $N^* < G$. Thus G/A and G/N^* generate different varieties. We have var $(G/A) = \mathfrak{Y}$: put var $(G/N^*) = \mathfrak{Y}_1$. If F is a free group of finite rank such that $Y(F)/Y_1(F)$ is non-cyclic, we have, using Lemma 2.7, that

$$F > Y(F) > Y_1(F) > [Y_1(F), Y(F)],$$

and $Y(F)/[Y_1(F), Y(F)]$ contains non-abelian factors of p-power order for some prime p dividing e. But

$$F/[Y_1(F), Y(F)] \in [\mathfrak{U}, \mathfrak{Y}] \leqq [\mathfrak{U}, \mathfrak{B}] = \mathfrak{X}\mathfrak{Y},$$

and so $Y(F)/[Y_1(F), Y(F)] \in \mathfrak{X}$, again giving that $\mathfrak{X}$ and $\mathfrak{Y}$ satisfy (b) of 3.1, a contradiction.

Thus $N = 1$, and 3.4 is proved.

From 3.4 it follows immediately that if either of G or H is infinite, $[\mathfrak{U}, \mathfrak{B}]$ is indecomposable, and so the proof of Theorem 2 is complete.

Now, with G and H as above, put $\mathfrak{U}_1 = \text{var}(Y(G))$, $\mathfrak{B}_1 = \text{var}(Y(H))$. Then $\mathfrak{U}_1$ and $\mathfrak{B}_1$ are locally finite of exponents prime to e. The next step is to prove

3.5 $[\mathfrak{U}_1, \mathfrak{B}_1] \leqq \mathfrak{X}$.

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PROOF. Since $[\mathfrak{U}_1, \mathfrak{V}_1]$ is generated by its finite groups, if $[\mathfrak{U}_1, \mathfrak{V}_1] \nless \mathfrak{X}$, there is a finite group A of minimal order such that $A \in [\mathfrak{U}_1, \mathfrak{V}_1]$, $A \notin \mathfrak{X}$. Observe that $[\mathfrak{U}_1, \mathfrak{V}_1] \leqq [\mathfrak{U}, \mathfrak{V}] = \mathfrak{X}\mathfrak{Y}$, and so $A \in \mathfrak{X}\mathfrak{Y}$. If A has order prime to e, then $A \in \mathfrak{X}$.

Since A is critical, if $U_1(A) \cap V_1(A) = 1$, then A has order prime to e, a contradiction. Now $N = U_1(A) \cap V_1(A)$ is an abelian normal subgroup of A: since A is critical it must be a p-group for some prime p, and since $A \notin \mathfrak{X}$, p must divide e. Further, if N_1 is the subgroup of N generated by the p^{th} powers of the elements of N, $A/N_1 \notin \mathfrak{X}$ and $N = \sigma A$. Also, $N \neq A$ since $A \notin \mathfrak{X}$. Since $Y(A) \neq 1$, $Y(A) \geqq N$. But then $Y(A)/N = Y(A/N)$, and since $A/N \in \mathfrak{U}_1 \cup \mathfrak{V}_1$, $Y(A/N) = A/N$. Hence $Y(A) = A$, and $A \in \mathfrak{X}$, a contradiction.

3.6 $[\mathfrak{U}_1\mathfrak{Y}, \mathfrak{V}_1\mathfrak{Y}] = [\mathfrak{U}, \mathfrak{V}]$.

PROOF. Using 3.5, [6] Theorem 21.23, and the fact that $\mathfrak{U} \leqq \mathfrak{U}_1\mathfrak{Y}$, $\mathfrak{V} \leqq \mathfrak{V}_1\mathfrak{Y}$, we have

$$\begin{aligned} [\mathfrak{U}, \mathfrak{V}] &= \mathfrak{X}\mathfrak{Y} \\ &\geqq [\mathfrak{U}_1, \mathfrak{V}_1]\mathfrak{Y} \\ &= [\mathfrak{U}_1\mathfrak{Y}, \mathfrak{V}_1\mathfrak{Y}] \\ &\geqq [\mathfrak{U}, \mathfrak{V}], \end{aligned}$$

and the result is proved.

We now want to show that in fact $\mathfrak{U} = \mathfrak{U}_1\mathfrak{Y}$, $\mathfrak{V} = \mathfrak{V}_1\mathfrak{Y}$. As a step in this direction, we prove:

3.7 $\mathfrak{U}_1$ *and* $\mathfrak{V}_1$ *are nilpotent.*

PROOF. Suppose that $\mathfrak{U}_1$ is not nilpotent. Then since $\mathfrak{U}_1$ is generated by its finite groups, we may conclude from Lemma 2.2 that $\mathfrak{A}_p\mathfrak{A}_q \leqq \mathfrak{U}_1$ for some distinct primes p, q. Also, if r is any prime dividing e, we have that $(p, r) = (q, r) = 1$, and $\mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r \leqq \mathfrak{U}_1\mathfrak{Y}$. Put $A_n = C(q, r)^n$: by a theorem of Gaschutz [3], A_n has a faithful irreducible representation over $GF(p)$. Let M_n be a vector space over $GF(p)$ on which A_n acts faithfully and irreducibly as a group of linear transformations. Put $B_n = M_nA_n$, the split extension of M_n by A_n: observe that $B_n \in \mathfrak{A}_p\mathfrak{A}_q\mathfrak{A}_r$. Then M_n is the unique minimal normal subgroup of B_n, and the minimal number of generators of M_n is at least 2^n. Hence, using Lemma 2.1 (f), there is an integer k such that $B_k \notin \mathfrak{U} \cup \mathfrak{V}$: $U(B_k) = M_k$, $V(B_k) \geqq M_k$, for $B_k/M_k \in \mathfrak{U}$, $B_k \notin \mathfrak{V}$, and M_k is the unique minimal normal subgroup of B_k.

Suppose that $V(B_k) > M_k$: then $C(q, r) \notin \mathfrak{V}$, but $\mathfrak{A}_r \leqq \mathfrak{V}$, and so $V(B_k) = M_kA'_k$. But then $[U(B_k), V(B_k)] \neq 1$. However, $U_1(Y(B_k)) = 1$, and so $[U_1(Y(B_k)), V_1(Y(B_k))] = 1$, contradicting 3.6.

Hence $V(B_k) = M_k$. If $|M_k| = p^t$, then $t \geqq 2$: let F be the free group

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of rank t of Q_p. On $F/\zeta(F)$, which is elementary abelian of order p^t, define the action of A_k by its action on M_k, and extend this action to give an automorphism group of F: put $D = FA_k$.

Then $V(D) = U(D) = F$, and so $[V(D), U(D)] \neq 1$. But

$$U_1(Y(D)) \leqq \zeta(F), \text{ and } V_1(Y(D)) \leqq F,$$

giving

$$[U_1(Y(D)), V_1(Y(D))] = 1,$$

contradicting 3.6 again.

Thus $\mathfrak{U}_1$ and similarly $\mathfrak{V}_1$ are nilpotent.

3.8 $\mathfrak{U} = \mathfrak{U}_1\mathfrak{Y}$, $\mathfrak{V} = \mathfrak{V}_1\mathfrak{Y}$.

Proof. Suppose that $\mathfrak{U} < \mathfrak{U}_1\mathfrak{Y}$. Then there is a group A of minimal order such that $A \in \mathfrak{U}_1\mathfrak{Y}$, $A \notin \mathfrak{U}$: A is critical, and $U(A) = \sigma A$. There are two cases to consider.

(i) $V(A) = 1$. Then let p be a prime which does not divide the exponent of $\mathfrak{U} \cup \mathfrak{V}$. By the theorem of Gaschutz [3], A has a faithful irreducible representation over $GF(p)$. Let M be a vector space over $GF(p)$ on which A acts faithfully and irreducibly as a group of linear transformations, and put $B = MA$, the split extension of M by A. Then

$$U_1(Y(B)) = V_1(Y(B)) = M,$$

and so

$$[U_1(Y(B)), V_1(Y(B))] = 1.$$

But $U(B) = M\sigma A$, and $V(B) = M$: since the centralizer of M in B is M, $[U(B), V(B)] \neq 1$, contradicting 3.6.

(ii) $V(A) \neq 1$. Observe that $Y(A)$ is nilpotent and

$$(|A/Y(A)|, |Y(A)|) = 1:$$

since A is critical, $Y(A)$ is a p-group for some prime p, and is the Fitting subgroup of A (that is, the maximal normal nilpotent subgroup of A), and so $\sigma A \leqq \zeta(Y(A))$. Also, since $\mathfrak{Y} \leqq \mathfrak{V}$, we have $\sigma A \leqq V(A) \leqq Y(A)$.

Now, let A_1 be isomorphic to the direct product of two copies of A. Let F be a free group of finite rank with a normal subgroup N such that $F/N \cong A_1$. Then if $M/N = U(F/N)$, M/N is a non-cyclic elementary abelian p-group. Also, put $Y/N = Y(F/N)$. As in Lemma 2.7, consider $Q_p(M)$: let $F_1 = F/Q_p(M)$, $N_1 = N/Q_p(M)$, $M_1 = M/Q_p(M)$, $Y_1 = Y/Q_p(M)$. Then $M_1/[M_1, N_1]$ is non-abelian. Further, it is easy to deduce from the fact that M_1/N_1 is central in Y_1/N_1, and $M_1/[M_1, N_1]$ is nilpotent of class 2 that $N_1/[M_1, N_1]$ is central in $Y_1/[M_1, N_1]$. Then we have that

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$$F_1/[M_1, N_1] \in [\mathfrak{U}_1\mathfrak{Y}, \mathfrak{V}_1\mathfrak{Y}],$$

but

$$F_1/[M_1, N_1] \notin [\mathfrak{U}, \mathfrak{V}],$$

again contradicting 3.6.

The proof of 3.8, and with it the proof of Theorem 1, is now finished.

Acknowledgement

I would like to thank the referee for several pertinent and helpful comments about the presentation of this paper.

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https://doi.org/10.1017/S1446788700007308 Published online by Cambridge University Press