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NEW REDUCTIONS AND LOGARITHMIC LOWER BOUNDS FOR THE NUMBER OF CONJUGACY CLASSES IN FINITE GROUPS

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Abstract

The unsolved problem of whether there exists a positive constant c such that the number $k(G)$ of conjugacy classes in any finite group G satisfies $k(G) \geq c \log_2 |G|$ has attracted attention for many years. Deriving bounds on $k(G)$ from (that is, reducing the problem to) lower bounds on $k(N)$ and $k(G/N)$, $N \trianglelefteq G$, plays a critical role. Recently Keller proved the best lower bound known for solvable groups:

$$k(G) > c_0 \frac{\log_2 |G|}{\log_2 \log_2 |G|} \quad (|G| \geq 4)$$

using such a reduction. We show that there are many reductions using $k(G/N) \geq \beta[G:N]^{\alpha}$ or $k(G/N) \geq \beta(\log[G:N])^t$ which, together with other information about G and N or $k(N)$, yield a *logarithmic* lower bound on $k(G)$.

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1. Introduction

Let $k(G)$ denote the number of conjugacy classes of the finite group G. Answering a question of Frobenius, E. Landau observed in 1903 that for a fixed k_0 only a finite number of finite groups G satisfy $k(G) = k_0$. In 1968 Erdős and Turán [ET] (and independently Newman [Ne]) made this explicit by proving that $k(G) > \log_2 \log_2 |G|$ always holds. Ongoing since around 1910, the classification of finite groups according to their number of conjugacy classes is now complete for $k \leq 14$ [VS, VV1, VV2]. Of the more than 350 nonisomorphic groups with $k(G) \leq 14$, 25 satisfy $k(G) < \log_2 |G|$. Exactly five of the latter are solvable, and these satisfy $k(G) > \frac{4}{5} \log_2 |G|$. In fact *all* groups with $k(G) \leq 14$ satisfy $k(G) > \log_3 |G|$, and thus $k(G) > \log_3 |G|$ whenever $|G| \leq 3^{15}$. Perhaps $k(G) > \log_3 |G|$ for all finite groups G.

A simple induction beginning with $k(G/Z(G))$ shows that $k(G) > \log_2 |G|$ whenever G is nilpotent. In 1985 Cartwright [Ca] proved that $k(G) \geq \frac{3}{5} \log_2 |G|$ when G is supersolvable, but there are important families of groups, for example Frobenius groups as well as G with $|G| = p^{\alpha}q^{\beta}$ or G' nilpotent, for which the best known bound so far is $k(G) > c \log_2 |G|/\log_2 \log_2 |G|$. On the other hand, for each prime p there is a p-group P of order p^p with $k(P) < (\log_2 |P|)^3$. But no collection $\{G\}$ is known with $|G| \to \infty$ and $k(G) < (\log_2 |G|)^2$. See [Be3] for a more complete history and bibliography.

Keller [Ke] proved in 2011 the best general lower bounds to date, improving on those of Pyber [Py] 20 years ago. Keller proved that:

(i) there exists an (explicitly computable) constant $\epsilon_1 > 0$ such that for *every* finite group G with $|G| \geq 4$,

$$k(G) > \frac{\epsilon_1 \log_2 |G|}{(\log_2 \log_2 |G|)^7}.$$

Moreover:

(ii) if G is *solvable* ($|G| \geq 4$), then

$$k(G) > \frac{\epsilon_1 \log_2 |G|}{\log_2 \log_2 |G|}.$$

Pyber had obtained like bounds with exponent 8 instead of 7 in (i), and a denominator of $(\log_2 \log_2 |G|)^3$ in (ii) (see [Be3]). One of the main results underlying these improvements is when G is solvable and the Frattini subgroup $\Phi(G) = 1$. Here Pyber proved that (when $|G| \geq 4$)

$$k(G) \geq |G|^{\gamma/(\log_2 \log_2 |G|)^2},$$

where γ is a positive constant ($\gamma < 2^{-12}$). Keller's improvement finds a polynomial lower bound when $\Phi(G) = 1$: $k(G) \geq |G|^{\beta}$, where β is a positive constant.

2. Background and preliminaries

Perhaps the simplest reduction arises when G itself is nilpotent. Here $Z(G) \neq 1$ and G/Z is nilpotent. For any group G and $N \trianglelefteq G$, $k(G) = k_G(G - N) + k_G(N) \geq k(G/N) + k_G(N) - 1$, where $k_G(S)$ is the number of G-conjugacy classes that partition the normal subset S, so $k(G) \geq k(G/Z) + |Z| - 1$. Thus if $k(G/Z) \geq \log_2[G : Z]$, then $k(G) \geq \log_2 |G|$. When G is supersolvable, Cartwright [Ca] began his proof that $k(G) \geq \frac{3}{5} \log_2 |G|$ with a reduction lemma having the hypothesis that $k(G/M) \geq \frac{3}{5} \log_2[G : M]$ and $k(G/N) \geq \frac{3}{5} \log_2[G : N]$, assuming the existence of certain normal subgroups M, N of G. Next is Pyber's reduction lemma, which also plays a key role in Keller's recent results mentioned above (here $\log(\cdot) = \log_2(\cdot)$):

Lemma 2.1 [Py, Lemma 2.2]. *Let G be any group ($|G| \geq 4$) and $N \trianglelefteq G$ with N nilpotent. If $k(G/N) \geq 2^{x(\log[G:N])^{1/t}}$ for constants $0 < x \leq 1$, $t \geq 1$, then*

$$k(G) \geq \left(\frac{x^t}{2}\right) \frac{\log |G|}{(\log \log |G|)^t}.$$

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From Pyber's lemma with $t = 1$ we conclude that when N is a nilpotent normal subgroup of G and $k(G/N) \geq [G : N]^{\alpha}$ $(0 < \alpha \leq 1)$, then $k(G) \geq \alpha/2 \log_2 |G|/\log_2 \log_2 |G|$. As we will see, there are general situations where $N \trianglelefteq G$ and $k(G/N) \geq \beta[G : N]^{\alpha}$ (or even $k(G/N) \geq \beta(\log[G : N])^{t}$) which, together with other information about N or $k(N)$, yield *logarithmic* lower bounds for $k(G)$.

In 2004 (see [Be3]) the author presented several general 'logarithmic reductions' including [Be3, Lemma 4.5]. Suppose that $N \trianglelefteq G$, $\alpha, \beta > 0$ and $(\beta/(1 + \alpha))^{\beta/(1+\alpha)} \leq b$ (the base of the logarithm). If:

(i) $k(N) \geq |N|^{\alpha}$;
(ii) $k(G/N) \geq \beta \log[G : N]$; and
(iii) $|G|^{\alpha-((1+\alpha)/\beta)} \geq \log |G|$,

then $k(G) \geq \log |G|$. But (iii) implies that $\beta/(1 + \alpha) > 1/\alpha$, so the smaller α is the larger the base b. Here we remove any relation between b and the other parameters (except in one useful situation). We know that when $|G| \leq 3^{15}$ then $k(G) > \log_3 |G|$, so assuming that $|G|$ is large is natural. But the requirement of (iii) that $\beta > (1 + \alpha)/\alpha$ and that $|G|$ be 'large enough' depending on α, β will be avoided in many important situations.

We often use the following lemma.

Lemma 2.2.

(a) *When G is solvable, $F'(G) \leq \Phi(G) < F(G)$* [Hu, III, 3.11, 4.2].
(b) *If $|G| = \prod p_i^{\alpha_i}$ and $s = \max\{\alpha_i\} \geq 3$, then the nilpotence class $c(\Phi) \leq (s - 1)/2$* [HP].
(c) *If G is nilpotent with nilpotence class c, then $k(G) \geq c|G|^{1/c} - c + 1$* [Sh].

We will also use results from [Be2, Be3]. The first part of Lemma 2.3(a) also appears in [Ca].

Lemma 2.3. *Suppose that $N \trianglelefteq G$. Then:*

(a) $k(G) \geq k(G/N) + (k(N) - 1)/[G : N]$; *(Note that equality occurs if and only if G is a Frobenius group with kernel N.)*
(b) *if $k(N) \geq |N|^{\alpha}$ and $k(G/N) \geq [G : N]^{\beta}$ $(\alpha, \beta > 0)$, then $k(G) \geq |G|^{\alpha\beta/(\alpha+\beta+1)}$;*
(c) $k(G/N \cap G') = [N : N \cap G']k(G/N)$.

As mentioned, it follows from the classification of finite groups according to their number k of conjugacy classes (now complete for $k \leq 14$) that $k(G) > \log_3 |G|$ when $|G| \leq 3^{15}$. Using this and Lemma 2.3(a), we have the following corollary.

Corollary 2.4. *Suppose that $N \trianglelefteq G$, together with* (i) $k(N) \geq |N|^{\alpha}$ $(0 < \alpha \leq 1)$ *and* (ii) $k(G/N) \geq (1 + \alpha) \log[G : N]$. *Then* (a) $k(G) \geq (\alpha - \log \log |G|/ \log |G|) \log |G|$; *and* (b) *when* $\log(\cdot) = \log_3(\cdot)$, (i), (ii) *and* (iii) $|N|^{\alpha-c} \geq \log |N|$ $(0 < c < \alpha)$ *imply that* $k(G) \geq \min\{c, 0.39\} \log |G|$.

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Proof. (a) If $|N|^{1+\alpha} \geq |G| \log |G|$, then assumption (i) and Lemma 2.3(a) imply that $k(G) > k(N)/[G : N] \geq |N|^{1+\alpha}/|G| \geq \log |G|$. Otherwise $|N|^{1+\alpha} < |G| \log |G|$ and thus $(1+\alpha) \log |G| - (1+\alpha) \log |N| > \alpha \log |G| - \log \log |G|$. Also, (ii) and Lemma 2.3(a) yield $k(G) > k(G/N) \geq (1+\alpha) \log[G : N] \geq \alpha \log |G| - \log \log |G|$, so

$$k(G) > \left(\alpha - \frac{\log \log |G|}{\log |G|}\right) \log |G|.$$

(b) Since $\log_3 \log_3 n/\log_3 n$ decreases for $n \geq 20$, when $|N| \geq 20$ then (iii) yields

$$\alpha - c \geq \frac{\log_3 \log_3 |N|}{\log_3 |N|} > \frac{\log_3 \log_3 |G|}{\log_3 |G|}$$

that is,

$$\alpha - \frac{\log_3 \log_3 |G|}{\log_3 |G|} > c,$$

so from (a) $k(G) > c \log_3 |G|$. For $|N| \leq 19$, N is abelian ($\alpha = 1$) except possibly when $|N| = 2n$ $(3 \leq n \leq 9)$. We check (for example, [VV1]) that for such N, $k(N) > |N|^{3/5}$ except when $N = \mathrm{Alt}(4)$, where $k(N) = 4 > |N|^{0.557}$. Since $|G| > 3^{15}$, $\log_3 \log_3 |G|/ \log_3 |G| < 0.165$. Thus for $|N| \leq 19$, $k(N) \geq |N|^{\alpha}$, where

$$\alpha - \frac{\log_3 \log_3 |G|}{\log_3 |G|} > 0.557 - 0.165 > 0.39.$$

The conclusion follows. □

If G is nilpotent-by-nilpotent (both N and G/N are nilpotent), Keller's general result gives $k(G) \geq \epsilon_1 \log_2 |G|/\log_2 \log_2 |G|$, where ϵ_1 is a small constant. Given more information about $k(N)$, we improve this in many cases. Recall, for example, that when N is nilpotent of nilpotence class c, then $k(N) \geq |N|^{1/c}$ [Sh].

Corollary 2.5. *If $N \trianglelefteq G$, G/N is nilpotent and $k(N) \geq |N|^{\alpha}$ $(0 < \alpha \leq 1)$ then $k(G) \geq \alpha \log_b |G| - \log_b \log_b |G|$ $(b = 2^{4/3})$.*

Proof. Since G/N is nilpotent, $k(G/N) \geq \frac{3}{2} \log_2[G : N]$ [Ca]. When $b = 2^{4/3}$, $\frac{3}{2} \log_2 n = 2 \log_b n \geq (1+\alpha) \log_b n$, so assumptions (i) and (ii) of Corollary 2.4 are met. Thus $k(G) \geq \alpha \log_b |G| - \log_b \log_b |G|$. □

We return to G/N nilpotent and $k(N) \geq |N|^{\alpha}$ in Corollary 3.11(c), finding a logarithmic lower bound with coefficient depending on α, but *without* having to assume that $|G|$ is large enough, depending on α.

As mentioned earlier, the best bound known when G' is nilpotent is [Ca] $k(G) \geq \log_2 |G|/\log_2 \log_2 |G|$, but the author knows of no such example with $k(G) < \log_2 |G|$. Taking into account the prime factorisations of $|G|$ and $|G'|$, we note the following improvements. Corollary 2.6(a) generalises [Be3, Proposition 4.10(b)], removing any restriction on how large $|G|$ is.

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COROLLARY 2.6. *Suppose that G' is nilpotent.*

(a) *If $|G| = \prod p_i^{\alpha_i}$ and $s := \max\{\alpha_i\}$, then $k(G) \geq |G|^{1/2s+1}$.*
(b) *If $|G'| = \prod p_i^{\beta_i}$ and $r := \max\{\beta_i\} \geq 2$, then $k(G) \geq |G|^{1/2r-1}$.*

PROOF. (a) Since G' is nilpotent, $G'' \leq F' \leq \Phi(G)$ (Lemma 2.2(a)), so G/Φ is metabelian and, by [Be2], $k(G/\Phi) \geq [G : \Phi]^{1/3}$. By Lemma 2.2(c), $k(\Phi) \geq |\Phi|^{1/c}$ where c is the nilpotence class of Φ. If $s \leq 2$, then all Sylow subgroups of G' are abelian, G' is abelian and $k(G) \geq |G|^{1/3}$. If $s \geq 3$, then by Lemma 2.2(b), $k(\Phi) \geq |\Phi|^{2/s-1}$. Using Lemma 2.3(b) with $\alpha = 2/(s-1)$ and $\beta = 1/3$ gives the result.

(b) The nilpotence class $c(G')$ is the maximum of the classes of its Sylow p-subgroups. The class of a group of order p^n, $n \geq 2$, is at most $n-1$, so $c(G') \leq r-1$. Again by Lemma 2.2(c), $k(G') \geq |G'|^{1/c} \geq |G'|^{1/r-1}$. From Lemma 2.3(b) with $N = G'$, $\alpha = 1/(r-1)$ and $\beta = 1$, the result follows. □

REMARK 2.7. When G is solvable, $|G| = \prod p_i^{\alpha_i}$ and $\alpha_i \leq 2$, each Sylow subgroup of G is abelian and the derived length $d(G) \leq 3$ [Ta]. Since $k(G) \geq |G|^{1/2^d-1}$ [Be2], here $k(G) > |G|^{1/7}$. Furthermore, when $n = \prod p_i^{\alpha_i}$ and $s(n) := \max\{\alpha_i\}$, Niven [Ni] proved that the *average order* of $s(n)$ lies between 1 and 2, that is, $\lim_{n\to\infty}(1/n)\sum_{j=1}^{n} s(j)$ is approximately 1.7.

3. New reductions

We begin with a general reduction related to Pyber's lemma above, when $t = 1$. Instead of assuming that N is nilpotent, we make an assumption on $k(N)$ which leads to the conclusion that $k(G) \geq \log|G|$ when $|G|$ is large enough. Unless otherwise noted, $\log(\cdot) = \log_b(\cdot)$, where $b \geq 2$. Note that Lemma 3.1 may be used when $N \geq G'$, putting $k(G/N) \geq \beta[G : N]^{\alpha}$ where for example $\alpha = 1/2$, $\beta = \sqrt{2}$ when $[G : N] = 2$ and $\alpha = 1 - 1/n$, $\beta = 1 + 1/n$ ($n \geq 2$) when $[G : N] \geq 3$.

LEMMA 3.1. *Suppose that $N \trianglelefteq G$, with*

(i) $k(G/N) \geq \beta[G : N]^{\alpha}$ *($0 < \alpha < 1 < \beta$) and*
(ii) $k(N) \geq (\log|N|)^{1+1/\alpha}$.

Then $k(G) \geq \log|G|$ for all $|G|$ large enough (depending only on α, β). In particular, when N is solvable and $\Phi(N)$ is abelian, together with (i), the conclusion follows for all $|G|$ large enough, depending only on α, β.

PROOF. Since $k(G) > \max\{k(G/N), k(N)/[G : N]\}$, the conclusion follows from hypothesis (i) when $\beta[G : N]^{\alpha} \geq \log|G|$, that is, when $|N| \leq \beta^{1/\alpha}|G|/(\log|G|)^{1/\alpha}$. So we may assume that $|N| > \beta^{1/\alpha}|G|/(\log|G|)^{1/\alpha}$. From $k(G) > k(N)/[G : N]$ it follows from hypothesis (ii) that

$$k(G) > \frac{\beta^{1/\alpha}(\log(\beta^{1/\alpha}) + \log|G| - \frac{1}{\alpha}\log\log|G|)^{1+1/\alpha}}{(\log|G|)^{1/\alpha}}.$$

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But $\beta^{1/\alpha} > \beta > 1$, so $k(G) > \log |G|$ as long as

$$\beta\Big(\log |G| - \frac{1}{\alpha}\log\log |G|\Big)^{1+\alpha} \geq (\log |G|)^{1+\alpha},$$

that is, when

$$\frac{\log |G|}{\log\log |G|} \geq \frac{1 + (\beta^{1/(\alpha+1)} - 1)^{-1}}{\alpha},$$

which is true for all sufficiently large $|G|$, depending only on α, β.

Now suppose that N is solvable and $\Phi(N)$ is abelian. From (i) we have $k(G) > k(G/N) > [G:N]^{\alpha} \geq \log |G|$, if $|N| \leq |G|/(\log |G|)^{1/\alpha}$. So assume that $|N| > |G|/(\log |G|)^{1/\alpha}$. Now $\Phi(N)$ is abelian, so [Be3, Proposition 2.3] $k(N) \geq (\log |N|)^{1+1/\alpha}$ (that is (ii) holds) when $|N|$ is large enough, and hence when $|G|$ is large enough, depending only on α. (This also follows from Keller's Theorem 3.1, applied to $N/\Phi(N)$, and Lemma 2.3(b) above with N replaced by $\Phi(N)$ and G replaced by N.) □

Corollary 3.2.

(a) *Suppose that $N \trianglelefteq G$ with G/N nilpotent of nilpotence class $c(G/N) \geq 2$ and $k(N) \geq (\log |N|)^{c+1}$. Then $k(G) \geq \log |G|$ as long as $|G|$ is large enough, depending only on c.*

(b) *Suppose that G' is nilpotent and $|G| \geq 2^{56}$.*
  - (i) *If $k(\Phi(G)) \geq (\log_2 |\Phi|)^4$, then $k(G) \geq \log_2 |G|$.*
  - (ii) *If $\Phi(G)$ has nilpotence class $c(\Phi) \leq \frac{1}{4}\log_2 |\Phi|/\log_2\log_2 |\Phi|$, then $k(G) \geq \log_2 |G|$.*

Proof. (a) Since G/N is nilpotent of class c, $k(G/N) \geq c[G:N]^{1/c} - (c-1)$ [Sh], and the latter is greater than or equal to $(1 + 1/c)[G:N]^{1/c}$ for $c \geq 2$. In Lemma 3.1 set $\beta = 1 + 1/c$ and $\alpha = 1/c$. From the proof we see that $k(G) > \log |G|$ as long as $\log |G|/\log\log |G| \geq (1 + (\beta^{1/(1+\alpha)} - 1)^{-1})/\alpha$, that is,

$$\frac{\log |G|}{\log\log |G|} \geq \bigg(1 + \bigg(\Big(1 + \frac{1}{c}\Big)^{1/(1+1/c)} - 1\bigg)^{-1}\bigg)c.$$

(b) (i) When G' is nilpotent $G'' \leq F'(G) \leq \Phi(G)$ (Lemma 2.2(a)), so $(G/\Phi)'' = \{1\}$. Thus $k(G/\Phi) \geq (\frac{9}{2}[G:\Phi])^{1/3}$ [Be1], and the conclusion follows from Lemma 3.1 with $N = \Phi$, $\alpha = 1/3$ and $\beta = (9/2)^{1/3}$, after checking that $|G|$ is large enough. (ii) Since $k(\Phi) \geq |\Phi|^{1/c}$ (Lemma 2.2(c)) the conclusion follows from the assumed upper bound on $c(\Phi)$, and (i). □

When G' is nilpotent, so far we only know that $k(G) \geq \log_2 |G|/\log_2\log_2 |G|$ [Ca]. It is thus worthwhile to record further reduction theorems in this area which conclude that $k(G) \geq \log |G|$. We may assume that G' is not abelian, since then $k(G) \geq (\frac{9}{2}|G|)^{1/3}$ [Be1], and we will often need the following lemma.

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Lemma 3.3 [Be3, Corollary 3.2(a)–(c)].

(a) *If* $N \trianglelefteq G$ *and* N *is nonabelian, then* $k_G(N) - 1 \geq 2|C_G(N)|/[G : N]$. *Thus for any* $N \trianglelefteq G$, $k_G(N) - 1 \geq (|N| - 1)/[G : C_G(N)]$.

(b) *If* G *is solvable and* N *is a minimal normal subgroup of* G *such that* (i) $k(G/N) \geq \log[G : N]$ *and* (ii) $[G : F] \leq (|N| - 1)/\log |N|$, *then* $k(G) \geq \log |G|$.

(c) *If* G' *is nilpotent and* N *is a minimal normal subgroup of* G *such that* $k(G/N) \geq \log[G : N]$ *and* $(|N| - 1)/\log |N| \geq \log |G|$, *then* $k(G) \geq \log |G|$.

Lemma 3.4. *Suppose that* $N \trianglelefteq G$, *with* $k(G/N) \geq (1 + \epsilon) \log[G : N]$ *and* $|N| \leq (\log |G|)^t$ $(\epsilon, t > 0)$. *Then* $k(G) \geq \log |G|$ *for* $|G|$ *large enough, depending only on* ϵ, t. *In fact, if also* $k(G) \geq t(1 + 1/\epsilon) \log \log |G|$ *then* $k(G) \geq \log |G|$, *without restriction on* $|G|$.

Proof. Since $k(G) > k(G/N) \geq (1 + \epsilon) \log[G : N]$, and $|N| \leq (\log |G|)^t$, we have $k(G) > (1 + \epsilon)(\log |G| - t \log \log |G|)$. The conclusion follows when the latter is greater than or equal to $\log |G|$, which is equivalent to $\log |G|/\log \log |G| \geq t(1 + 1/\epsilon)$. This is true for all large enough $|G|$, depending only on ϵ, t. And when it is false, assuming $k(G) \geq t(1 + 1/\epsilon) \log \log |G|$ yields $k(G) > \log |G|$. □

Theorem 3.5.

(a) *Suppose that* $C_G(G') \not\leq G'$ *and* $k(G/C_G(G')) \geq \log[G : C_G(G')]$. *Then* $k(G) \geq \log |G|$ *when* $|G| \geq 2^{13}$ *(in base 3 we may assume that* $|G| > 3^{15}$*).*

(b) *Given* $\epsilon > 0$, *for all large enough solvable groups* G *(depending only on* ϵ*), if* $k(G/C_G(G')) \geq (1 + \epsilon) \log[G : C_G(G')]$ *then* $k(G) \geq \log |G|$.

Proof. (a) Since $C_G(G') \trianglelefteq G$, when $|C_G(G')| \leq |G|^{1/2}$ we are done using our assumptions on $C_G(G')$ and [Be3, Corollary 3.9(a)]. On the other hand, when $|C_G(G')| \geq |G|^{1/2}$, by Lemma 3.3(a),

$$k(G) > k_G(G') - 1 \geq \frac{2|C_G(G')|}{[G : G']} \geq \frac{2|G|^{1/2}}{[G : G']}.$$

But $k(G) \geq [G : G'] + 1$, so we may assume that $[G : G'] < \log |G| - 1$. Since $2|G|^{1/2} \geq \log_2^2 |G| - 1$ for $|G| \geq 2^{13}$, we conclude that $k(G) > \log |G|$.

(b) Using Lemma 3.4, if $|C_G(G')| \leq (\log |G|)^2$ then our hypothesis yields $k(G) \geq \log |G|$. Next assume that $|C_G(G')| \geq (\log |G|)^2$. From Lemma 3.3(a),

$$k_G(C_G(G')) - 1 \geq \frac{|C_G(C_G(G'))|(|C_G(G')| - 1)}{|G|} \geq \frac{|C_G(G')| - 1}{[G : G']} \geq \frac{(\log |G|)^2 - 1}{[G : G']}.$$

Again, we may assume that $[G : G'] < \log |G| - 1$, so $k(G) > \log |G| + 1$. □

We have remarked that no collection of groups is known for which $|G| \to \infty$ and $k(G) < (\log |G|)^2$. If $0 < \delta < 1$, then for all $|G|$ large enough (depending only on δ) $k(G') > (\log |G'|)^2$ implies that $k(G) > \delta \log |G|$ [Be3, Lemma 3.5]. In Corollary 3.7 (another application of Lemma 3.1) we prove that, for each $n \geq 2$ and all $|G|$ large enough depending only on n, if $k(G^{(n)}) \geq (\log |G^{(n)}|)^{2^n}$ then $k(G) \geq \log |G|$. We must

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first extend a result of [Be2] that if G has derived length d, then $k(G) \geq |G|^{1/2^d-1}$. Lemma 3.6 is a slight improvement over a result of M. Herzog communicated to the author.

Lemma 3.6. *If G is a finite solvable group of derived length d, then*

$$k(G) \geq \left(\frac{3}{2} - \frac{1}{2^d}\right)|G|^{1/(2^d-1)}. \tag{3.1}$$

Proof. If $d = 1$, then (3.1) holds with equality. In [Be1] we proved that $k(G) \geq (\frac{9}{2}|G|)^{\frac{1}{3}}$ when G is metabelian, and since $(\frac{9}{2})^{\frac{1}{3}} > 5/4$, (3.1) is true when $d = 2$. Thus we may suppose that $d \geq 3$ and (3.1) holds with $d - 1$ replacing d. Using our inductive assumption,

$$k(G') \geq \left(\frac{3}{2} - \frac{1}{2^{d-1}}\right)|G'|^{1/(2^{d-1}-1)}.$$

Lemma 2.3(a) with $N = G'$ yields

$$k(G) \geq [G : G'] + \frac{k(G')|G'|}{|G|} - \frac{1}{2}.$$

Setting $|G'| = x$, $|G| = g$ and

$$a = 1 + \frac{1}{2^{d-1} - 1}, \quad b = \frac{3}{2} - \frac{1}{2^{d-1}},$$

we arrive at

$$k(G) \geq \frac{g}{x} + \frac{b}{g}x^a - \frac{1}{2}. \tag{3.2}$$

Let $f(x) = (g/x) + (b/g)x^a$. Then $f'(x) = -(g/x^2) + (ab/g)x^{a-1}$, and since $f''(x) > 0$ for $x > 0$ the solution x_0 to $f'(x) = 0$ corresponds to a minimum for $f(x)$. From $f'(x_0) = 0$ we obtain $(g/x_0)^2 = abx_0^{a-1}$, that is, $x_0 = (g^2/ab)^{1/(a+1)}$. Thus $g/x_0 = (ab)^{1/(a+1)}g^{1-2/(a+1)} = (ab)^{1/(a+1)}g^{1/(2^d-1)}$. Furthermore,

$$\frac{b}{g}x_0^a = \frac{b}{g}\left(\frac{g^2}{ab}\right)^{a/(a+1)} = \frac{(ab)^{1/(a+1)}}{a}g^{1-2/(a+1)},$$

so, from (3.2), $k(G) \geq (ab)^{1/(a+1)}(1 + (1/a) - (1/2))g^{1/(2^d-1)}$.

It remains only to show that when $d \geq 3$, $(ab)^{1/(a+1)}(\frac{1}{2} + (1/a)) \geq \frac{3}{2} - 1/2^d$. First check that

$$ab = \left(1 + \frac{1}{2^{d-1} - 1}\right)\left(\frac{3}{2} - \frac{1}{2^{d-1}}\right) = \frac{(\frac{3}{2})2^{d-1} - 1}{2^{d-1} - 1},$$

which decreases to $\frac{3}{2}$ as $d \to \infty$. Also $1/(a + 1)$ increases as $d \to \infty$. Thus $(ab)^{1/(a+1)} > (\frac{3}{2})^{\frac{3}{7}} > \frac{9}{8}$, and finally

$$(ab)^{1/(a+1)}\left(\frac{1}{2} + \frac{1}{a}\right) > \frac{9}{8}\left(\frac{1}{2} + \frac{1}{a}\right) = \frac{9}{8}\left(\frac{3}{2} - \frac{1}{2^{d-1}}\right) > \frac{3}{2} - \frac{1}{2^d},$$

since $d \geq 3$. □

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Corollary 3.7. *For each* $n \geq 2$*, let* $\{G\}_n$ *denote the class of solvable groups* G *for which* $k(G^{(n)}) \geq (\log|G^{(n)}|)^{2^n}$*. If* $G \in \{G\}_n$ *and* $|G|$ *is large enough (depending only on* n*), then* $k(G) \geq \log|G|$*.*

Proof. Since $G/G^{(n)}$ has derived length n, by Lemma 3.6 we have $k(G/G^{(n)}) \geq (3/2 - 1/2^n)[G : G^{(n)}]^{1/2^n - 1}$. In Lemma 3.1 set $N = G^{(n)}$, $\alpha = 1/2^n - 1$, and $\beta = 3/2 - 1/2^n$ (which is greater than 1 since $n \geq 2$). Also $1 + 1/\alpha = 2^n$, and hypotheses (i) and (ii) are satisfied. Thus $\log|G|/\log\log|G| \geq (2^n - 1)(1 + ((3/2 - 1/2^n)^{1-1/2^n} - 1)^{-1})$ yields $k(G) \geq \log|G|$. □

The logarithmic reductions in [Be3, Lemma 4.5 and Theorem 4.8], while assuming that $k(N) \geq |N|^\alpha$ and $k(G/N) \geq \beta \log[G : N]$, also require that $|G|$ be 'large enough', depending on the parameters involved. Theorem 3.9 below shows that by relating α, β and $[N : N \cap G']$ in a single inequality, the requirement that $|G|$ is large can be avoided. This has important consequences. First we need the following lemma.

Lemma 3.8. *If* $k(G/N) \geq \beta \log[G : N]$ *and* $\beta \geq \log|G|/\log\log|G|$*, then* $k(G) \geq \log|G|$*.*

Proof. We always have $k(G) \geq k(G/N)$, so we may assume (using our hypothesis) that $\beta \leq k(G/N)/\log[G : N] < \log|G|/\log[G : N]$. If $[G : N] \geq \log|G|$, it follows that $\beta < \log|G|/\log\log|G|$, contradicting our assumption. If $[G : N] < \log|G|$ then $\beta \leq [G : N]/\log[G : N] < \log|G|/\log\log|G|$, since $x/\log x$ increases for $x \geq 3$ and we may assume that $\log|G| > k(G) \geq 4$. Again $\beta < \log|G|/\log\log|G|$, contradicting our assumption. □

Theorem 3.9. *Suppose that* $N \trianglelefteq G$*, with*

(i) $k(N) \geq |N|^\alpha$ $(0 < \alpha \leq 1)$ *and*
(ii) $k(G/N) \geq \beta \log[G : N]$ $(\beta > 0)$.

If also either

(iii) $(\beta\alpha - 1)[N : N \cap G'] \geq 1 + \alpha$ *or*
(iv) $|G|^{\alpha - (1+\alpha)/\beta[N:N\cap G']} \geq \log|G|$,

then $k(G) \geq \log|G|$*.*

Proof. From Lemma 2.3(a),

$$k(G) \geq k(G/N) + \frac{k(N) - 1}{[G : N]} > \frac{k(N)}{[G : N]},$$

so (i) yields $k(G) > |N|^{1+\alpha}/|G|$. By Lemma 3.8 and (ii) we may assume that $|G|^{1/\beta} \geq \log|G|$, so if $|N|^{1+\alpha}/|G| \geq |G|^{1/\beta}$ we are done. If $|N|^{1+\alpha} < |G|^{1+1/\beta}$

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then $[G:N] > |G|^{(\alpha-1/\beta)/(\alpha+1)}$, so by Lemma 2.3(c) and (iii),

$$\begin{aligned} k(G) &\geq k(G/N \cap G') = [N : N \cap G']k(G/N) \\ &\geq \beta[N : N \cap G'] \log[G : N] \\ &> \beta[N : N \cap G']\Big(1 - \frac{1 + 1/\beta}{1+\alpha}\Big) \log |G| \\ &= \frac{(\beta\alpha - 1)[N : N \cap G']}{1+\alpha} \log |G| \geq \log |G|. \end{aligned}$$

Concerning (iv), note that as before (i) yields $k(G) > |N|^{1+\alpha}/|G|$, and we may assume that $|N|^{1+\alpha}/|G| < \log |G|$, that is, $[G : N] > (|G|^{\alpha}/ \log |G|)^{1/(1+\alpha)}$. From (ii) we obtain

$$k(G/N) \geq \frac{\beta}{1+\alpha}(\alpha \log |G| - \log\log |G|).$$

By Lemma 2.3(c),

$$\begin{aligned} k(G) \geq k(G/N \cap G') &= [N : N \cap G']k(G/N) \\ &\geq \Big(\frac{\beta}{1+\alpha}\Big)[N : N \cap G'](\alpha \log |G| - \log\log |G|). \end{aligned}$$

The latter is greater than or equal to $\log |G|$ when

$$\Big(\frac{\alpha\beta[N : N \cap G']}{1+\alpha} - 1\Big) \log |G| \geq \frac{\beta[N : N \cap G']}{1+\alpha} \log\log |G|,$$

which is (iv). □

Note that G' is nilpotent in Corollary 2.6(a), where we used $k(G/\Phi) \geq [G : \Phi]^{1/3}$ along with $s = \max\{\alpha_i\}$ in $|G| = \prod p_i^{\alpha_i}$ to conclude that $k(G) \geq |G|^{1/2s+1}$. In particular, $k(G) \geq \log |G|$ when $(2s + 1) \log\log |G| \leq \log |G|$. We always have $k(G) \geq \log\log |G|$ [ET], but here if we *also* know that $k(G) \geq (2s + 1) \log\log |G|$, then again $k(G) \geq \log |G|$.

Assuming only that G is solvable, Keller [Ke, Theorem 3.1] has proved that when $\Phi(G) = 1$ then $k(G) \geq |G|^{\beta}$ for some universal constant $\beta > 0$ (a specific value for β is not provided). In Corollary 3.10 we use $k(G/\Phi)$ and $s = \max\{\alpha_i\}$ to conclude that $k(G)$ has a logarithmic lower bound, in three different ways. As discussed in Remark 2.7, if $s \leq 2$ then $k(G) > |G|^{1/7}$ when G is solvable.

Corollary 3.10. *Suppose that G is solvable, $|G| = \prod p_i^{\alpha_i}$ (p_i distinct primes, $\alpha_i \geq 1$) and $s = \max\{\alpha_i\} \geq 3$.*

(a) *If $k(G/\Phi) \geq [G : \Phi]^{\alpha}$, $\alpha > 0$ and $k(G) \geq (s + 1) \log\log |G|$, then $k(G) \geq \alpha \log |G|$.*
(b) *If $k(G/\Phi) \geq [G : \Phi]^{\alpha}$ and $k(G) \geq s^{1+\epsilon}$ ($\epsilon > 0$), then $k(G) \geq \log |G|$ when G is sufficiently large (depending only on α, ϵ).*
(c) *If $k(G/\Phi) \geq s \log[G : \Phi]$, then $k(G) \geq \log |G|$.*

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Proof. Since $s \geq 3$, we use Lemma 2.2(b) and (c) as in the proof of Corollary 2.6(a) to conclude that $k(\Phi) \geq |\Phi|^{2/(s-1)}$.

(a) From Lemma 2.3(a),

$$k(G) \geq k(G/\Phi) + \frac{k(\Phi) - 1}{[G : \Phi]} > \max\left\{k(G/\Phi), \frac{k(\Phi)}{[G : \Phi]}\right\}.$$

If $|\Phi|$ is 'large', that is, $|\Phi| \geq |G|^{1-1/(s+1)}$, then

$$k(G) > \frac{k(\Phi)}{[G : \Phi]} \geq \frac{|\Phi|^{2/(s-1)}}{[G : \Phi]} = \frac{|\Phi|^{(s+1)/(s-1)}}{|G|} \geq |G|^{1/(s-1)}.$$

When $s - 1 \leq \log|G|/\log\log|G|$ we conclude that $k(G) > \log|G|$. If $s - 1 \geq \log|G|/\log\log|G|$ then by assumption $k(G) \geq (s + 1)\log\log|G| > \log|G|$. Finally, if $|\Phi|$ is 'small', that is, $[G : \Phi] \geq |G|^{1/s+1}$, then $k(G) > k(G/\Phi) \geq [G : \Phi]^{\alpha} \geq |G|^{\alpha/s+1}$, and the latter is greater than or equal to $\log|G|$ as long as $\alpha/s + 1 \geq \log\log|G|/\log|G|$. Otherwise, $\alpha/(s + 1) \leq \log\log|G|/\log|G|$ and it follows from our assumption that $k(G) \geq (s + 1)\log\log|G| \geq \alpha\log|G|$.

(b) If $|\Phi| \geq |G|^{1-1/s}$, then $k(G) > |\Phi|^{(s+1)/(s-1)}/|G| \geq |G|^{1/s}$. If $s \leq \log|G|/\log\log|G|$ then $k(G) > \log|G|$. Otherwise

$$s > \frac{\log|G|}{\log\log|G|} \quad \text{and} \quad k(G) \geq s^{1+\epsilon} > \left(\frac{\log|G|}{\log\log|G|}\right)^{1+\epsilon} \geq \log|G|,$$

when $\log|G| \geq (\log\log|G|)^{1+1/\epsilon}$. On the other hand, if $|\Phi| < |G|^{1-1/s}$, then $k(G) > k(G/\Phi) \geq [G : \Phi]^{\alpha} > |G|^{\alpha/s}$. Here if $s \leq \alpha(\log|G|/\log\log|G|)$ then $k(G) > \log|G|$. Otherwise

$$s > \alpha\left(\frac{\log|G|}{\log\log|G|}\right) \quad \text{and} \quad k(G) \geq s^{1+\epsilon} > \alpha^{1+\epsilon}\left(\frac{\log|G|}{\log\log|G|}\right)^{1+\epsilon},$$

so $k(G) > \log|G|$ when $\log|G| \geq (\log\log|G|/\alpha)^{1+1/\epsilon}$.

(c) Since $k(\Phi) \geq |\Phi|^{2/(s-1)}$ and $k(G/\Phi) \geq s\log[G : \Phi]$, set $N = \Phi$, $\alpha = 2/(s - 1)$ and $\beta = s$ in Theorem 3.9. Then $\alpha(\beta - 1) = 2$, that is, $\beta\alpha - 1 = 1 + \alpha$ so (i)–(iii) are satisfied and $k(G) \geq \log|G|$. □

Comment. Theorem 3.9 implies that if (i) $k(N) \geq |N|^{\alpha}$, (ii) $k(G/N) \geq \beta\log[G : N]$ $(\beta > 0)$ *and* $N \nleq G'$, then *either* (iii) *or* (iv) yield $k(G) \geq \log|G|$: (iii) $\beta \geq (1 + 3/\alpha)/2$ (≥ 2), (iv) N is abelian, $\beta > 1$ and $|G|^{1-1/\beta} \geq \log|G|$ (or $k(G) \geq \beta/(\beta - 1)$ $\log\log|G|$). But whether or not $N \leq G'$, $|G|$ need not be large, as we see next.

Corollary 3.11. *Suppose that* $N \trianglelefteq G$ *and* $k(N) \geq |N|^{\alpha}$ $(0 < \alpha \leq 1)$*.*

(a) *If* $k(G/N) \geq \beta\log[G : N]$, $\beta \geq 1 + 2/\alpha$ (≥ 3)*, then* $k(G) \geq (\beta/(1 + 2/\alpha))\log|G|$.

(b) *Suppose that* $k(G/N) \geq (1 + \alpha)\log_a[G : N]$, $(a := 1/\alpha > 1)$*. Then*

$$k(G) \geq \left(\frac{a + 1}{2a^2 + a}\right)\log_a|G| > \frac{\alpha}{2}\log_a|G|.$$

(Note that $(a + 1)/(2a^2 + a) < 2\alpha/3$*.)*

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(c) *Suppose also that G/N is nilpotent. If $\alpha = 1$ then $k(G) \geq \frac{3}{4}\log_2|G|$ [Ca]. If $\alpha = \frac{1}{2}$ then $k(G) \geq \frac{3}{10}\log_2|G|$. In general, let $n \geq 1$ be the smallest integer such that $k(N) \geq |N|^{1/2^n}$. Then $k(G) \geq (1/n2^{n+1})\log_2|G|$.*

Proof. (a) Suppose $k(G/N) \geq \beta\log_b[G:N]$. Choose c such that $\beta\log_b[G:N] = (1+2/\alpha)\log_c[G:N]$, that is, $\beta/(1+2/\alpha) = \log_c[G:N]/\log_b[G:N] = \log_c b$. Then hypotheses (i)–(iii) of Theorem 3.9 are satisfied (whether $N \leq G'$ or not) where 'β' in the Theorem equals $1+2/\alpha$, and the base of the logarithm is c. Thus $k(G) \geq \log_c|G| = (\beta/(1+2/\alpha))\log_b|G|$.

(b) Here we set $b := (a^a)^{(2a+1)/(a+1)}$. Since $a := 1/\alpha$,

$$\frac{1+\alpha}{1+2/\alpha} = \frac{a+1}{2a^2+a} = \log_b a.$$

Thus

$$\begin{aligned} k(G/N) &\geq (1+\alpha)\log_a[G:N] \\ &= \left(\left(1+\frac{2}{\alpha}\right)(\log_b a)\right)\log_a[G:N] \\ &= \left(1+\frac{2}{\alpha}\right)\log_b[G:N]. \end{aligned}$$

With $\beta := 1+2/\alpha$, hypotheses (i)–(iii) of Theorem 3.9 are satisfied, so

$$k(G) \geq \log_b|G| = \frac{\log_a|G|}{\log_a b} = \left(\frac{a+1}{2a^2+a}\right)\log_a|G|.$$

(c) Since G/N is nilpotent, $k(G/N) \geq \frac{3}{2}\log_2[G:N]$ [Ca], so we set $a = 2$ and $\alpha = \frac{1}{2}$ in (b), obtaining $k(G) \geq \frac{3}{10}\log_2|G|$ when $k(N) \geq |N|^{1/2}$. If $k(N) \geq |N|^{1/2^n}$ we use $a = 2^n$ and $\alpha = 1/2^n$ in (b). Thus

$$k(G/N) \geq \tfrac{3}{2}\log_2[G:N] \geq (1+\alpha)\log_a[G:N]$$

so, again using (b), we conclude that $k(G) > (\alpha/2)\log_a|G|$. Finally, $\log_{2^n}|G| = (1/n)\log_2|G|$ and $\alpha/2 = 1/2^{n+1}$. □

It follows from Theorem 3.9 that when $N \trianglelefteq G$, $N \not\leq G'$ and N is abelian, with $k(G/N) \geq (1+\epsilon)\log[G:N]$ $(\epsilon > 0)$, then $k(G) \geq \log|G|$ for $|G|$ large enough (depending only on ϵ). But what if $N \leq G'$? Generally, when G is solvable and N is a minimal normal subgroup of G, Theorem 3.12 gives the same conclusion.

Theorem 3.12. *For each $\epsilon > 0$ and all solvable groups G with $|G|$ large enough (depending only on ϵ), if N is a minimal normal subgroup of G and $k(G/N) \geq (1+\epsilon)\log[G:N]$, then $k(G) \geq \log|G|$.*

Proof. For ease of presentation we give a proof when $\log(\cdot) = \log_2(\cdot)$, but a careful examination of the proof shows that Theorem 3.12 holds in any base at least 2.

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Among solvable groups we first consider G for which $[G:\Phi] \geq |G|^{1/\sqrt{\log_2 |G|}}$.[1] It is always true that $F'(G) \leq \Phi(G)$, so $[G:F'] \geq [G:\Phi] \geq |G|^{1/\sqrt{\log_2 |G|}}$. Among such G, and with γ the constant from Pyber's theorem, suppose G large enough so that $(\log_2 \log_2 |G|)^3 < \gamma(\log_2 |G|)^{1/2}$, and thus $[G:\Phi] > |G|^{(\log_2 \log_2 |G|)^3/\gamma \log_2 |G|}$. Since $\Phi(G/\Phi) = \{1\}$, by Pyber's theorem,

$$\begin{aligned} k(G) > k(G/\Phi) &\geq [G:\Phi]^{\gamma/(\log_2 \log_2 [G:\Phi])^2} \\ &> |G|^{(\log_2 \log_2 |G|/\log_2 \log_2 [G:\Phi])^2 (\log_2 \log_2 |G|/\log_2 |G|)} > \log_2 |G|, \end{aligned}$$

the desired result.

Next we consider those solvable groups G satisfying $[G:\Phi] < |G|^{1/\sqrt{\log_2 |G|}}$, and hence $[G:F] < |G|^{1/\sqrt{\log_2 |G|}}$. By assumption, N is a minimal normal subgroup of G and $k(G/N) \geq (1+\epsilon)\log_2 [G:N]$. If $(|N|-1)/\log_2 |N| \geq |G|^{1/\sqrt{\log_2 |G|}}$, then

$$\frac{|N|-1}{\log_2 |N|} > [G:F],$$

and from Lemma 3.3(b) we conclude that $k(G) \geq \log_2 |G|$. So finally we assume that $(|N|-1)/\log_2 |N| < |G|^{1/\sqrt{\log_2 |G|}}$. If $|N| \leq 25$, then

$$k(G) > k(G/N) \geq (1+\epsilon)\log_2 [G:N] \geq (1+\epsilon)(\log_2 |G| - \log_2 25),$$

and the latter is greater than or equal to $\log_2 |G|$ if $|G| \geq 5^{2(1+1/\epsilon)}$. If $|N| \geq 25$, then

$$|N|^{1/2} \leq \frac{|N|-1}{\log_2 |N|} < |G|^{1/\sqrt{\log_2 |G|}},$$

which implies that $[G:N] > |G|^{1-2/\sqrt{\log_2 |G|}}$, and

$$k(G) > k(G/N) \geq (1+\epsilon)\log_2 [G:N] > (1+\epsilon)(\log_2 |G| - 2\sqrt{\log_2 |G|}).$$

Here $k(G) > \log_2 |G|$ when $|G| \geq 2^{4(1+1/\epsilon)^2}$. □

As mentioned earlier, Theorem 3.12 holds when the base of the logarithm is 2 or greater. For example, we have the following corollary.

COROLLARY 3.13. *For all solvable groups G with $|G|$ large enough, if N is a minimal normal subgroup of G and $k(G/N) \geq \frac{3}{4}\log_2 [G:N]$, then $k(G) \geq \log_3 |G|$.*

PROOF. As in the theorem, first consider solvable G for which $[G:\Phi] \geq |G|^{1/\sqrt{\log_3 |G|}}$. Since G may be assumed nonnilpotent, $[G:\Phi] \geq 6$ so

$$\left(\frac{\log_2 \log_2 [G:\Phi]}{\log_3 \log_3 [G:\Phi]}\right)^2 < 10.$$

[1] With considerably more effort (see Proposition 2.3 and its corollary in [Be3]), we have shown that under the latter condition on $|\Phi|$, for all large enough $|G|$ (depending only on $t > 0$) $k(G) > (\log_2 |G|)^t$. Even more follows from Theorem 3.1 of Keller [Ke], but the proof here is much simpler and this is all we need.

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(Note that $\log_2 \log_2 n = \log_2 \log_2 3 + (\log_2 3) \log_3 \log_3 n$ always holds.) Set $\beta_0 := \gamma/10$ (γ being Pyber's constant), so by Pyber's theorem,

$$\begin{aligned} k(G) > k(G/\Phi) &\geq [G:\Phi]^{\gamma/(\log_2 \log_2 [G:\Phi])^2} \\ &> [G:\Phi]^{\beta_0/(\log_3 \log_3 [G:\Phi])^2} \\ &> [G:\Phi]^{\beta_0/(\log_3 \log_3 |G|)^2} \\ &\geq |G|^{\beta_0/(\log_3 \log_3 |G|)^2 \sqrt{\log_3 |G|}}. \end{aligned}$$

If $|G|$ is so large that $(\log_3 \log_3 |G|)^3 \leq \beta_0 \sqrt{\log_3 |G|}$, then

$$k(G) > |G|^{\beta_0/\sqrt{\log_3 |G|}(\log_3 \log_3 |G|)^2} \geq |G|^{\log_3 \log_3 |G|/\log_3 |G|} = \log_3 |G|.$$

Working in base 3, when $[G:\Phi] < |G|^{1/\sqrt{\log_3 |G|}}$ the remainder of the proof goes through, since Lemma 3.3(b) makes no reference to the base. □

Example 3.14.

(a) Let G be solvable, with $N \leq M \trianglelefteq G$, N minimal normal in G, M/N abelian and G/M nilpotent. Then G/N is abelian-by-nilpotent so $k(G/N) \geq \frac{3}{4} \log_2 [G:N]$ [Ca]. By Corollary 3.13, $k(G) \geq \log_3 |G|$ for $|G|$ large enough.

(b) Suppose that G is solvable, N is a minimal normal subgroup of G and G/N is supersolvable. Then $k(G/N) \geq \frac{3}{5} \log_2 [G:N] = (1 + \frac{1}{5}) \log_4 [G:N]$ [Ca]. By Theorem 3.9, for $|G|$ large enough, $k(G) \geq \log_4 |G| = \frac{1}{2} \log_2 |G|$.

In [Be3, Proposition 2.3] we proved that for all solvable groups G with *abelian* Frattini subgroup $\Phi(G)$, if $|G|$ is large enough (depending only on $t > 0$) then $k(G) > (\log_2 |G|)^t$, and Keller [Ke, Theorem 4.1] proved that $k(G) > |G|^{\beta/2+\beta}$. Here we obtain a $(\log |G|)^t$ lower bound for $k(G)$ assuming only that the nilpotence class of $\Phi(G)$ is 'small enough' with respect to $\log |G|$, and $|G|$ is large enough, depending only on t.

Theorem 3.15. *For all solvable groups G with $|G|$ large enough (depending only on $t \geq 1$), if the class $c(\Phi)$ satisfies $c \leq \sqrt{\log |G|}(1 - 1/\log \log |G|)$ then $k(G) > (\log |G|)^t$.*

Proof. As in the proofs of Proposition 2.3 and its corollary in [Be3], when $[G:\Phi] \geq |G|^{1/\sqrt{\log |G|}}$ we use Pyber's theorem to prove that when $|G|$ is large enough, depending only on t, $k(G) > (\log |G|)^t$.

Suppose on the other hand that $[G:\Phi] < |G|^{1/\sqrt{\log |G|}}$. By assumption, $\Phi(G)$ has nilpotence class c, so

$$k(\Phi) \geq |\Phi|^{1/c} > |G|^{(1/c)(1-1/\sqrt{\log |G|})}.$$

Now

$$k(G) \geq k(G/\Phi) + \frac{k(\Phi) - 1}{[G:\Phi]} > \frac{k(\Phi)}{[G:\Phi]} > \frac{|G|^{(1/c)(1-1/\sqrt{\log |G|})}}{[G:\Phi]}.$$

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Again using our assumption that $[G:\Phi] < |G|^{1/\sqrt{\log|G|}}$,

$$k(G) > |G|^{(1/c)(1-1/\sqrt{\log|G|})-1/\sqrt{\log|G|}}.$$

Thus $k(G) > (\log|G|)^t$ as long as

$$\frac{1}{c}\Big(1-\frac{1}{\sqrt{\log|G|}}\Big) > \frac{t\log\log|G|}{\log|G|} + \frac{1}{\sqrt{\log|G|}},$$

that is, as long as

$$c(\sqrt{\log|G|} + t\log\log|G|) < \log|G| - \sqrt{\log|G|}. \tag{3.3}$$

Finally, for all large enough $|G|$ (depending only on t), $(t\log\log|G|)^2 < \sqrt{\log|G|}$, that is,

$$(t\log\log|G| - 1)(t\log\log|G| + \sqrt{\log|G|}) < (\sqrt{\log|G|} - 1)(t\log\log|G|),$$

which is equivalent to

$$1 - \frac{1}{t\log\log|G|} < \frac{\sqrt{\log|G|}-1}{t\log\log|G| + \sqrt{\log|G|}}.$$

By hypothesis,

$$c \le \sqrt{\log|G|}\Big(1-\frac{1}{\log\log|G|}\Big) \le \sqrt{\log|G|}\Big(1-\frac{1}{t\log\log|G|}\Big),$$

since $t \ge 1$. But the latter is less than $(\log|G| - \sqrt{\log|G|})/(\sqrt{\log|G|} + t\log\log|G|)$ so (3.3) is indeed satisfied, and $k(G) > (\log|G|)^t$ in each case. □

Remark 3.16. Keller [Ke, Theorem 3.1] proved that when $\Phi(G) = 1$, $k(G) \ge |G|^\beta$, where $\beta < 1$ is a positive constant. Thus $k(G) > k(G/\Phi) \ge [G:\Phi]^\beta$, and if $|\Phi| \le |G|^{1-1/\sqrt{\log|G|}}$ we have $k(G) > |G|^{\beta/\sqrt{\log|G|}} > (\log|G|)^t$ for all sufficiently large $|G|$ (depending only on t). On the other hand, if $|\Phi| > |G|^{1-1/\sqrt{\log|G|}}$, then (as shown in the proof above) $k(G) > |G|^{(1/c)(1-1/\sqrt{\log|G|})-1/\sqrt{\log|G|}}$. For $c \le \frac{2}{3}\sqrt{\log|G|}$ it is straightforward to show that this lower bound for $k(G)$ is (for all large enough $|G|$) greater than $((c/2)|G|^{1/c})^{\beta/3}$, the lower bound given in [Ke, Theorem 4.1].

We are now able to generalise the last statement of Lemma 3.1, no longer assuming that $\Phi(N)$ is abelian.

Corollary 3.17. *Suppose N is solvable and $N \trianglelefteq G$, with*

(i) $k(G/N) \ge \beta[G:N]^\alpha$ *$(0 < \alpha < 1 < \beta)$ and*
(ii) *the nilpotence class* $c(\Phi(N)) \le \sqrt{\log|N|}(1 - 1/\log\ \log|N|)$.

Then $k(G) \ge \log|G|$ when $|G|$ is large enough (depending only on α, β).

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PROOF. We will show that hypothesis (ii) of Lemma 3.1 is also satisfied for the pair (G, N), and thus the conclusion follows. By hypothesis (i), $k(G) > k(G/N) > [G : N]^{\alpha}$, and the latter is greater than or equal to $\log |G|$ when $|N| \le |G|/(\log |G|)^{1/\alpha}$. So suppose that $|N| \ge |G|/(\log |G|)^{1/\alpha}$. According to the proof of Theorem 3.15 (with N replacing G and $t = 1 + 1/\alpha$), we only need $\sqrt{\log |N|}/(\log \log |N|)^2 > (1 + 1/\alpha)^2$ to ensure that $k(N) \ge (\log |N|)^{1+1/\alpha}$ and hence that hypothesis (ii) of Lemma 3.1 is also satisfied. Since $|N| \ge |G|/(\log |G|)^{1/\alpha}$ and $\sqrt{\log x}/(\log \log x)^2$ is an increasing function for $\log \log x > 4$, we conclude that when $|G|$ is large enough (depending on α), hypothesis (ii) of Lemma 3.1 is satisfied along with hypothesis (i), and the desired conclusion follows. □

4. $k(G/N) \ge (\log[G : N])^t$

Up to this point we have assumed that either $k(G/N) \ge \beta[G : N]^{\alpha}$ or $k(G/N) \ge \beta \log[G : N]$, β a positive constant. But sometimes (the best) we may assume is that $k(G/N) \ge (\log[G : N])^t$, $t \ge 2$. (Again, we note that no collection $\{G\}$ is known with $|G| \to \infty$ and $k(G) < (\log |G|)^2$.)

LEMMA 4.1. *Let $N \trianglelefteq G$, N nilpotent and $k(G/N) \ge (\log[G : N])^t$, $t \ge 2$. If N has nilpotence class $c \ge 1$, then $k(G) > (\log |G|)^{t-1}$ for all such G with $|G|$ large enough, depending only on c, t.*

PROOF. We prove that with these hypotheses $k(G) > (\log |G|)^{t-1}$ as long as $\{|G|, c, t\}$ satisfy

$$(\log |G|)^{1-1/t}((\log |G|)^{1/t} - (c + 1)) \ge c(t - 1) \log \log |G|. \tag{4.1}$$

With $\log(\cdot) = \log_b(\cdot)$, we first note that $(\log[G : N])^t > (\log |G|)^{t-1}$ if and only if $[G : N] > b^{(\log |G|)^{1-1/t}}$. So we assume that $|N| \ge |G|/b^{(\log |G|)^{1-1/t}}$, which is equivalent to

$$\frac{|N|^{1+1/c}}{|G|} \ge \frac{|G|^{1/c}}{b^{(1+1/c)(\log |G|)^{1-1/t}}}.$$

But $k(N) \ge |N|^{1/c}$, and hence, by Lemma 2.3(a), $k(G) > |G|^{1/c}/b^{(1+1/c)(\log |G|)^{1-1/t}}$. Thus $k(G) > (\log |G|)^{t-1}$ as long as $|G| \ge (\log |G|)^{c(t-1)} b^{(c+1)(\log |G|)^{1-1/t}}$, which is equivalent to (4.1). □

Note. Suppose that N is abelian ($c = 1$). It is easy to check that (4.1) follows from $|G| \ge b^{3^t}$ and

$$1 - \frac{1}{t} \ge \frac{\log \log \log |G| + \log(t - 1)}{\log \log |G|}.$$

If $b = 3$, the latter follows from $|G| \ge 3^{3^t}$ and $t \ge 2$. If $b = 2$, (4.1) follows from $|G| \ge 2^{2^{2t}}$ and $t \ge 2$.

COROLLARY 4.2. *Suppose that N is a nilpotent normal subgroup of G and the nilpotence class c of N satisfies $2c + 1 \le (\log |G|)^{1/2}$. If also $k(G/N) \ge (\log[G : N])^2$, then $k(G) > \log |G|$ for all such G with $|G|$ large enough.*

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Proof. Our assumption on c yields (4.1) of Lemma 4.1, with $t = 2$. Hence $k(G) > \log |G|$. □

Question 4.3. When $\Phi(G) = 1$ (or more generally when $F(G)$ is abelian) does $k(G/F) \geq (\log[G : F])^2$ hold? If so, then $k(G/F) \geq (\log[G : F])^2$ *always*, since $\Phi(G/\Phi) = 1$ and $F(G/\Phi) = F(G)/\Phi(G)$ (is abelian) so $G/F(G) \cong G/\Phi/F(G/\Phi)$. In general, Corollary 4.2 implies that when $|G|$ is large enough, $k(G/F) \geq (\log[G : F])^2$ and the nilpotence class $c(F)$ satisfies $c \leq ((\log |G|)^{1/2} - 1)/2$, then $k(G) \geq \log |G|$.

Corollary 4.4. *If $k(G/N) \geq (\log[G : N])^t$ $(t \geq 2)$, then $k(G/N') \geq \log[G : N']^{t-1}$, whenever $[G : N']$ is large enough, depending only on t.*

Proof. In Lemma 4.1, replace G by G/N' and N by N/N'. The conclusion follows as long as $[G : N']$ satisfies (4.1) with respect to t, when $c = 1$. □

Lemma 4.5. *Let $y > x \geq b^e$, $t \geq 2$, and*

(i) $(\log x)^{1-1/t}((\log x)^{1/t} - 2) \geq (t - 1) \log \log x$, *where* $\log(\cdot) = \log_b(\cdot)$.

Then

(ii) $(\log y)^{1-1/(t-1)}((\log y)^{1/(t-1)} - 2) \geq (t - 2) \log \log y$.

Proof. Note that (ii) is automatically satisfied when $t = 2$, since $y > b^2$. So assume that $t \geq 3$, and we first check that (i) $\Longrightarrow$ (ii) follows from

$$\frac{(\log y)(1 - 2(\log y)^{-1/t-1})}{(t - 2) \log \log y} > \frac{(\log x)(1 - 2(\log x)^{-1/t})}{(t - 1) \log \log x} \geq 1. \tag{4.2}$$

Since $\log x/\log \log x$ is an increasing function for $x \geq b^e$,

$$\frac{\log y}{\log \log y} > \frac{\log x}{\log \log x}.$$

Also, $\log y > (\log x)^{(t-1)/t}$ implies that $1 - 2(\log y)^{-1/t-1} > 1 - 2(\log x)^{-1/t}$, and (4.2) follows. □

Theorem 4.6. *Suppose that G is solvable, $N \trianglelefteq G$ and $k(G/N) \geq (\log[G : N])^{d(N)+1}$, $d(N)$ the derived length of N. Then $k(G) \geq \log |G|$, as long as $[G : N']$ is large enough, depending only upon $d(N)$.*

Proof. We will prove that $k(G) \geq \log |G|$ as long as (4.1) of Lemma 4.1 is satisfied, with $[G : N']$ replacing $|G|$ and $d(N) + 1$ replacing t, always with $c = 1$.

When N is abelian, the conclusion follows from Lemma 4.1, with $c = 1$ and $t = 2$. When $d(N) = 2$ the assumption is that $k(G/N) \geq (\log[G : N])^3$. If $[G : N']$ satisfies (4.1) with $t = 3$, then from Corollary 4.4 $k(G/N') \geq (\log[G : N'])^2$. Here N' is abelian so we may again apply Lemma 4.1 with $c = 1$, $t = 2$ and conclude that $k(G) \geq \log |G|$ as long as $|G|$ satisfies (4.1) with $t = 2$. From Lemma 4.5 with $t = 3$, $[G : N']$ replacing x and $[G : N''] = |G|$ replacing y, we see that $|G|$ indeed satisfies (4.1) with $t = 2$.

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Assume for an inductive proof that the theorem is true whenever $d(N) = n$. Now let $d(N) = n + 1$ and $k(G/N) \geq (\log[G : N])^{d(N)+1} = (\log[G : N])^{n+2}$. Suppose also that $[G : N']$ satisfies (4.1) with $t = n + 2$. From Corollary 4.4,

$$k(G/N') \geq (\log[G : N'])^{n+1} = (\log[G : N'])^{d(N')+1}.$$

From our inductive hypothesis ($d(N') = n$), $k(G) \geq \log |G|$ as long as $[G : N'']$ satisfies (4.1) with $t = n + 1$. But Lemma 4.5, with $t = n + 2$, $[G : N']$ replacing x, and $[G : N'']$ replacing y, guarantees that $[G : N'']$ indeed satisfies (4.1) with $t = n + 1$. Thus the theorem is also true when $d(N) = n + 1$. □

As mentioned, Keller [Ke, Theorem 3.1] proved that when G is solvable and $\Phi(G) = 1$, $k(G) \geq |G|^{\beta}$, where $\beta < 1$ is a positive constant. We now use this to significantly improve the result of [Be2, Theorem 1] that if G has derived length $d(G)$, then $k(G) \geq |G|^{1/2^d-1}$, shifting attention to $d(F(G))$.

Theorem 4.7. *Suppose that G is a solvable group with Fitting subgroup $F(G)$. Then for each $n \geq 1$,*

$$k(G/F^{(n)}(G)) \geq [G : F^{(n)}(G)]^{1/(1+1/\beta)2^n-1} \tag{4.3}$$

where β is the constant from Keller's theorem. In particular,

$$k(G) \geq |G|^{1/(1+1/\beta)2^d-1}$$

where $d = d(F)$ is the derived length of $F(G)$.

Proof. From Keller's theorem, $k(G/\Phi) \geq [G : \Phi]^{\beta}$. If $F(G)$ is abelian, so is $\Phi(G)$, and using Lemma 2.3(b) with $N = \Phi$ and $\alpha = 1$ we obtain the inequality for $k(G)$ when $d = 1$. If $N \trianglelefteq G$ and $N \leq \Phi(G)$, then $\Phi(G/N) = \Phi(G)/N$ and $F(G/N) = F(G)/N$ [Hu, III. 3.4, 4.2]. Thus

$$k((G/F')/\Phi(G/F')) = k(G/\Phi) \geq [G : \Phi]^{\beta} = [G/F' : \Phi(G/F')]^{\beta},$$

and $\Phi(G/F')$ is abelian (Lemma 2.3(a)). As before, now with $N = \Phi(G/F')$, we conclude that $k(G/F') \geq [G : F']^{1/(1+2/\beta)}$, and thus inequality (4.3) with $n = 1$. If, in addition, $F'(G)$ is abelian, another use of Lemma 2.3(b) with $N = F'(G)$, $\alpha = 1$ and β replaced by $(1 + 2/\beta)^{-1}$ yields the desired inequality when $d = 2$.

To complete the proof of (4.3) by induction, we assume that $n \geq 2$, and for all solvable groups G,

$$k(G/F^{(n-1)}(G)) \geq [G : F^{(n-1)}(G)]^{1/(1+1/\beta)2^{n-1}-1}. \tag{4.4}$$

First note that $F'(G/F^{(n)}) = (F/F^{(n)})' = F'/F^{(n)}$ so $F''(G/F^{(n)}) = F''/F^{(n)} \ldots$ and finally $F^{(n-1)}(G/F^{(n)}) = F^{(n-1)}/F^{(n)}$ is abelian. Next substitute $G/F^{(n)}(G)$ for G in (4.4), and use $G/F^{(n-1)} \cong (G/F^{(n)})/F^{(n-1)}(G/F^{(n)})$ to obtain

$$k((G/F^{(n)})/F^{(n-1)}(G/F^{(n)})) \geq [(G/F^{(n)}) : F^{(n-1)}(G/F^{(n)})]^{1/(1+1/\beta)2^{n-1}-1}.$$

Since $F^{(n-1)}(G/F^{(n)})$ is abelian we use Lemma 2.3(b) with $N = F^{(n-1)}(G/F^{(n)})$, $\alpha = 1$ and β replaced by $1/(1 + 1/\beta)2^{n-1} - 1$ to obtain inequality (4.3). □

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Setting $\beta_0 = \beta/(\beta + 1)$ immediately leads to the following corollary.

Corollary 4.8. *If* $2^{d(F)} \le \beta_0(\log |G|/\log\log |G| + 1)$*, then* $k(G) \ge \log |G|$.

Remark 4.9. If G is a nilpotent group of nilpotence class c, then $d(G) \le \lfloor \log_2 c \rfloor + 1$ [Hu, III. 2.12], so $k(G) \ge \log_2 |G|$ when

$$c(F(G)) \le \frac{\beta_0}{2}\left(\frac{\log |G|}{\log\log |G|} + 1\right).$$

This may be compared to Corollary 4.2, and more importantly to Theorem 3.15, and Corollary 3.2(b)(ii).

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