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# ASYMPTOTIC CONDITIONAL DISTRIBUTION OF EXCEEDANCE COUNTS

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**Abstract**

We investigate the asymptotic distribution of the number of exceedances among $d$ identically distributed but not necessarily independent random variables (RVs) above a sequence of increasing thresholds, conditional on the assumption that there is at least one exceedance. Our results enable the computation of the *fragility index*, which represents the expected number of exceedances, given that there is at least one exceedance. Computed from the first $d$ RVs of a strictly stationary sequence, we show that, under appropriate conditions, the reciprocal of the fragility index converges to the extremal index corresponding to the stationary sequence as $d$ increases.

*Keywords:* Exceedance over high threshold; fragility index; multivariate extreme value theory; peaks-over-threshold approach; copula; generalized Pareto distribution (GPD); GPD copula; D-norm; extremal index

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## 1. Introduction

Since the pioneering papers of Balkema and de Haan (1974) and Pickands (1975), it is well known that the distribution of a random exceedance above a high threshold can be reasonably approximated by a *generalized Pareto distribution* (GPD). This led to the *peaks-over-threshold approach* (POT approach), where a GPD is fitted to the exceedances above a high threshold in a given sample, which is by now quite common in statistical analyses; see, for example, Reiss and Thomas (2007, Chapter 5), Beirlant *et al.* (2004, Chapter 5), and Embrechts *et al.* (1997, Chapter 6).

Much less seems to be known about the (random) *number of exceedances*, unless the observations are independent and identically distributed, in which case the number of exceedances above a high threshold obviously follows a binomial distribution with a small probability of success and, thus, can be approximated by a Poisson distribution (see Barbour *et al.* (1992)).

We consider in this paper a random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$, whose components $X_i$ are identically distributed but not necessarily independent. Keeping the dimension $d$ fixed, we are interested in the asymptotic conditional distribution of exceedance counts given that there is at

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least one exceedance (ACDEC). Specifically, choose a threshold $s \in \mathbb{R}$ and denote by

$$N_s := \sum_{i=1}^{d} 1_{(s,\infty)}(X_i)$$

the number of exceedances among $X_1, \dots, X_d$. We want to study the asymptotic conditional distribution of $N_s$ as the threshold increases, i.e.

$$p_k := \lim_{s\nearrow} \mathrm{P}(N_s = k \mid N_s > 0) = \lim_{s\nearrow} \frac{\mathrm{P}(N_s = k)}{\mathrm{P}(N_s > 0)}, \qquad 1 \le k \le d,$$

if it exists.

Note that we keep the number $d$ fixed. If $X_1, \dots, X_d$ is a block of random variables (RVs) taken from a stationary process satisfying some mixing condition, and the block size $d = d(n)$ satisfies $d(n) \to \infty$ and $d(n)/n \to 0$ as $n \to \infty$, then the asymptotic *cluster size distribution*

$$\pi_k := \lim_{n\to\infty} \mathrm{P}(N_{s(n)} = k \mid N_{s(n)} > 0), \qquad k \in \mathbb{N},$$

exists under suitable regularity conditions (see Hsing *et al.* (1988, Theorems 4.1, 4.2)). We refer the reader to Embrechts *et al.* (1997, Section 8.1) for a discussion. An investigation of the asymptotic distribution of general *cluster functionals* is provided in Yun (2000) and Segers (2003), among others; we refer the reader to Beirlant *et al.* (2004, Section 10.3.2) for an overview.

If the ACDEC actually exists then we can define the *fragility index* (FI) corresponding to $\{X_1, \dots, X_d\}$ as the asymptotic expectation of the number of exceedances given that there is at least one exceedance:

$$\mathrm{FI} := \lim_{s\nearrow} \mathrm{E}(N_s \mid N_s > 0) = \sum_{k=1}^{d} k p_k.$$

The fragility index was introduced in Geluk *et al.* (2007) to measure the stability of the stochastic system $\{X_1, \dots, X_d\}$. The system is called *stable* if $\mathrm{FI} = 1$, otherwise it is called *fragile*. The collapse of a bank, symbolized by an exceedance $X_i > s$, would be a typical example, illustrating the fragility index as a measure of joint stability among a portfolio of $d$ banks.

Using tools from multivariate extreme value theory, we show in this paper that the ACDEC exists, if the copula of the random vector $\boldsymbol{X}$ is in the domain of attraction of a multivariate extreme value distribution. In this case, the ACDEC can be represented in terms of a norm on $\mathbb{R}^d$. In particular, for the usual $L_\lambda$-norm with $\lambda \in [1, \infty]$, the ACDEC turns out to be quite simple and, in addition, enables the computation of the *asymptotic* ACDEC with an increasing dimension $d$. The asymptotic ACDEC is in this case the distribution of a stopping rule. This will be done in Section 3. The fragility index will be computed under quite general conditions in Section 4.

Computed from the first $d$ RVs of a strictly stationary sequence $(X_k)_{k\in\mathbb{N}}$, we show that, under appropriate conditions, the reciprocal of the fragility index converges to the extremal index associated with $(X_k)_{k\in\mathbb{N}}$ as $d$ increases. This will be shown in Section 5.

Our approach immediately enables the computation of the *extended* fragility index, defined as the asymptotic expected number of exceedances, given that there are at least $m \ge 1$ exceedances:

$$\mathrm{FI}(m) := \lim_{s\nearrow} \mathrm{E}(N_s \mid N_s \ge m) = \frac{\sum_{k=m}^{d} k p_k}{\sum_{k=m}^{d} p_k}, \qquad 1 \le m \le d.$$

But now we encounter the problem that the denominator $\sum_{k=m}^{d} p_k$ in the definition of $\mathrm{FI}(m)$ may vanish, although the ACDEC exists.

Take, for example, independent components $X_1, \ldots, X_d$. Then $N_s$ follows a binomial distribution $B(d, p(s))$ with $p(s) = \mathrm{P}(X_1 > s)$ and, thus,

$$p_k = \lim_{s \nearrow} \frac{\binom{d}{k} p(s)^k (1-p(s))^{d-k}}{\sum_{j=1}^{d} \binom{d}{j} p(s)^j (1-p(s))^{d-j}} = \begin{cases} 1, & k = 1, \\ 0, & 2 \le k \le d. \end{cases}$$

In this case the $\mathrm{FI}(m)$ would not be defined for $m \ge 2$, but $\mathrm{FI} = \mathrm{FI}(1) = 1$.

If, on the other hand, $X_1 = \cdots = X_d$ almost surely then, clearly,

$$p_k = \begin{cases} 1, & k = d, \\ 0, & 1 \le k \le d-1, \end{cases}$$

and $\mathrm{FI}(m)$ is defined for any $1 \le m \le d$ with $\mathrm{FI}(m) = d$. In Section 6 we provide a precise characterization of the case $\sum_{k=m}^{d} p_k = 0$ in terms of multivariate extreme value theory.

The mathematical results established in Section 6 enable the characterization of the case of no exceedance $\mathrm{P}(X_k > s,\ k \in K) = 0$ for a subset $K \subset \{1, \ldots, d\}$, although $\mathrm{P}(X_k > s) > 0$, $k \in K$. This will be achieved in Section 7.

By Sklar's theorem (see, for example, Nelson (2006, Theorem 2.10.9)) we can assume the representation

$$(X_1, \ldots, X_d) = (F^{-1}(U_1), \ldots, F^{-1}(U_d)),$$

where $F$ is the (univariate) distribution function (DF) of $X_1$, and the random vector $\boldsymbol{U} = (U_1, \ldots, U_d)$ follows a *copula* on $\mathbb{R}^d$, i.e. each $U_i$ is distributed uniformly on $(0, 1)$. By $F^{-1}(q) := \inf\{t \in \mathbb{R} \colon F(t) \ge q\}$, $q \in (0, 1)$, we denote the generalized inverse of $F$.

From the equivalence $F^{-1}(q) > t \Leftrightarrow q > F(t)$, $q \in (0, 1)$, $t \in \mathbb{R}$, we obtain

$$N_s = \sum_{i=1}^{d} 1_{(s,\infty)}(F^{-1}(U_i)) = \sum_{i=1}^{d} 1_{(F(s),1]}(U_i).$$

Throughout this paper, we therefore consider a random vector $\boldsymbol{U}$ following an arbitrary copula $C$ on $\mathbb{R}^d$, denoted by $\boldsymbol{U} \sim C$; $1 - c < 1$ will be a threshold converging to 1 and

$$N_{1-c} = \sum_{i=1}^{d} 1_{(1-c,1]}(U_i)$$

is the number of exceedances among $U_1, \ldots, U_d$ above $1 - c$.

## 2. Auxiliary results and tools

It turns out that multivariate extreme value theory provides the tools to investigate the ACDEC

$$p_k = \lim_{c \downarrow 0} \mathrm{P}(N_{1-c} = k \mid N_{1-c} > 0) = \lim_{c \downarrow 0} \frac{\mathrm{P}(N_{1-c} = k)}{\mathrm{P}(N_{1-c} > 0)}, \qquad 1 \le k \le d.$$

In this section we compile several definitions and results from multivariate extreme value theory. For the general theory, we refer the reader to the books de Haan and Ferreira (2006), Resnick (1987), (2007), Beirlant *et al.* (2004), and Falk *et al.* (2004), among others.

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A copula $C$ on $\mathbb{R}^d$ is said to be in the domain of attraction of a multivariate extreme value DF (EVDF) $G$, denoted by $C \in \mathcal{D}(G)$, if and only if

$$C^n\left(\left(1+\frac{x_1}{n}, \dots, 1+\frac{x_d}{n}\right)\right) \to G(\boldsymbol{x}) \quad \text{as } n \to \infty$$

for any $\boldsymbol{x} = (x_1, \dots, x_d) \le \mathbf{0} \in \mathbb{R}^d$. All operations on vectors are meant componentwise. The EVDF $G$ is characterized by its max-stability

$$G^n\left(\frac{\boldsymbol{x}}{n}\right) = G(\boldsymbol{x}), \qquad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d,\ n \in \mathbb{N},$$

and it has standard negative exponential margins $G(x\boldsymbol{e}_i) = \exp(x),\ x \le 0,\ 1 \le i \le d$, where $\boldsymbol{e}_i$ denotes the $i$th unit vector in $\mathbb{R}^d$. More precisely, there exists a norm $\|\cdot\|_D$ on $\mathbb{R}^d$ with $\|\boldsymbol{e}_i\|_D = 1,\ 1 \le i \le d$, such that

$$G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D), \qquad \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d;$$

see Falk *et al.* (2004, Section 4.3).

The following result, which essentially goes back to Deheuvels (1978), (1984), is established in Aulbach *et al.* (2012).

**Theorem 2.1.** *We have $C \in \mathcal{D}(G)$ if and only if there exists a norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that*

$$\lim_{\boldsymbol{y}\uparrow\mathbf{1}} \frac{C(\boldsymbol{y}) - (1 - \|\boldsymbol{y}-\mathbf{1}\|_D)}{\|\boldsymbol{y}-\mathbf{1}\|_D} = 0. \tag{2.1}$$

*In this case $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D),\ \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$.*

Viewed as a function from $[0,\infty)^d$ to $[0,\infty)$, $\|\cdot\|_D$ is also known as the *stable tail dependence function* (see Huang (1992), Drees and Huang (1998), and Beirlant *et al.* (2004)).

In the bivariate case with $\boldsymbol{x} = (1,1)$, the number $2 - \|(1,1)\|_D$ is the *tail dependence parameter*:

$$\lim_{c\downarrow 0} \mathrm{P}(U_2 > 1-c \mid U_1 > 1-c) = 2 - \|(1,1)\|_D;$$

see Reiss and Thomas (2007, Chapter 13) and Beirlant *et al.* (2004, Section 8.3.2).

A $D$-norm $\|\cdot\|_D$ is in general monotone, i.e.

$$\|\boldsymbol{x}\|_D \le \|\boldsymbol{y}\|_D, \qquad \mathbf{0} \le \boldsymbol{x} \le \boldsymbol{y},$$

and always between the maximum norm and the $L_1$-norm, i.e.

$$\|\boldsymbol{x}\|_\infty = \max(x_1, \dots, x_d) \le \|\boldsymbol{x}\|_D \le \sum_{i\le d} |x_i|, \qquad \mathbf{0} \le \boldsymbol{x} \in \mathbb{R}^d;$$

see Falk *et al.* (2004, Section 4.3). A complete characterization of a $D$-norm and, thus, an answer to the question of when an arbitrary norm is a $D$-norm is given in Hofmann (2009).

The following result is an immediate consequence of Theorem 2.1.

**Corollary 2.1.** *Suppose that $\boldsymbol{U} \sim C \in \mathcal{D}(G)$. Then there exists a norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that, for any nonempty subset $K \subset \{1, \dots, d\}$,*

$$\mathrm{P}(U_k \le 1-c,\ k \in K) = 1 - c\left\|\sum_{k\in K} \boldsymbol{e}_k\right\|_D + o(c)$$

*as $c \downarrow 0$. In this case $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D),\ \boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$.*

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Note that we have equality in the preceding result, i.e.

$$\mathrm{P}(U_k \le 1-c,\ k \in K) = 1 - c\left\| \sum_{k\in K} \boldsymbol{e}_k \right\|_D,$$

for $c$ close to 0 if $C$ is a *GPD copula*, i.e. if $C$ has the representation

$$C(\boldsymbol{u}) = 1 - \|(1-u_1, \dots, 1-u_d)\|_D \tag{2.2}$$

for $\boldsymbol{u} \in (0,1]^d$ close to $(1,\dots,1)$; we refer the reader to Aulbach *et al.* (2012) for details.

## 3. Computation of the ACDEC

In this section we establish the ACDEC. By $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, we denote an arbitrary EVDF on $\mathbb{R}^d$ with standard exponential margins and corresponding $D$-norm $\|\cdot\|_D$; $\boldsymbol{U} = (U_1, \dots, U_d)$ denotes a random vector that follows a copula $C$ on $\mathbb{R}^d$. In the next lemma we compute the *unconditional* asymptotic distribution of exceedance counts.

**Lemma 3.1.** *Suppose that $C \in \mathcal{D}(G)$. Then we have*

(i) $\displaystyle \mathrm{P}(N_{1-c} = 0) = 1 - c\left\| \sum_{1\le j\le d} \boldsymbol{e}_j \right\|_D + o(c),$

(ii) $\displaystyle \mathrm{P}(N_{1-c} = k) = c \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subset\{1,\dots,d\}\\ |T|=d-j}} \left\| \sum_{i\in T} \boldsymbol{e}_i \right\|_D + o(c), \qquad 1 \le k \le d,$

*as $c \downarrow 0$.*

*Proof.* Corollary 2.1 immediately implies that

$$\mathrm{P}(N_{1-c} = 0) = \mathrm{P}(U_j \le 1-c,\ 1 \le j \le d) = 1 - c\left\| \sum_{j=1}^d \boldsymbol{e}_j \right\|_D + o(c)$$

for $c \downarrow 0$. For $1 \le k \le d$, we obtain, by the well-known additivity formula,

$$\begin{aligned}
&\mathrm{P}(N_{1-c} = k)\\
&\quad= \sum_{\substack{S\subset\{1,\dots,d\}\\ |S|=k}} \mathrm{P}(U_i > 1-c,\ i \in S,\ U_j \le 1-c,\ j \in S^{\complement})\\
&\quad= \sum_{\substack{S\subset\{1,\dots,d\}\\ |S|=k}} \mathrm{P}(U_i > 1-c,\ i \in S \mid U_j \le 1-c,\ j \in S^{\complement})\,\mathrm{P}(U_j \le 1-c,\ j \in S^{\complement})\\
&\quad= \sum_{\substack{S\subset\{1,\dots,d\}\\ |S|=k}} \Bigg[\Bigg(1 - \sum_{1\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\ |K|=r}} \mathrm{P}(U_i \le 1-c,\ i \in K \mid U_j \le 1-c,\ j \in S^{\complement})\Bigg)\\
&\qquad\qquad \times \mathrm{P}(U_j \le 1-c,\ j \in S^{\complement})\Bigg]\\
&\quad= \sum_{\substack{S\subset\{1,\dots,d\}\\ |S|=k}} \Bigg[\Bigg(1 - \sum_{1\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\ |K|=r}} \frac{\mathrm{P}(U_i \le 1-c,\ i \in K \cup S^{\complement})}{\mathrm{P}(U_j \le 1-c,\ j \in S^{\complement})}\Bigg)\\
&\qquad\qquad \times \mathrm{P}(U_j \le 1-c,\ j \in S^{\complement})\Bigg]
\end{aligned}$$

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$$= \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \Big[ \mathrm{P}(U_j \le 1-c,\ j \in S^{\complement}) \\ - \sum_{1\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \mathrm{P}(U_i \le 1-c,\ i \in K \cup S^{\complement}) \Big].$$

Corollary 2.1, together with the equality $\sum_{1\le r\le|S|}(-1)^{r+1}\sum_{K\subset S,\,|K|=r} 1 = 1$, now implies that

$$\begin{aligned}
\mathrm{P}(N_{1-c}=k) &= \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \Big[ 1 - c\Big\| \sum_{j\in S^{\complement}} \boldsymbol{e}_j \Big\|_D + o(c) \\
&\qquad - \sum_{1\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \Big(1 - c\Big\| \sum_{j\in K\cup S^{\complement}} \boldsymbol{e}_j \Big\|_D\Big)\Big] \\
&= \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \Big[ c\Big( \sum_{1\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \Big\| \sum_{j\in K\cup S^{\complement}} \boldsymbol{e}_j \Big\|_D - \Big\| \sum_{j\in S^{\complement}} \boldsymbol{e}_j \Big\|_D \Big) + o(c) \Big] \\
&= \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \Big[ o(c) + c \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \Big\| \sum_{j\in K\cup S^{\complement}} \boldsymbol{e}_j \Big\|_D \Big] \\
&= c \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \Big\| \sum_{j\in K\cup S^{\complement}} \boldsymbol{e}_j \Big\|_D + o(c).
\end{aligned}$$

With a suitable index transformation we obtain

$$\begin{aligned}
\mathrm{P}(N_{1-c}=k) &= c \sum_{\substack{S\subset\{1,\dots,d\}\\|S|=k}} \sum_{0\le r\le |S|} (-1)^{r+1} \sum_{\substack{K\subset S\\|K|=r}} \Big\| \sum_{\substack{j\in K\cup S^{\complement}=:T\\|T|=r+d-k}} \boldsymbol{e}_j \Big\|_D + o(c) \\
&= c \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{K\subset\{1,\dots,d\}\\|K|=r}} \sum_{\substack{T\supset K\\|T|=r+d-k}} \Big\| \sum_{i\in T} \boldsymbol{e}_i \Big\|_D + o(c) \\
&= c \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{T\subset\{1,\dots,d\}\\|T|=r+d-k}} \sum_{\substack{K\subset T\\|K|=r}} \Big\| \sum_{i\in T} \boldsymbol{e}_i \Big\|_D + o(c) \\
&= c \sum_{0\le r\le k} (-1)^{r+1} \sum_{\substack{T\subset\{1,\dots,d\}\\|T|=r+d-k}} \binom{r+d-k}{r} \Big\| \sum_{i\in T} \boldsymbol{e}_i \Big\|_D + o(c) \\
&= c \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subset\{1,\dots,d\}\\|T|=d-j}} \Big\| \sum_{i\in T} \boldsymbol{e}_i \Big\|_D + o(c),
\end{aligned}$$

which completes the proof of Lemma 3.1.

The next result is just a reformulation of Lemma 3.1.

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**Corollary 3.1.** *Suppose that $C \in \mathcal{D}(G)$. Then*

$$a_k := \lim_{c\downarrow 0} \frac{\mathrm{P}(N_{1-c} = k)}{c} = \sum_{0\le j\le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T\subset\{1,\dots,d\}\\ |T|=d-j}} \Big\| \sum_{i\in T} \boldsymbol{e}_i \Big\|_D$$

*for* $1 \le k \le d$, *and*

$$a_0 := \lim_{c\downarrow 0} \frac{1-\mathrm{P}(N_{1-c} = 0)}{c} = \Big\| \sum_{1\le j\le d} \boldsymbol{e}_j \Big\|_D.$$

Note that we have the equalities $a_k = \mathrm{P}(N_{1-c} = k)/c,\ 1 \le k \le d$, and $a_0 = (1-\mathrm{P}(N_{1-c} = 0))/c$ for $c$ close to 0 if $C$ is a GPD copula as in (2.2).

The next result is the main result of this section. It is an obvious consequence of Corollary 3.1.

**Theorem 3.1.** (ACDEC.) *Suppose that $C \in \mathcal{D}(G)$. Then*

$$p_k := \lim_{c\downarrow 0} \mathrm{P}(N_{1-c} = k \mid N_{1-c} > 0) = \frac{a_k}{a_0}, \qquad 1 \le k \le d,$$

*defines a probability distribution on* $\{1, \dots, d\}$.

Take, for example, $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_\lambda = (\sum_{i\le d} |x_i|^\lambda)^{1/\lambda}$ for $1 \le \lambda < \infty$ and $\|\boldsymbol{x}\|_\infty = \max_{i\le d} |x_i|$. Then, for $1 \le k \le d$,

$$p_k = \binom{d}{k} \sum_{0\le j\le k} (-1)^{k-j+1} \binom{k}{j} \Big(1 - \frac{j}{d}\Big)^{1/\lambda}. \tag{3.1}$$

The *Marshall–Olkin D*-norm is the convex combination of the maximum norm and the $L_1$-norm:

$$\|\boldsymbol{x}\|_{\mathrm{MO}} := \vartheta \|\boldsymbol{x}\|_1 + (1-\vartheta)\|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^d,\ \vartheta \in [0,1];$$

see Falk *et al.* (2004, Example 4.3.2). In this case we obtain

$$p_1 = \frac{\vartheta d}{\vartheta d + 1 - \vartheta}, \qquad p_d = \frac{1-\vartheta}{\vartheta d + 1 - \vartheta}, \qquad p_k = 0, \quad 2 \le k \le d-1.$$

In the particular case where the $D$-norm is the usual $L_\lambda$-norm with $\lambda \in [1,\infty]$, we can derive the limit

$$\lim_{d\to\infty} p_k = \lim_{d\to\infty} p_k(d) \tag{3.2}$$

of the ACDEC as the dimension $d$ increases. Since

$$p_k = \begin{cases} 1, & k = 1, \\ 0, & 2 \le k \le d, \end{cases}$$

in the $\lambda = 1$ case and

$$p_k = \begin{cases} 0, & 1 \le k \le d-1, \\ 1, & k = d, \end{cases}$$

in the $\lambda = \infty$ case, the limit behavior of $p_k$ in (3.2) is clear for $\lambda \in \{1, \infty\}$. We therefore restrict ourselves in the following to $\lambda \in (1, \infty)$.

The following auxiliary result will be crucial. It can be shown by induction.

**Lemma 3.2.** *We have, for $k \in \mathbb{N}$,*

$$\sum_{0\le j\le k}(-1)^j\binom{k}{j}j^i=\begin{cases}0, & 0\le i\le k-1,\\ (-1)^k k!, & i=k.\end{cases}$$

The next proposition provides the asymptotic ACDEC for the $L_\lambda$-norm.

**Proposition 3.1.** (Asymptotic ACDEC.) *Suppose that the underlying $D$-norm is the $L_\lambda$-norm with $1<\lambda<\infty$. Then we have, for $k\in\mathbb{N}$,*

$$p_k^*(\lambda):=\lim_{d\to\infty}p_k=\frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right).$$

*Proof.* Recall that

$$p_k=p_k(d)=\binom{d}{k}\sum_{0\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda},\qquad 1\le k\le d.$$

Set $f(x):=x^{1/\lambda}$, $x\ge 0$. Taylor's expansion of length $k$ implies that, for $\varepsilon\in(0,1)$,

$$f(1-\varepsilon)=f(1)+\sum_{1\le i\le k-1}\frac{f^{(i)}(1)}{i!}(-\varepsilon)^i+\frac{f^{(k)}(\xi)}{k!}(-\varepsilon)^k,$$

where $\xi\in(1-\varepsilon,1)$ and

$$f^{(i)}(x)=x^{1/\lambda-i}\prod_{0\le r\le i-1}\left(\frac{1}{\lambda}-r\right).$$

We thus obtain, for $1\le j\le k<d$ with $\varepsilon=j/d$,

$$\begin{aligned}\left(1-\frac{j}{d}\right)^{1/\lambda}&=1+\sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^i\frac{\prod_{0\le r\le i-1}(1/\lambda-r)}{i!}\\&\quad+\xi_j^{1/\lambda-k}\left(-\frac{j}{d}\right)^k\frac{\prod_{0\le r\le k-1}(1/\lambda-r)}{k!},\end{aligned}$$

where $\xi_j\in(1-j/d,1)$. This implies that, for fixed $1\le k<d$,

$$\begin{aligned}p_k&=\binom{d}{k}\sum_{0\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda}\\&=\binom{d}{k}\left((-1)^{k+1}+\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(1-\frac{j}{d}\right)^{1/\lambda}\right)\\&=\binom{d}{k}\Bigg((-1)^{k+1}+\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\Bigg\{1+\sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^i\frac{\prod_{0\le r\le i-1}(1/\lambda-r)}{i!}\\&\qquad+\xi_j^{1/\lambda-k}\left(-\frac{j}{d}\right)^k\frac{\prod_{0\le r\le k-1}(1/\lambda-r)}{k!}\Bigg\}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \binom{d}{k} \sum_{1\le j\le k} (-1)^{k-j+1}\binom{k}{j}\Bigg\{\sum_{1\le i\le k-1}\left(-\frac{j}{d}\right)^i \frac{\prod_{0\le r\le i-1}(1/\lambda - r)}{i!} \\
&\qquad\qquad\qquad\qquad + \xi_j^{1/\lambda-k}\left(-\frac{j}{d}\right)^k \frac{\prod_{0\le r\le k-1}(1/\lambda - r)}{k!}\Bigg\} \\
&= \binom{d}{k} \sum_{1\le i\le k-1} \frac{\prod_{0\le r\le i-1}(1/\lambda - r)}{i!}\left(\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(-\frac{j}{d}\right)^i\right) \\
&\quad + \binom{d}{k}\frac{\prod_{0\le r\le k-1}(1/\lambda - r)}{k!}\sum_{1\le j\le k}(-1)^{k-j+1}\binom{k}{j}\left(-\frac{j}{d}\right)^k \xi_j^{1/\lambda-k}.
\end{aligned}
$$

The first term on the right-hand side of this equation vanishes by Lemma 3.2. For fixed $k$ and $d \to \infty$, the second term converges to

$$
\begin{aligned}
\frac{\prod_{0\le r\le k-1}(1/\lambda - r)}{(k!)^2}\sum_{1\le j\le k}(-1)^{-j+1}\binom{k}{j} j^k &= (-1)^{k-1}\frac{\prod_{0\le r\le k-1}(1/\lambda - r)}{k!} \\
&= \frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right)
\end{aligned}
$$

by Lemma 3.2.

Note that $p_k^*(\lambda) = 1/(\lambda k)\prod_{j=1}^{k-1}(1-1/(j\lambda))$, $k \in \mathbb{N}$, is the distribution of a stopping time. Let $X_1, X_2, \dots$ be independent RVs with values in $\{0, 1\}$, and let

$$
\mathrm{P}(X_j = 0) = 1 - \frac{1}{j\lambda} = 1 - \mathrm{P}(X_j = 1), \qquad j \in \mathbb{N}.
$$

Set

$$
\tau(\lambda) := \min\{j \in \mathbb{N} \colon X_j = 1\}.
$$

Then, obviously,

$$
\mathrm{P}(\tau(\lambda) = k) = \frac{1}{\lambda k}\prod_{j=1}^{k-1}\left(1-\frac{1}{j\lambda}\right) = p_k^*(\lambda), \qquad k \in \mathbb{N}.
$$

Note that $\mathrm{P}(\tau(\lambda) < \infty) = 1$, $1 \le \lambda < \infty$, whereas $\mathrm{P}(\tau(\infty) = \infty) = 1$, if we include $\lambda \in \{1, \infty\}$ in our considerations.

Denote by $P_\lambda$ the ACDEC on $\mathbb{N}$ as in (3.1), i.e. $P_\lambda(k) = p_k(d)$, $k \in \mathbb{N}$. Then Proposition 3.1 can be formulated as follows, where '$\xrightarrow{\mathrm{w}}$' denotes weak convergence.

**Proposition 3.2.** *We have, for $\lambda \in [1, \infty)$, as $d \to \infty$,*

$$
P_\lambda \xrightarrow{\mathrm{w}} \tau(\lambda).
$$

## 4. Computation of the fragility index

In this section we compute the fragility index under the condition that $C \in D(G)$. The following theorem is the main result of this section.

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**Theorem 4.1.** *Suppose that* $C \in \mathcal{D}(G)$, $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$. *Then*

$$\mathrm{FI} = \frac{d}{\|\sum_{1\le j\le d} \boldsymbol{e}_j\|_D}.$$

*Proof.* We have

$$\begin{aligned}
\mathrm{E}(N_{1-c} \mid N_{1-c} > 0) &= \sum_{i=1}^{d} \mathrm{E}(1_{(1-c,1]}(U_i) \mid N_{1-c} > 0) \\
&= \sum_{i=1}^{d} \frac{\mathrm{P}(U_i > 1-c)}{1 - \mathrm{P}(N_{1-c} = 0)} \\
&= d\frac{c}{1 - \mathrm{P}(N_{1-c} = 0)} \\
&\to \frac{d}{\|\sum_{i=1}^{d} \boldsymbol{e}_i\|_D} \quad \text{as } c \downarrow 0
\end{aligned}$$

by Corollary 3.1.

The number

$$\varepsilon := \left\| \sum_{1\le j\le d} \boldsymbol{e}_j \right\|_D = \|(1,\dots,1)\|_D \in [1,d]$$

measures the dependence structure of the margins of $G$, and we have in particular, by Takahashi's (1988) theorem,

$$\varepsilon = 1 \quad \iff \quad \|\cdot\|_D = \|\cdot\|_\infty \quad \iff \quad \text{complete dependence of the margins}$$

and

$$\varepsilon = d \quad \iff \quad \|\cdot\|_D = \|\cdot\|_1 \quad \iff \quad \text{independence of the margins.}$$

The number $\varepsilon$ was introduced in Smith (1990) as the *extremal coefficient* of $G^*$, defined as that constant which satisfies

$$G^*(x,\dots,x) = F^{\varepsilon}(x), \qquad x \in \mathbb{R}, \tag{4.1}$$

where $G^*$ is an *arbitrary* $d$-dimensional EVDF with identical margins $G_j^* = F$, $j \le d$.

We have thus established in Theorem 4.1 the fact that $\varepsilon/d \in [1/d, 1]$ equals the reciprocal of the fragility index. This is in complete accordance with the extremal coefficient for stationary processes, which can be interpreted as the reciprocal of the mean cluster size of the limiting compound Poisson process; we refer the reader to Embrechts *et al.* (1997, Section 8.1). In Section 5 we will show that the reciprocal $1/\mathrm{FI}$ actually converges to the extremal index, if $(U_1,\dots,U_d)$ is a clipping from a stationary process and $d \to \infty$.

Using the $D$-norm representation of an EVDF, property (4.1) of $\varepsilon$ can easily be seen as follows. Transforming the margins of $G^*$ to the negative exponential distribution $F(x) = \exp(x)$, $x \le 0$, we can assume without loss of generality that $G^*(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$. Then we have, for $x \le 0$,

$$\begin{aligned}
G^*(x,\dots,x) &= \exp(-\|(x,\dots,x)\|_D) \\
&= \exp(x\|(1,\dots,1)\|_D) \\
&= \exp(x\varepsilon) \\
&= \exp(x)^{\varepsilon}.
\end{aligned}$$

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In the case where the $D$-norm is the $L_\lambda$-norm with $\lambda \in [1, \infty]$, we have $\varepsilon = d^{1/\lambda}$ and, thus, the fragility index is given by

$$\mathrm{FI} = d^{1-1/\lambda} = \begin{cases} 1, & \lambda = 1, \\ d, & \lambda = \infty. \end{cases}$$

Using Lemma 3.2, it is straightforward to also compute the variance corresponding to the fragility index for a general $D$-norm:

$$\begin{aligned} \sigma^2(\mathrm{FI}) &:= \lim_{c \downarrow 0} \mathrm{E}((N_{1-c} - \mathrm{FI})^2 \mid N_{1-c} > 0) \\ &= \sum_{k=1}^{d} k^2 p_k - \left(\sum_{k=1}^{d} k p_k\right)^2 \\ &= \frac{2d^2 - d - 2\sum_{1 \le i \ne j \le d} \|\boldsymbol{e}_i + \boldsymbol{e}_j\|_D}{\|\sum_{i=1}^{d} \boldsymbol{e}_i\|_D} - \left(\frac{d}{\|\sum_{i=1}^{d} \boldsymbol{e}_i\|_D}\right)^2. \end{aligned}$$

The variance vanishes, of course, for the $L_1$-norm and the maximum norm.

For the Marshall–Olkin $D$-norm $\|\boldsymbol{x}\|_\vartheta = \vartheta\|\boldsymbol{x}\|_1 + (1-\vartheta)\|\boldsymbol{x}\|_\infty$, $\vartheta \in [0, 1]$, we obtain $\varepsilon = d - (1-\vartheta)(d-1)$ as well as

$$\mathrm{FI}(\vartheta) = \frac{d}{d - (1-\vartheta)(d-1)}, \qquad \sigma^2(\mathrm{FI}(\vartheta)) = \vartheta(1-\vartheta)\frac{d(d-1)^2}{d - (1-\vartheta)(d-1)}.$$

## 5. Extremal index

In what follows we show that the reciprocal of the fragility index $\mathrm{FI}^{(d)}$ as a function of the dimension $d$ converges to the extremal index of a strictly stationary sequence. To adjust to the common notation of stationary processes, we switch in this chapter from the uniformly on $(0, 1)$ distributed RV $U_k$ to the initial $X_k$.

Let $(X_d)_{d \in \mathbb{N}}$ be a strictly stationary sequence of RVs, and let $\theta$ be a number in $[0, 1]$. Assume that, for every $\tau > 0$, there exists a sequence $(u_d)_{d \in \mathbb{N}}$ of numbers such that

$$\lim_{d \to \infty} d(1 - F(u_d)) = \tau, \tag{5.1}$$

where $F$ is the DF of $X_1$, and

$$\lim_{d \to \infty} \mathrm{P}\Big(\max_{1 \le k \le d} X_k \le u_d\Big) = \exp(-\theta\tau). \tag{5.2}$$

Then $\theta$ is called the *extremal index* of the sequence $(X_d)_{d \in \mathbb{N}}$. We refer the reader to Embrechts *et al.* (1997, Section 8.1) for a discussion of the extremal index. It is in particular well known (see Hsing *et al.* (1988)) that the extremal index is the reciprocal of the mean cluster size of the limiting compound process associated with the point process of the exceedances among $X_1, \ldots, X_d$ above $u_d$ for $d \to \infty$.

The following result links the fragility index with the extremal index.

**Theorem 5.1.** *Let $(X_d)_{d \in \mathbb{N}}$ be a strictly stationary sequence with extremal index $\theta$. Suppose that the copula $C^{(d)}$ associated with the vector $\boldsymbol{X}^{(d)} = (X_1, \ldots, X_d)$ satisfies the expansion*

$$C^{(d)}(\boldsymbol{y}) = 1 - \|\boldsymbol{1} - \boldsymbol{y}\|_{D^{(d)}} + o(d|1 - y|) \tag{5.3}$$

*with* $\boldsymbol{y} = (y, \dots, y)$ *uniformly for* $y \in [0, 1]$ *and* $d \in \mathbb{N}$*, where* $\|\cdot\|_{D^{(d)}}$ *is a* $D$*-norm on* $\mathbb{R}^d$*. Then the fragility index* $\mathrm{FI} = \mathrm{FI}^{(d)}$ *exists for* $\boldsymbol{X}^{(d)}$ *for each* $d \in \mathbb{N}$*, i.e.*

$$\mathrm{FI}^{(d)} = \frac{d}{\|\mathbf{1}\|_{D^{(d)}}},$$

*and we have*

$$\lim_{d\to\infty} \frac{1}{\mathrm{FI}^{(d)}} = \theta.$$

Note that condition (5.3) is derived from condition (2.1) in a natural way using the fact that every $D$-norm is bounded above by the $L_1$-norm.

*Proof of Theorem 5.1.* We have

$$\begin{aligned}\mathrm{FI}^{(d)} &= \lim_{s\nearrow} \sum_{k=1}^{d} \mathrm{E}\Big(1_{(s,\infty)}(X_k) \;\Big|\; \max_{1\le k\le d} X_k > s\Big)\\ &= \lim_{s\nearrow} \frac{d(1-F(s))}{1-\mathrm{P}(X_k \le s,\ 1\le k\le d)}\\ &= \lim_{s\nearrow} \frac{d(1-F(s))}{1-C^{(d)}(F(s),\dots,F(s))}\\ &= \frac{d}{\|\mathbf{1}\|_{D^{(d)}}}\end{aligned}$$

by condition (5.3). We have, moreover, by the same condition,

$$\begin{aligned}\mathrm{P}\Big(\max_{1\le k\le d} X_k \le u_d\Big) &= C^{(d)}(F(u_d),\dots,F(u_d))\\ &= 1-(1-F(u_d))\|\mathbf{1}\|_{D^{(d)}} + o(d(1-F(u_d)))\\ &= 1-\frac{d(1-F(u_d))}{\mathrm{FI}^{(d)}} + o(d(1-F(u_d))),\end{aligned}$$

and, thus, by conditions (5.1) and (5.2),

$$\exp(-\theta\tau) + o(1) = 1 - \frac{\tau + o(1)}{\mathrm{FI}^{(d)}} + o(\tau)$$

as $d\to\infty$. This implies that

$$\lim_{d\to\infty}\frac{1}{\mathrm{FI}^{(d)}} = \frac{1-\exp(-\theta\tau)+o(\tau)}{\tau}.$$

Letting $\tau$ converge to 0 yields the assertion.

The preceding result enables a further interpretation of the extremal index. Take again the Marshall–Olkin $D$-norm, i.e. the convex combination of the $L_1$- and the maximum norm, which are the two extremal $D$-norms representing independence and complete dependence of the margins of the associated EVDF:

$$\|\cdot\|_{\mathrm{MO}} = \vartheta\|\cdot\|_1 + (1-\vartheta)\|\cdot\|_\infty,$$

where $\vartheta \in [0, 1]$ (see Section 4.3 of Falk *et al.* (2004)). Take an arbitrary $D$-norm $\|\cdot\|_{D^{(d)}}$ on $\mathbb{R}^d$. Since every $D$-norm is bounded above by the $L_1$-norm and bounded below by the maximum norm, there exists a unique $\vartheta_d \in [0, 1]$ such that $\|\mathbf{1}\|_{D^{(d)}}$ coincides with the pertaining Marshall–Olkin norm of $\mathbf{1}$, i.e.

$$\|\mathbf{1}\|_{D^{(d)}} = \vartheta_d \|\mathbf{1}\|_1 + (1 - \vartheta_d)\|\mathbf{1}\|_\infty = 1 + (d-1)\vartheta_d.$$

We thus find that the sequence of reciprocals $\|\mathbf{1}\|_{D^{(d)}}/d$ of the fragility index $\mathrm{FI}^{(d)}$ converges as $d \to \infty$ if and only if $\lim_{d\to\infty} \vartheta_d \in [0, 1]$ exists. Theorem 5.1 now yields $\lim_{d\to\infty} \vartheta_d = \theta$, the extremal index.

The extremal index can therefore be considered as the 'proportion of tail independence' contained in the vector $\boldsymbol{X}^{(d)}$ for large $d$, as the $L_1$-norm represents the case of independence of the margins of the limiting extreme value distribution $G^{(d)}(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_{D^{(d)}})$, $\boldsymbol{x} \le \mathbf{0} \in \mathbb{R}^d$, of the copula $C^{(d)}$ associated with $\boldsymbol{X}^{(d)}$.

**Example 5.1.** (*GPD process.*) Let $(Z_k)_k \in \mathbb{N}$ be a strictly stationary process with $0 \le Z_1 \le c$ almost surely for some $c > 1$ and $\mathrm{E}(Z_1) = 1$. Let $U$ be a uniformly on $(0, 1)$ distributed RV, which is independent of the process $(Z_k)_{k\in\mathbb{N}}$, and set

$$X_k := 1 - \frac{U}{Z_k}, \qquad k \in \mathbb{N}.$$

Then the process $(X_k)_{k\in\mathbb{N}}$ is a *GPD process* (see Buishand *et al.* (2008)). It is obviously strictly stationary and the copula $C^{(d)}$ corresponding to $(X_1, \dots, X_d)$ is a GPD copula.

We show in the following that $C^{(d)}$ satisfies condition (5.3) and that the extremal index corresponding to $(X_k)_{k\in\mathbb{N}}$ is 0.

Note that we have, for $1 - 1/c \le x_k \le 1$, $k \le d$,

$$\begin{aligned}
\mathrm{P}(X_k \le x_k, 1 \le k \le d) &= 1 - \int \max_{1\le k\le d}((1-x_k)z_k)(\mathrm{P} * (Z_1, \dots, Z_d))(\mathrm{d}\boldsymbol{z}) \\
&= 1 - \mathrm{E}\Big(\max_{1\le k\le d}((1-x_k)Z_k)\Big) \\
&= 1 - \|(1-x_1, \dots, 1-x_d)\|_{D^{(d)}},
\end{aligned}$$

where

$$\|\boldsymbol{y}\|_{D^{(d)}} := \mathrm{E}\Big(\max_{1\le k\le d}(|y_k| Z_k)\Big), \qquad \boldsymbol{y} \in \mathbb{R}^d,$$

defines a $D$-norm on $\mathbb{R}^d$ for each $d \in \mathbb{N}$. Condition (5.3) is therefore automatically satisfied.

Next we show that the extremal index of $(X_k)_{k\in\mathbb{N}}$ exists and that it is equal to 0. With $d = 1$ we obtain, for $1 - 1/c \le x \le 1$,

$$\mathrm{P}(X_1 \le x) = 1 - (1-x)\,\mathrm{E}(Z_1) = x,$$

and, thus, with $u_d := 1 - \tau/d$, $\tau > 0$, we have

$$d(1 - \mathrm{P}(X_1 \le u_d)) = \tau$$

for large $d$.

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Finally, we obtain

$$\begin{aligned}
\mathrm{P}\Big(\max_{1\le k\le d} X_k \le u_d\Big) &= C^{(d)}(u_d,\dots,u_d)\\
&= 1-\|(1-u_d,\dots,1-u_d)\|_{D^{(d)}}\\
&= 1-\frac{\tau}{d}\|(1,\dots,1)\|_{D^{(d)}}\\
&\underset{d\to\infty}{\to} 1\\
&= \exp(-\theta\tau),
\end{aligned}$$

as $\|(1,\dots,1)\|_{D^{(d)}} = \mathrm{E}(\max_{1\le k\le d} Z_k) \le c$ and, thus, the extremal index of $(X_k)_{k\in\mathbb{N}}$ is $\theta = 0$.

## 6. The extended fragility index

The *extended fragility index* $\mathrm{FI}(m)$ is the obvious extension of the fragility index by the condition that there are at least $m$ exceedances, i.e.

$$\mathrm{FI}(m) := \lim_{c\downarrow 0} \mathrm{E}(N_{1-c} \mid N_{1-c} \ge m) = \frac{\sum_{k=m}^{d} kp_k}{\sum_{k=m}^{d} p_k}$$

for $m \in \{1,\dots,d\}$ and $p_k = \lim_{c\downarrow 0} \mathrm{P}(N_{1-c} = k \mid N_{1-c} > 0)$, $1 \le k \le d$, if these limits exist.

We call the system $\{U_1,\dots,U_d\}$ *m-stable* if $\mathrm{FI}(m) = m$ and *fragile* if $\mathrm{FI}(m) > m$.

We now encounter the problem that we might divide by 0 in the definition of $\mathrm{FI}(m)$ for $m \ge 2$, i.e. $\sum_{k=m}^{d} p_k = 0$, which is, for example, the case for the $L_1$-norm; see the discussion after Theorem 3.1. When does this occur in general? In this section we develop a precise characterization.

This characterization will be formulated in terms of multivariate extreme value theory. The following well-known representations of an EVDF $G$ on $\mathbb{R}^d$ with standard negative exponential margins $G(x\boldsymbol{e}_i) = \exp(x)$, $x \le 0$, $1 \le i \le d$, will be crucial. We have, for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$,

$$\begin{aligned}
G(\boldsymbol{x}) &= \exp(-\|\boldsymbol{x}\|_D) \quad \text{(Hofmann)}\\
&= \exp\Big(-\int_{S_d} \max(-u_i x_i)\mu(\mathrm{d}\boldsymbol{u})\Big) \quad \text{(Pickands–de Haan–Resnick)}\\
&= \exp(-\nu([-\infty,\boldsymbol{x}]^{\complement})) \quad \text{(Balkema–Resnick)},
\end{aligned}$$

where $\mu$ is the *angular measure* on the unit simplex $S_d = \{\boldsymbol{u} \in [0,1]^d : \sum_{i\le d} u_i = 1\}$, satisfying $\mu(S_d) = d$ and $\int_{S_d} u_i\,\mu(\mathrm{d}\boldsymbol{u}) = 1$, $1 \le i \le d$, and $\nu$ is the $\sigma$-finite *exponent measure* on $[-\infty,0]^d \backslash \{\infty\}$; for details, see Falk *et al.* (2004).

The following auxiliary result is of interest in its own right. It implies in particular the general inequality

$$\sum_{\varnothing\ne T\subset\{1,\dots,d\}} (-1)^{|T|-1} \Big\|\sum_{i\in T} x_i\boldsymbol{e}_i\Big\|_D \ge 0, \qquad \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d \text{ or } \boldsymbol{x} \ge \boldsymbol{0} \in \mathbb{R}^d.$$

**Lemma 6.1.** *Let $G$ be an EVDF on $\mathbb{R}^d$ with corresponding D-norm $\|\cdot\|_D$ and exponent measure $\nu$. Then we have, for $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$,*

$$\nu(\boldsymbol{x},\boldsymbol{0}] = \sum_{\varnothing\ne T\subset\{1,\dots,d\}} (-1)^{|T|-1} \Big\|\sum_{i\in T} x_i\boldsymbol{e}_i\Big\|_D.$$

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*Proof.* Since $\nu$ is $\sigma$-finite, there exists a sequence of measurable subsets $B_1 \subset B_2 \subset \cdots$ of $\Omega := [-\boldsymbol{\infty}, \mathbf{0}] \setminus \{-\boldsymbol{\infty}\}$ with $\bigcup_{n\in\mathbb{N}} B_n = \Omega$ and $\nu(B_n) =: b_n < \infty$, $n \in \mathbb{N}$.

Set

$$\nu_n(\cdot) := \nu(\cdot \cap B_n), \qquad n \in \mathbb{N}.$$

Then $\nu_n$, $n \in \mathbb{N}$, defines a sequence of finite measures on $\Omega$, $\nu_n(\Omega) = b_n$, $n \in \mathbb{N}$, with

$$\lim_{n\to\infty} \nu_n(B) = \nu(B)$$

for any measurable subset $B$ of $\Omega$.

The $\Delta$-monotonicity of an arbitrary finite measure implies that

$$\nu_n(\boldsymbol{x}, \boldsymbol{y}] = \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]\right) \ge 0$$

for any $-\boldsymbol{\infty} < \boldsymbol{x} \le \boldsymbol{y} \le \mathbf{0}$, and, thus, switching to complements,

$$\begin{aligned}\nu_n(\boldsymbol{x}, \boldsymbol{y}] &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} \left(b_n - \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right)\right)\\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right)\end{aligned}$$

for any $n \in \mathbb{N}$; note that

$$\sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d-\sum_{j\le d} m_j} = \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{\sum_{j\le d} m_j} = \sum_{k=0}^{d} (-1)^k \binom{d}{k} = 0$$

and that

$$\begin{aligned}\nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right) &\underset{n\to\infty}{\to} \nu\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right)\\ &= \left\| \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right\|_D.\end{aligned}$$

We thus obtain

$$\begin{aligned}\nu(\boldsymbol{x}, \boldsymbol{y}] &= \lim_{n\to\infty} \nu_n(\boldsymbol{x}, \boldsymbol{y}]\\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \lim_{n\to\infty} \nu_n\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right)\\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \nu\left(\left[-\boldsymbol{\infty}, \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i\right]^{\complement}\right)\\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \left\| \sum_{i\le d} y_i^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right\|_D.\end{aligned}$$

Setting $\boldsymbol{y}=\mathbf{0}$ and replacing $m_i$ by $1-m_i$ we obtain

$$\begin{aligned}\nu(\boldsymbol{x},\mathbf{0}] &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{d+1-\sum_{j\le d} m_j} \left\| \sum_{i\le d} 0^{m_i} x_i^{1-m_i} \boldsymbol{e}_i \right\|_D \\ &= \sum_{\boldsymbol{m}\in\{0,1\}^d} (-1)^{1+\sum_{j\le d} m_j} \left\| \sum_{i\le d} 0^{1-m_i} x_i^{m_i} \boldsymbol{e}_i \right\|_D \\ &= \sum_{\varnothing\ne T\subset\{1,\dots,d\}} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D .\end{aligned}$$

The following characterization is the main result of this section.

**Proposition 6.1.** *Suppose that the random vector* $\boldsymbol{U}=(U_1,\dots,U_d)$ *follows a copula* $C\in\mathcal{D}(G)$. *Choose* $m\in\{2,\dots,d\}$. *Then we have, for the ACDEC,* $\sum_{k=m}^d p_k=0$ *if and only if we have, for any subset* $K\subset\{1,\dots,d\}$ *with at least* $m$ *elements,*

$$\lim_{c\downarrow 0}\frac{\mathrm{P}(U_k>1-c,\ k\in K)}{c}=0$$

$$\iff \sum_{T\subset K}(-1)^{|T|-1}\left\|\sum_{i\in T}x_i\boldsymbol{e}_i\right\|_D=0 \quad \textit{for all } \boldsymbol{x}\ge\mathbf{0}\in\mathbb{R}^d \tag{6.1}$$

$$\iff \sum_{T\subset K}(-1)^{|T|-1}\left\|\sum_{i\in T}\boldsymbol{e}_i\right\|_D=0 \tag{6.2}$$

$$\iff \mu\Big(\Big\{\boldsymbol{u}\in S_d:\ \min_{i\in K}u_i>0\Big\}\Big)=0 \tag{6.3}$$

$$\iff \nu\Big(\mathop{\times}_{k\in K}(-\infty,0]\mathop{\times}_{i\notin K}[-\infty,0]\Big)=0, \tag{6.4}$$

*i.e. the projection* $\nu_K:=\nu*(\pi_i,\ i\in K)$ *of the exponent measure* $\nu$ *onto its components* $i\in K$ *is the null measure on* $(-\infty,0]^{|K|}$.

*Proof.* We have, by Corollary 3.1,

$$\begin{aligned} &\sum_{k=m}^d p_k=0\\ \iff\ &\lim_{c\downarrow 0}\mathrm{P}(N_{1-c}\ge m\mid N_{1-c}>0)=0\\ \iff\ &\lim_{c\downarrow 0}\frac{1}{c}\mathrm{P}\Bigg(\bigcup_{\substack{K\subset\{1,\dots,d\}\\|K|\ge m}}\{U_k>1-c,\ k\in K\}\Bigg)=0\\ \iff\ &\lim_{c\downarrow 0}\frac{1}{c}\mathrm{P}(U_k>1-c,\ k\in K)=0, \qquad K\subset\{1,\dots,d\},\ |K|\ge m\\ \iff\ &\text{condition (6.1) is satisfied,}\end{aligned}$$

where the final equivalence is an immediate consequence of Corollary 2.1 and the well-known additivity formula.

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Note that $\tilde{\mu} := \mu/d$ defines a probability measure on $S_d$, and let $\boldsymbol{T} = (T_1, \dots, T_d)$ be a random vector with values in $S_d$, whose distribution is $\tilde{\mu}$. Set $\boldsymbol{Z} = (Z_1, \dots, Z_d) := d\boldsymbol{T}$. Then we have

$$Z_i \in [0, d], \quad i \le d; \qquad \sum_{i \le d} Z_i = d \sum_{i \le d} T_i = d; \qquad \mathrm{E}(Z_i) = d\,\mathrm{E}(T_i) = 1, \quad i \le d.$$

Let $V$ be an RV that is independent of $\boldsymbol{Z}$ and uniformly on (0,1) distributed, and set

$$\boldsymbol{Q} = (Q_1, \dots, Q_d) := \frac{1}{V}\boldsymbol{Z}.$$

Note that $1/V$ follows a standard Pareto distribution on $[1, \infty)$.

We have, for $\boldsymbol{x} \ge (d, \dots, d) \in \mathbb{R}^d$,

$$\begin{aligned}
\mathrm{P}\Big(\frac{1}{V}\boldsymbol{Z} \le \boldsymbol{x}\Big) &= \mathrm{P}\Big(V \ge \max_{i \le d} \frac{1}{x_i} Z_i\Big) \\
&= \int_{S_d} \mathrm{P}\Big(V \ge \max_{i \le d} \frac{d t_i}{x_i}\Big) \tilde{\mu}(\mathrm{d}\boldsymbol{t}) \\
&= 1 - \int_{S_d} \mathrm{P}\Big(V \le \max_{i \le d} \frac{d t_i}{x_i}\Big) \tilde{\mu}(\mathrm{d}\boldsymbol{t}) \\
&= 1 - \int_{S_d} \max_{i \le d} \frac{d t_i}{x_i} \tilde{\mu}(\mathrm{d}\boldsymbol{t}) \\
&= 1 - \int_{S_d} \max_{i \le d} \frac{t_i}{x_i} \mu(\mathrm{d}\boldsymbol{t}) \\
&= 1 - \Big\| \Big(\frac{1}{x_1}, \dots, \frac{1}{x_d}\Big) \Big\|_D .
\end{aligned}$$

From the well-known additivity formula, for $\gamma_k \le 1/d,\ k \in K$, we obtain

$$\begin{aligned}
\mathrm{P}\Big(Q_k > \frac{1}{\gamma_k},\ k \in K\Big) &= 1 - \mathrm{P}\Big(\bigcup_{k \in K} \Big\{Q_k \le \frac{1}{\gamma_k}\Big\}\Big) \\
&= 1 - \sum_{\varnothing \ne T \subset K} (-1)^{|T|-1} \mathrm{P}\Big(Q_i \le \frac{1}{\gamma_i},\ i \in T\Big) \\
&= 1 - \sum_{\varnothing \ne T \subset K} (-1)^{|T|-1} \Big(1 - \Big\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \Big\|_D\Big) \\
&= \sum_{\varnothing \ne T \subset K} (-1)^{|T|-1} \Big\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \Big\|_D
\end{aligned}$$

as $\sum_{\varnothing \ne T \subset K} (-1)^{|T|-1} = 1$.

Choosing identical $\gamma_k = \gamma \le 1/d$, $k \in K$, we obtain

$$\mathrm{P}\Big(Q_k > \frac{1}{\gamma},\ k \in K\Big) = 0 \quad \iff \quad \sum_{T \subset K} (-1)^{|T|-1} \Big\| \sum_{i \in T} \boldsymbol{e}_i \Big\|_D = 0$$

and, thus, the equivalence of condition (6.1) and (6.2).

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Moreover, condition (6.1) is satisfied if and only if $\mathrm{P}(Q_k > x_k,\ k \in K) = 0$ for all $x_k \geq d$, $k \in K$, i.e.

$$\begin{aligned}
0 = \mathrm{P}(Q_k > x_k,\ k \in K) &= \mathrm{P}\Big(V < \min_{k\in K} \frac{1}{x_k} Z_k\Big) = \int_0^1 \mathrm{P}\Big(v < \min_{k\in K} \frac{1}{x_k} Z_k\Big)\mathrm{d}v \\
\iff \quad & \mathrm{P}\Big(\min_{k\in K} \frac{1}{x_k} Z_k > v\Big) = 0, \qquad 0 < v < 1 \\
\iff \quad & \mathrm{P}\Big(\min_{k\in K} \frac{1}{x_k} Z_k = 0\Big) = 1 \\
\iff \quad & \mathrm{P}(\min_{k\in K} Z_k = 0) = 1 \\
\iff \quad & \mu\Big(\Big\{\boldsymbol{u} \in S_d : \min_{k\in K} u_k = 0\Big\}\Big) = d,
\end{aligned}$$

which is condition (6.3).

Denote by $\pi_K : [-\infty, 0]^d \ni \boldsymbol{x} \mapsto (x_k)_{k\in K} \in [-\infty, 0]^{|K|}$ the projection of a vector in $[-\infty, 0]^d$ onto the vector of its coordinates given by the subset $K \subset \{1, \ldots, d\}$. Then the measure induced by the exponent measure $\nu$ and the projection $\pi_K$ is the angular measure of the EVDF $G_K$, defined as the marginal distribution of $G$ given by $K$ with $|K| = m$:

$$\begin{aligned}
G_K(y_1, \ldots, y_m) &= G\Big(\sum_{k\in K} y_{i_k} \boldsymbol{e}_k\Big) \\
&= \exp\Big(-\nu\Big(\Big(\mathop{\times}_{k\in K} [-\infty, y_{i_k}] \times [-\infty, 0]^{d-m}\Big)^{\complement}\Big)\Big) \\
&= \exp\Big(-(\nu * \pi_K)\Big(\Big(\mathop{\times}_{i=1}^{m} [-\infty, y_i]\Big)^{\complement}\Big)\Big) \\
&= \exp\Big(-\nu_K\Big(\Big(\mathop{\times}_{i=1}^{m} [-\infty, y_i]\Big)^{\complement}\Big)\Big), \qquad y_1, \ldots, y_m \leq 0.
\end{aligned}$$

From Lemma 6.1, it follows that condition (6.1) is equivalent to $\nu_K((\boldsymbol{y}, \boldsymbol{0}]) = 0,\ \boldsymbol{y} \in \mathbb{R}^m$, which is condition (6.4).

To summarize, the preceding considerations imply that, for an arbitrary copula $C$ in the domain of attraction of an EVDF $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D),\ \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, the index

$$m^* := \max\{1 \leq m \leq d : \mathrm{FI}(m) \text{ is well defined}\}$$

exists, providing the maximum range $\{1, \ldots, m^*\}$ on which the extended $\mathrm{FI}(m)$ is defined:

$$\mathrm{FI}(m) = \lim_{c\downarrow 0} \mathrm{E}(N_{1-c} \mid N_{1-c} > 0) = \frac{\sum_{k=m}^d k p_k}{\sum_{k=m}^d p_k}, \qquad 1 \leq m \leq m^*.$$

Moreover,

$$\begin{aligned}
m^* &= \max\Big\{1 \leq m \leq d : \sum_{k=m}^{d} p_k > 0\Big\} \\
&= \max\Big\{1 \leq m \leq d : \text{ there exists } K \subset \{1, \ldots, d\},\ |K| = m : \\
&\qquad \sum_{\varnothing \neq T \subset K} (-1)^{|T|-1} \Big\|\sum_{i\in T} \boldsymbol{e}_i\Big\|_D > 0\Big\}
\end{aligned}$$

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$$= \max\Big\{1 \le m \le d\colon \text{ there exists } K \subset \{1,\dots,d\},\ |K| = m\colon$$
$$\sum_{\emptyset \ne T \subset K} (-1)^{|T|-1} \Big\| \sum_{i \in T} x_i \boldsymbol{e}_i \Big\|_D > 0 \text{ for all } \boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d \Big\}$$
$$= \max\Big\{1 \le m \le d\colon \text{ there exists } K \subset \{1,\dots,d\},\ |K| = m\colon$$
$$\mu\Big(\Big\{\boldsymbol{u} \in S_d\colon \min_{k \in K} u_k > 0\Big\}\Big) > 0\Big\}$$
$$= \max\{1 \le m \le d\colon \text{ there exists } K \subset \{1,\dots,d\}, |K| = m : \nu_K((-\infty,0]^m) > 0\}.$$

For the Marshall–Olkin $D$-norm $\|\cdot\|_\vartheta = \vartheta\|\cdot\|_\infty + (1-\vartheta)\|\cdot\|_1$, we obtain, for example,

$$\mathrm{FI} = \mathrm{FI}(1) = \frac{d}{d - \vartheta(d-1)}; \qquad \mathrm{FI}(m) = d, \quad 2 \le m \le d.$$

## 7. No exceedance above a high threshold

The considerations in Section 6 also enable the characterization of those copulas $C$ such that $\mathrm{P}(\boldsymbol{U} > \boldsymbol{c}_0) = 0$ for some $\boldsymbol{c}_0 \in (0,1)^d$, where the random vector $\boldsymbol{U}$ follows the copula $C$, i.e. there will be no exceedance above a high threshold.

Let $U$ be uniformly on $(0,1)$ distributed, and set $\boldsymbol{U} := (U_1, U_2) := (U, 1-U)$. Then $\boldsymbol{U}$ follows a bivariate copula and satisfies $U_1 + U_2 = 1$, i.e.

$$\mathrm{P}(\boldsymbol{U} > \boldsymbol{c}) = 0, \qquad \boldsymbol{c} = (c_1, c_2) \in (0,1)^2,\ c_1 + c_2 > 1,$$

which is illustrated in Figure 1.

Note that

$$\mathrm{P}(\boldsymbol{U} \le \boldsymbol{c}) = 1 - \|(1-c_1, 1-c_2)\|_1, \qquad 0 \le c_1, c_2 \le 1,\ c_1 + c_2 \ge 1,$$

i.e. $\boldsymbol{U}$ follows a bivariate GPD copula whose $D$-norm is the $L_1$-norm.

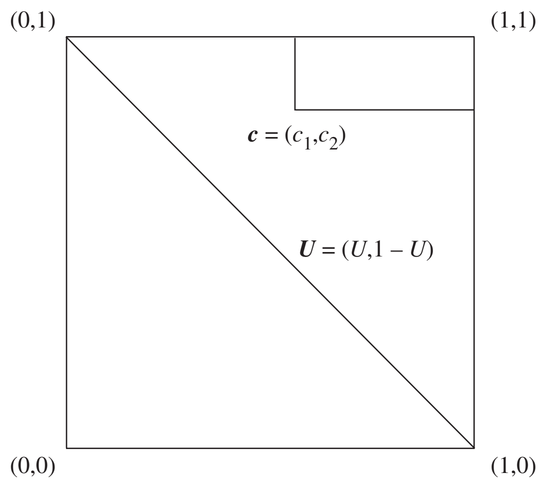


FIGURE 1: Support line of the random vector $\boldsymbol{U} = (U, 1-U)$.

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Now let $\boldsymbol{U} = (U_1, U_2)$ follow an *arbitrary* bivariate copula $C$ such that $\mathrm{P}(\boldsymbol{U} > \boldsymbol{c}_0)$ for some $\boldsymbol{c}_0 \in (0, 1)^2$. Then we obtain, for $\boldsymbol{c}_0 \le \boldsymbol{c} \le (1, 1)$,

$$\begin{aligned} 0 &= \mathrm{P}(U_1 > c_1,\ U_2 > c_2) \\ &= 1 - (\mathrm{P}(U_1 \le c_1) + \mathrm{P}(U_2 \le c_2) - \mathrm{P}(U_1 \le c_1,\ U_2 \le c_2)) \\ &= 1 - c_1 - c_2 + C(\boldsymbol{c}), \end{aligned}$$

and, thus,

$$C(\boldsymbol{c}) = 1 - \|(1 - c_1, 1 - c_2)\|_1,$$

i.e. in the bivariate case we have no exceedance above a high threshold if and only if the underlying copula is a GPD copula, whose $D$-norm is the $L_1$-norm.

Also, in higher dimensions, a GPD copula

$$C(\boldsymbol{c}) = 1 - \|(1 - c_1, \dots, 1 - c_d)\|_1, \qquad \boldsymbol{c}_0 \le \boldsymbol{c} \le (1, \dots, 1) \in \mathbb{R}^d,$$

whose $D$-norm is the $L_1$-norm yields no exceedance $\mathrm{P}(\boldsymbol{U} > \boldsymbol{u}) = 0$ above a high threshold $\boldsymbol{u}$ close to $(1, \dots, 1)$. This is immediate from the additivity formula.

In dimension $d \ge 3$, however, the $L_1$-norm is no longer the only $D$-norm that entails no exceedance above a high threshold. Take, for example, the angular measure $\mu$ which puts equal weight 1 on each of the set of $d$ points

$$\begin{aligned} &\left\{\left(0, \frac{1}{d-1}, \dots, \frac{1}{d-1}\right), \dots, \left(\frac{1}{d-1}, \dots, \frac{1}{d-1}, 0\right)\right\} \\ &\quad = \left\{\frac{1}{d-1} \sum_{j \le d,\, j \ne i} \boldsymbol{e}_j,\ 1 \le i \le d\right\} \subset S_d. \end{aligned}$$

The corresponding $D$-norm is

$$\begin{aligned} \|\boldsymbol{x}\|_D &= \int_{S_d} \max_{k \le d}(|x_k| u_k) \mu(\mathrm{d}\boldsymbol{u}) \\ &= \sum_{i \le d} \int_{\{(1/(d-1)) \sum_{j \le d,\, j \ne i} \boldsymbol{e}_j\}} \max_{k \le m}(|x_k| u_k) \mu(\mathrm{d}\boldsymbol{u}) \\ &= \sum_{i \le d} \frac{1}{d-1} \max_{j \le d,\, j \ne i} |x_j| \\ &= \frac{1}{d-1} \sum_{i \le d} \Big(\max_{j \le d,\, j \ne i} |x_j|\Big), \qquad \boldsymbol{x} \in \mathbb{R}^d. \end{aligned}$$

Note that $\|\cdot\|_D = \|\cdot\|_1 \Leftrightarrow d = 2$.

Now choose a random vector $\boldsymbol{U}$ that follows the above GPD copula $C(\boldsymbol{u}) = 1 - \|\mathbf{1} - \boldsymbol{u}\|_D$, $\boldsymbol{u}_0 \le \boldsymbol{u} \le \mathbf{1} \in \mathbb{R}^d$. Then we obtain, for $\boldsymbol{u} = u \sum_{i \le m} \boldsymbol{e}_i \in [\boldsymbol{u}_0, (1, \dots, 1)]$,

$$\begin{aligned} \mathrm{P}(\boldsymbol{U} > \boldsymbol{u}) &= 1 - \mathrm{P}\Big(\bigcup_{i \le m} \{Y_i \le u\}\Big) \\ &= \sum_{\varnothing \ne T \subset \{1, \dots, d\}} (-1)^{|T|-1} \Big\| \sum_{i \in T} u \boldsymbol{e}_i \Big\|_D \\ &= 0, \end{aligned}$$

where the final equation is established by induction.

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From the fact that

$$\mathrm{P}(U_k \leq 1-c,\ k \in K) = 1 - c\left\| \sum_{k\in K} \boldsymbol{e}_k \right\|_D$$

for $c$ close to 0, if $\boldsymbol{U} = (U_1, \dots, U_d)$ follows an arbitrary GPD copula with $D$-norm $\|\cdot\|_D$, we find that the characterization of $\sum_{k=m}^{d} p_k = 0$ in Proposition 6.1 provides the following characterization of the case of no exceedance among $U_k$, $k \in K \subset \{1, \dots, d\}$ above a high threshold. Simply repeat its proof. By $\mu$ and $\nu$, we denote the angular and exponent measures corresponding to $\|\cdot\|_D$.

**Proposition 7.1.** *Suppose that the random vector $\boldsymbol{U} = (U_1, \dots, U_d)$ follows a GPD copula $C$ with corresponding $D$-norm $\|\cdot\|_D$. Then we have, for an arbitrary subset $K \subset \{1, \dots, d\}$ with at least two elements,*

$$\begin{aligned}
& \mathrm{P}(U_k > 1-c,\ k \in K) = 0 \quad \textit{for some } c \textit{ close to } 0 \\
\iff\ & \sum_{T\subset K} (-1)^{|T|-1} \left\| \sum_{i\in T} x_i \boldsymbol{e}_i \right\|_D = 0 \quad \textit{for all } \boldsymbol{x} \geq \boldsymbol{0} \in \mathbb{R}^d \\
\iff\ & \sum_{T\subset K} (-1)^{|T|-1} \left\| \sum_{i\in T} \boldsymbol{e}_i \right\|_D = 0 \\
\iff\ & \mu\Big(\Big\{\boldsymbol{u} \in S_d \colon \min_{i\in K} u_i > 0\Big\}\Big) = 0 \\
\iff\ & \nu\Big(\mathop{\times}_{k\in K} (-\infty, 0] \mathop{\times}_{i\notin K} [-\infty, 0]\Big) = 0,
\end{aligned}$$

*i.e. the projection $\nu_K := \nu * (\pi_i,\ i \in K)$ of the exponent measure $\nu$ onto its components $i \in K$ is the null measure on $(-\infty, 0]^{|K|}$.*

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