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THE MODIFIED QUANTUM WIGNER SYSTEM IN WEIGHTED L^2-SPACE

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Abstract

This paper is concerned with the modified Wigner (respectively, Wigner–Fokker–Planck) Poisson equation. The quantum mechanical model describes the transport of charged particles under the influence of the modified Poisson potential field without (respectively, with) the collision operator. Existence and uniqueness of a global mild solution to the initial boundary value problem in one dimension are established on a weighted L^2-space. The main difficulties are to derive *a priori* estimates on the modified Poisson equation and prove the Lipschitz properties of the appropriate potential term.

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1. Introduction and main results

In this paper, we study a quantum mechanical model for extremely small devices in semiconductor nanostructures (such as quantum wires) where the quantum effects are important, as in the case of microstructures [16]. This model can be described by the time-dependent self-consistent modified Schrödinger–Poisson system (see, for example, [6]), which is a set of equations

$$i\partial_t u = -\tfrac{1}{2}\Delta u + \phi u, \quad -\mathrm{div}[(a + b|\nabla\phi|^2)\nabla\phi] = |u|^2, \tag{1.1}$$

where a, b are positive field dependent dielectric constants.

The Wigner transform [20] of the Schrödinger wave function $u(t, x)$ is

$$f(t, x, v) = \frac{1}{(2\pi)^n}\int_{R^n} u\Big(t, x - \frac{1}{2}y\Big)\overline{u}\Big(t, x + \frac{1}{2}y\Big)\exp(iv\cdot y)\,dy,$$

where $\overline{u}$ denotes the complex conjugate of u. A direct calculation by applying the Wigner transform to (1.1) shows that $f(t, x, v)$ satisfies the so-called (collisionless) modified Wigner–Poisson equations (called mWP in the subsequent work, for simplicity)

$$f_t + (v\cdot\nabla_x)f + \Theta[\phi]f = 0, \quad -\mathrm{div}[(a + b|\nabla\phi|^2)\nabla\phi] = \rho. \tag{1.2}$$

The charge density $\rho(t, x)$ is derived from the Schrödinger wave function $u(t, x)$ by

$$\rho(t,x) = |u(t,x)|^2 = \int_{R^n} f(t,x,v)\,dv. \tag{1.3}$$

The operator $\Theta[\phi]f$ in (1.2) is a pseudodifferential operator defined by

$$\Theta[\phi]f(t,x,v) = -\frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\delta[\phi](t,x,\eta)f(t,x,v')e^{i(v-v')\eta}\,dv'\,d\eta, \tag{1.4}$$

with the potential difference

$$\delta[\phi](t,x,\eta) = \phi\Big(t, x+\frac{\eta}{2}\Big) - \phi\Big(t, x-\frac{\eta}{2}\Big). \tag{1.5}$$

If $a = 1, b = 0$ then (1.2)–(1.5) reduce to the classical Wigner–Poisson (WP) system. There are many investigations of this system. It has been studied in the whole space [7, 11], in a bounded spatial domain with periodic [5], absorbing [1], or time-dependent inflow [13, 14] boundary conditions and on a discrete lattice [9]. However, no rigorous results for the initial boundary value problem of the mWP equations (1.2)–(1.5) have been obtained. The first aim in this work is to investigate the initial boundary value problem for the one-dimensional mWP equations (1.2)–(1.5) and to establish the existence of global mild solutions with initial boundary conditions

$$f(0,x,v) = f_0(x,v), \quad (x,v) \in \Omega = I \times R, I = [0,1], \tag{1.6}$$
$$f(t,0,v) = f(t,1,v), \quad \phi(t,0) = \phi(t,1). \tag{1.7}$$

Analysis of the classical WP problem is based on a self-adjointness property or a reformulation of the WP problem as a system of a countable number of Schrödinger equations coupled to a Poisson equation for the potential (see, for example, [7, 11, 15]). There is still a need for a purely kinetic analysis of the classical WP problem in the Wigner framework. Towards this goal, [13, 14] gave a local-in-time well-posedness result for the three-dimensional nonlinear WP problem and a global-in-time well-posedness result for the n-dimensional linearised problem and the one-dimensional nonlinear problem in a bounded spatial domain, with nonhomogeneous and time-dependent inflow boundary conditions. However, the approach relies on the explicit solution of the linear Poisson equation. There is no straightforward generalisation to the problem (1.2)–(1.7) because of the nonlinear Poisson equation in (1.2).

Consequently, in order to carry out our analysis in the Wigner framework, we have to use different mathematical techniques. One of our main ideas is to use new estimates for the nonlinear Poisson equation and nonlinear operator $\Theta[\phi]f$ (see Lemmas 2.1–2.3) in the weighted L^2-space

$$X_1 = L^2(I \times R, (1 + |v|^2)\,dx\,dv). \tag{1.8}$$

These estimates are crucial for our local-in-time analysis. By the law of conservation of mass, the global-in-time mild solution can be established by recovering *a priori* estimates of f in X_1. With this notation, our first result is the following theorem.

THEOREM 1.1. *For $n = 1$ and $f_0 \in X_1$, the initial boundary problem (1.2)–(1.7) has a unique mild solution $f \in C([0, \infty), X_1)$.*

REMARK 1.2. Such an approach could possibly allow for a partial extension to the mWP problem with time-dependent inflow boundary conditions, as in [13, 14] in the Wigner framework.

For realistic device simulations, scattering (between electrons and impurities or with phonons, that is, thermal vibrations of the crystal lattice), a heat bath of oscillators in thermal equilibrium and the effects of system–environment interactions must be included in the model. Hence, the right-hand side of the first equation in (1.2) has to be augmented by a scattering term. Of particular practical interest are interaction mechanisms that can be described by quantum Fokker–Planck scattering terms

$$Q(f) = \beta \operatorname{div}_v(vf) + \sigma \Delta_v f + 2\gamma \operatorname{div}_x(\nabla_v f) + \alpha \Delta_x f.$$

The collision operator $Q(f)$ models diffusive effects (for example, the electron–phonon interaction), as discussed in the papers cited below.

The combination of the quantum Fokker–Planck operator with the first equation in (1.2) gives the Wigner–Fokker–Planck equation

$$f_t + (v \cdot \nabla_x) f + \Theta[\phi] f = \beta \operatorname{div}_v(vf) + \sigma \Delta_v f + 2\gamma \operatorname{div}_x(\nabla_v f) + \alpha \Delta_x f. \tag{1.9}$$

Equation (1.9) coupled with (1.3)–(1.7) gives a modified Wigner–Poisson–Fokker–Planck system (called mWPFP in the subsequent work, for simplicity). Here, $\beta \geq 0$ is the friction parameter and the parameters $\sigma > 0$, $\alpha \geq 0$, $\gamma \geq 0$ satisfy $\alpha\sigma \geq \gamma^2 + \beta^2/16$, which guarantees that the system is quantum mechanically correct (see [4]). The subsequent mathematical analysis holds for $\alpha\sigma \geq \gamma^2$.

If $a = 1$, $b = 0$, the mWPFP problem reduces to the classical Wigner–Poisson–Fokker–Planck (WPFP) problem. By adapting L^1-techniques from the classical Vlasov–Poisson–Fokker–Planck equation, the global-in-time solution for the WPFP problem was established in [8]. However, the weighted L^2-space is increasingly favoured because the choice of the weight can be used to control the L^2-norm of the density $\rho(t, x)$. For example, in [3], the WPFP equation with periodic boundary conditions in one dimension was studied, and the unique global classical solution was constructed. In three dimensions, the global mild solution of the WPFP equation was established in [4]. A survey of these results and other quantum physical problems can be found in [2].

The results for the classical WPFP problem rely on the explicit solution of the linear Poisson equation (see [2]). Again, these approaches cannot be generalised to the mWPFP problem because of the nonlinearity of the modified Poisson equation.

With the aforementioned notation, the second result of this paper is the following theorem.

THEOREM 1.3. *For $n = 1$ and $f_0 \in X_1$, the mWPFP system with the initial boundary value conditions (1.6)–(1.7), has a unique mild solution $f \in C([0, \infty), X_1)$.*

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Remark 1.4. In one dimension, Theorems 1.1 and 1.3 succeed because we can work with the Sobolev embedding inequality $W_0^{1,2} \hookrightarrow L^\infty$ (see Lemmas 2.2 and 2.3) and the nonlinear Poisson equation in (1.2) yields a potential ϕ whose extension to the whole real line belongs to $W^{1,\infty}(R)$. However, in three dimensions, we need the state space,

$$X_3 = L^2(Q \times R_v^3, (1+|v|^2)^2\, dx\, dv) \quad \text{where } Q = [0,1]^3,$$

and require the extension of ϕ to belong to $W^{2,\infty}(R^3)$, but this is, in general, false. Moreover, the Sobolev embedding inequality $W_0^{1,2} \hookrightarrow L^\infty$ does not hold in three dimensions. More precisely, we cannot establish the Lipschitz property of the nonlinear term $\Theta[\phi]f$ (as in Lemma 2.3) on the n-dimensional space for $n \geq 2$.

Remark 1.5. Under the semiclassical limit, $\Theta[\phi]f \to \nabla_x\phi \cdot \nabla_v f$ as $\hbar \to 0$ (see [4]). This is the nonlinear term of the Vlasov or Vlasov–Fokker–Planck equation. The generalised Vlasov model

$$f_t + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f = k\Delta_x f + k\Delta_v f, \quad -\nabla\cdot[\mu(|\nabla\phi|)\nabla\phi] = \rho(t,x), \tag{1.10}$$

with

$$\mu(x) = \left[\frac{\epsilon x^2+1}{x^2+\epsilon}\right]^{(2-p)/2} \quad \text{for } 1<p<2 \quad \text{and} \quad \mu(x) = (x^2+\epsilon)^{(p-2)/2} \quad \text{for } p \geq 2,$$

was studied in [10, 12]. Here $\epsilon > 0$ and ϕ is the so-called non-Newtonian potential. Equations (1.10), preserve the essential structure of (1.9) and are motivated by (1.9) with $\beta = \gamma = 0$ and $\sigma = \alpha = k$. Therefore, we can extend Theorem 1.3 to a generalised Poisson potential field, such as the non-Newtonian potential. We can obtain existence and uniqueness of the local mild solution for $1 < p < 2$ and the global mild solution for $p \geq 2$ for the mWPFP problem coupled with the non-Newtonian potential. However, the analysis of [10, 12] cannot be generalised to the mWPFP problem because the nonlinear term $\Theta[\phi]f$ is a pseudodifferential operator.

2. Global existence for mWP

To begin, we discuss the functional analytic preliminaries for studying the mWP problem and the global existence result from [18, Theorem 6.1.4].

2.1. The functional setting and preliminaries. The weighted L^2 space X_1, defined in (1.8), is endowed with the scalar product

$$\langle f, g\rangle_{X_1} := \int_I \int_R f(t,x,v)\overline{g(x,v)}(1+|v|^2)\, dv\, dx.$$

In our calculations, we shall use the norm

$$\|f\|_{X_1}^2 := \|f\|_{L^2(\Omega)}^2 + \|vf\|_{L^2(\Omega)}^2. \tag{2.1}$$

The first lemma gives estimates for the potential ϕ defined in (1.2) and the nonlinear operator $\Theta[\phi]f$ on X_1 defined in (1.4).

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Lemma 2.1.

(i) *For $n = 1$, if $f \in X_1$, then $\rho(t, x)$ belongs to $L^2(I)$ and satisfies*

$$\|\rho\|_{L^2(I)} \le C\|f\|_{X_1}.$$

(ii) *For $n = 1$, if $f \in X_1$, then $\partial_x\phi, \partial_x^2\phi$ belong to $L^2(I)$ and satisfy*

$$\|\partial_x\phi\|_{L^2(I)} \le C\|f\|_{X_1}, \quad \|\partial_x^2\phi\|_{L^2(I)} \le C\|f\|_{X_1}.$$

Moreover, ϕ belongs to $W^{1,\infty}(I)$ and satisfies

$$\|\phi\|_{W^{1,\infty}}(I) \le C\|\phi\|_{W^{2,2}(I)} \le C\|f\|_{X_1}.$$

Proof. The first assertion follows directly by applying the Cauchy–Schwartz inequality in the v-integral (see also [4]). For the second assertion, by (1.2) and the first assertion,

$$|\rho| = |-\partial_x[(a + b|\partial_x\phi|^2)\partial_x\phi]| = (a + 3b|\partial_x\phi|^2)|\partial_x^2\phi| \ge a|\partial_x^2\phi|.$$

By squaring both sides and integrating over I in x, using the first assertion,

$$\|\partial_x^2\phi\|_{L^2(I)} \le C\|\rho\|_{L^2(I)} \le C\|f\|_{X_1}.$$

By the Hölder inequality and the Poincaré inequality $\|\phi\|_{L^2(I)} \le C\|\partial_x\phi\|_{L^2(I)}$,

$$\int_I (a + b|\partial_x\phi|^2)b|\partial_x\phi|^2\,dx = \int_I \rho\phi\,dx \le \|\rho\|_{L^2(I)}\|\phi\|_{L^2(I)} \le C\|\rho\|_{L^2(I)}\|\partial_x\phi\|_{L^2(I)}.$$

Hence,

$$\|\partial_x\phi\|^2_{L^2(I)} \le C\|\rho\|_{L^2(I)}\|\partial_x\phi\|_{L^2(I)}$$

and, consequently,

$$\|\partial_x\phi\|_{L^2(I)} \le C\|\rho\|_{L^2(I)} \le C\|f\|_{X_1}.$$

Finally, using the Sobolev embedding and the Poincaré inequalities,

$$\begin{aligned}\|\phi\|^2_{W^{1,\infty}(I)} &\le C\|\phi\|^2_{W^{2,2}(I)} = C(\|\partial_x^2\phi\|^2_{L^2(I)} + \|\partial_x\phi\|^2_{L^2(I)} + \|\phi\|^2_{L^2(I)}) \\ &\le C(\|f\|^2_{X_1} + \|\partial_x\phi\|^2_{L^2(I)}) \le C\|f\|^2_{X_1},\end{aligned}$$

and the second assertion is proved. □

Next, we shall analyse the properties of the pseudodifferential operator $\Theta[\phi]f$ defined in (1.4). By its definition, the potential has to be appropriately extended to R: that is, ϕ should be a periodically extended solution of the nonlinear Poisson equation in (1.2). The operator $\Theta[\phi]f$ can be rewritten in more compact form (see [4]) as

$$\mathcal{F}_{v\to\eta}(\Theta[\phi]f)(x, \eta) = i(\delta\phi)(x, \eta)\mathcal{F}_{v\to\eta}f(x, \eta),$$

where the symbol $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v,

$$\mathcal{F}_{v\to\eta}[f(x, \cdot)](\eta) = \int_{R^n} f(x, v)e^{iv\cdot\eta}\,dv$$

for a suitable function f. We have the following result.

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LEMMA 2.2. *The nonlinear operator $\Theta[\phi]f$ maps X_1 into itself.*

PROOF. The assertion follows directly from Lemma 2.1 and [14, Lemma 4.2]. □

In the subsequent work, we shall establish the Lipschitz property of the nonlinear term $\Theta[\phi]f$. This is one of the key ingredients for proving the uniqueness theorem. It is not a straightforward extension of the results of [3, 14] because those results rely on the explicit solution of the linear Poisson equation $-\Delta V = \rho$, which satisfies

$$V_1 - V_2 = \frac{c}{|x|^{n-2}} * (\rho_1 - \rho_2). \tag{2.2}$$

An equation such as (2.2) is not possible for the nonlinear Poisson equation in (1.2).

LEMMA 2.3. *For $n = 1$, the operator $\Theta[\phi]f$ satisfies*

$$\|\Theta[\phi_1]f_1 - \Theta[\phi_2]f_2\|_{X_1} \le C(\|f_1\|_{X_1} + \|f_2\|_{X_1})\|f_1 - f_2\|_{X_1}$$

for every $f_i \in X_1$, where $\phi_i = \phi[f_i]$, $i = 1, 2$.

PROOF. From (2.1),

$$\begin{aligned}\|\Theta[\phi_1]f_1 - \Theta[\phi_2]f_2\|_{X_1}^2 &= \|\Theta[\phi_1]f_1 - \Theta[\phi_2]f_2\|_{L^2(\Omega)}^2 + \|v\Theta[\phi_1]f_1 - v\Theta[\phi_2]f_2\|_{L^2(\Omega)}^2 \\ &= \Pi_1^2 + \Pi_2^2.\end{aligned}$$

For Π_1, by Lemmas 2.1, 2.2 and the Sobolev embedding inequality $W^{1,2}(I) \hookrightarrow L^\infty(I)$,

$$\begin{aligned}\Pi_1 &\le \|\Theta[\phi_1]f_1 - \Theta[\phi_2]f_1\|_{L^2(\Omega)} + \|\Theta[\phi_2]f_1 - \Theta[\phi_2]f_2\|_{L^2(\Omega)} \\ &\le C(\|\phi_1 - \phi_2\|_{L^\infty(I)}\,\|f_1\|_{L^2(\Omega)} + \|\phi_2\|_{L^\infty(I)}\,\|f_1 - f_2\|_{L^2(\Omega)}) \\ &\le C(\|\phi_1 - \phi_2\|_{W^{1,2}(I)}\,\|f_1\|_{L^2(\Omega)} + \|\phi_2\|_{W^{1,2}(I)}\,\|f_1 - f_2\|_{L^2(\Omega)}) \\ &\le C(\|\phi_1 - \phi_2\|_{W^{1,2}(I)}\,\|f_1\|_{L^2(\Omega)} + \|f_2\|_{X_1}\,\|f_1 - f_2\|_{L^2(\Omega)}).\end{aligned}$$

Consider the auxiliary function

$$u(x) = (a + bx^2)x, \quad a, b > 0,$$

where $u'(x)$ satisfies $u'(x) = a + 3bx^2 > 0$. By the mean value theorem for integrals,

$$\int_{x_1}^{x_2} u'(x)\,dx = u'((1-\theta)x_1 + \theta x_2)(x_1 - x_2) \ge a(x_1 - x_2),$$

where $0 < \theta < 1$ and $x_1 \ge x_2$.

Multiplying the nonlinear Poisson equation by $\phi_1 - \phi_2$ and integrating over I in x gives

$$\begin{aligned}&\int_I (\rho_1 - \rho_2)(\phi_1 - \phi_2)\,dx \\ &\quad= \int_I ((a + b(\partial_x\phi_1)^2)\partial_x\phi_1 - (a + b(\partial_x\phi_2)^2)\partial_x\phi_2)(\partial_x(\phi_1 - \phi_2))\,dx.\end{aligned}$$

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On the one hand, by Lemma 2.1 and the Hölder and Poincaré inequalities,

$$\begin{aligned}\int_I(\rho_1-\rho_2)(\phi_1-\phi_2)\,dx &\le \|\rho_1-\rho_2\|_{L^2(I)}\|\phi_1-\phi_2\|_{L^2(I)}\\ &\le C\|f_1-f_2\|_{X_1}\|\partial_x(\phi_1-\phi_2)\|_{L^2(I)}.\end{aligned}$$

On the other hand, assuming, without loss of generality, that $\phi_1-\phi_2\ge 0$,

$$\begin{aligned}\|\partial_x(\phi_1-\phi_2)\|^2_{L^2(I)} &\le \frac{1}{a}\int_I u'(\theta\phi_2+(1-\theta)\phi_1)[\partial_x(\phi_1-\phi_2)]^2\,dx\\ &\le C\int_I((a+b(\partial_x\phi_1)^2)\partial_x\phi_1-(a+b(\partial_x\phi_2)^2)\partial_x\phi_2)(\partial_x(\phi_1-\phi_2))\,dx.\end{aligned}$$

Hence, we obtain $\|\partial_x(\phi_1-\phi_2)\|_{L^2(I)}\le C\|f_1-f_2\|_{X_1}$. Thus,

$$\begin{aligned}\Pi_1 &\le C(\|f_1-f_2\|_{X_1}\|f_1\|_{L^2(\Omega)}+\|f_2\|_{X_1}\|f_1-f_2\|_{L^2(\Omega)})\\ &\le C(\|f_1\|_{X_1}+\|f_2\|_{X_1})\|f_1-f_2\|_{X_1}.\end{aligned}$$

Similarly, we can also deduce that

$$\begin{aligned}\Pi_2 &\le \|v\Theta[\phi_1]f_1-v\Theta[\phi_2]f_1\|_{L^2(\Omega)}+\|v\Theta[\phi_2]f_1-v\Theta[\phi_2]f_2\|_{L^2(\Omega)}\\ &\le 2\pi(\|\partial_\eta(\delta(\phi_1-\phi_2)\mathcal{F}_vf_1)\|_{L^2(\Omega)}+\|\partial_\eta(\delta\phi_2\mathcal{F}_v(f_1-f_2))\|_{L^2(\Omega)})\\ &\le 2\pi(\|\partial_\eta(\delta(\phi_1-\phi_2))\mathcal{F}_vf_1\|_{L^2(\Omega)}+\|\delta(\phi_1-\phi_2)\partial_\eta(\mathcal{F}_vf_1)\|_{L^2(\Omega)})\\ &\quad+2\pi(\|\partial_\eta(\delta\phi_2)(\mathcal{F}_v(f_1-f_2)\|_{L^2(\Omega)}+\|(\delta\phi_2)\partial_\eta(\mathcal{F}_v(f_1-f_2)\|_{L^2(\Omega)})\\ &\le C(\|\partial_x(\phi_1-\phi_2)\|_{L^2(I)}\|(1+v)f_1\|_{L^2(\Omega)}+\|\phi_1-\phi_2\|_{L^\infty(I)}\|vf_1\|_{L^2(\Omega)})\\ &\quad+C(\|\partial_x\phi_2\|_{L^\infty(I)}\|f_1-f_2\|_{L^2(\Omega)}+\|\phi[f_2]\|_{L^\infty(I)}\|v(f_1-f_2)\|_{L^2(\Omega)})\\ &\le C(\|f_1-f_2\|_{X_1}\|(1+v)f_1\|_{L^2(\Omega)}+\|\phi_1-\phi_2\|_{W^{1,2}(I)}\|vf_1\|_{L^2(\Omega)})\\ &\quad+C(\|\phi_2\|_{W^{2,2}(I)}\|f_1-f_2\|_{L^2(\Omega)}+\|\phi[f_2]\|_{W^{1,2}(I)}\|v(f_1-f_2)\|_{L^2(\Omega)})\\ &\le C(\|f_1-f_2\|_{X_1}\|(1+v)f_1\|_{L^2(\Omega)}+\|f_1-f_2\|_{X_1}\|vf_1\|_{L^2(\Omega)})\\ &\quad+C(\|f_2\|_{X_1}\|f_1-f_2\|_{L^2(\Omega)}+\|f_2\|_{X_1}\|v(f_1-f_2)\|_{L^2(\Omega)})\\ &\le C\|f_1-f_2\|_{X_1}(\|f_1\|_{X_1}+\|f_2\|_{X_1}).\end{aligned}$$

This concludes the proof of the lemma. □

2.2. Proof of Theorem 1.1. In this subsection, we will prove Theorem 1.1 by means of [18, Theorem 6.1.4]. First, we consider the linear operator $A:D(A)\to X_1$ defined by $Af=-v\cdot\nabla_xf$ and its domain

$$D(A)=\{f\in X_1 \mid v\cdot\nabla_xf\in X_1, f(0,v)=f(1,v), f_x(0,v)=f_x(1,v), v\in R\}.$$

Then A is an infinitesimal generator of a C_0 semigroup of isometries $\{S(t),t\in R\}$ on X_1 given by $S(t)f(x,v)=\widetilde{f}(x-vt,v)$, where $\widetilde{f}$ denotes the 1-periodic extension of f. Moreover, $D(A)$ is dense in X_1 and $\langle Af,g\rangle_{X_1}=\langle f,A^*g\rangle_{X_1}$ with $A^*g=v\cdot\nabla_xg$ (see [18, Corollary 1.2.5]).

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We may rewrite the mWP system as

$$f_t = Af + \Theta[\phi]f, \quad f(t=0) = f_0. \tag{2.3}$$

We consider $\Theta[\phi]f$ as a bounded perturbation of the generator A. Since $\Theta[\phi]f$ is bounded and locally Lipschitz continuous (see Lemmas 2.2 and 2.3), [18, Theorem 6.1.4] shows that the problem (2.3) has a unique mild solution for every $f_0 \in X_1$ on some time interval $[0, t_{\max})$.

To prove $t_{\max} = +\infty$, we shall now derive an *a priori* estimate for $\|f(t)\|_{X_1}$. By computing the time derivative and taking into account (2.3),

$$\frac{1}{2}\frac{d}{dt}\|f\|_{X_1}^2 = \langle Af, f\rangle_{X_1} + \langle \Theta[\phi]f, f\rangle_{X_1}.$$

Using the dissipativity of A, we conclude that

$$\frac{1}{2}\frac{d}{dt}\|f\|_{X_1}^2 \le \langle \Theta[\phi]f, f\rangle_{X_1}.$$

The operator $\Theta[\phi]f$ is skew-symmetric on $L^2(\Omega)$, that is, $\langle \Theta[\phi]f, f\rangle_{L^2(\Omega)} = 0$. Consequently, by [3, Proposition 2.3] and Lemma 2.1,

$$\frac{1}{2}\frac{d}{dt}\|f\|_{X_1}^2 \le \langle \Theta[\phi]f, f\rangle_{L^2(\Omega,|v|^2\,dx\,dv)} \le C\|\partial_x\phi\|_{L^\infty(I)}\|vf\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \le C\|f\|_{X_1}^2\|f\|_{L^2(\Omega)}.$$

On the other hand, by the law of conservation of mass, $\|f\|_{L^2(\Omega)} = \|f_0\|_{L^2(\Omega)}$. Hence,

$$\frac{d}{dt}\|f\|_{X_1}^2 \le C\|f_0\|_{L^2(\Omega)}\|f\|_{X_1}^2.$$

Finally, by applying Gronwall's inequality, we obtain $t_{\max} = \infty$.

3. Global existence for mWPFP

In the ensuing discussion, we define $L : D(L) \to X_1$ by

$$Lf = -vf_x + \beta(vf)_v + \sigma f_{vv} + 2\gamma f_{xv} + \alpha f_{xx}, \tag{3.1}$$

where

$$D(L) = \{f \in X_1 \mid vf_x, vf_v, f_{vv}, f_{xv}, f_{xx} \in X_1, f(0,v) = f(1,v), f_x(0,v) = f_x(1,v)\}.$$

Further, we define the mapping ψ on $R \times X_1$ by

$$\psi(\lambda, f) = e^{Lt}f_0 + \lambda\int_0^t e^{L(t-s)}\Theta[\phi]f(s)\,ds,$$

where ϕ is the $H_0^2(I)$ solution of the nonlinear Poisson equation with f given. Obviously, (f,ϕ) is a mild solution of mWPFP, subject to the initial boundary value conditions (1.6)–(1.7), if f is a fixed point of ϕ, that is, $\psi(\lambda, f) = f$. We prove the following local existence result.

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Lemma 3.1. *For $f_0 \in X_1$, there is a unique solution (λ, f) of $\psi(\lambda, f) = f$ in $[0, \lambda) \times X_1$ emanating from $(0, e^{Lt} f_0)$.*

Lemma 3.1 is a consequence of the following two lemmas.

Lemma 3.2. *For $f_0 \in X_1$, the Leray–Schauder degree of the mapping $I - \psi(0, f)$ satisfies $\deg_{LS}[I - \psi(0, f), X_1, 0] \neq 0$.*

Proof. By [3, Lemmas 2.1 and 2.2], the mapping $\psi(0, \cdot)$ has a unique fixed point in X_1. □

Lemma 3.3 [17, Theorem 6.4]. *Let $\psi : [0, \infty) \times X_1 \to X_1$ be a completely continuous mapping with Leray–Schauder degree $\deg_{LS}[I - \psi(0, f), X_1, 0] \neq 0$. Then there exists an unbounded continuum ι of fixed points $\{(\lambda, f) \in [0, \infty) \times X_1; \psi(\lambda, f) = f\}$ with $\iota \cap (\{0\} \times X_1) \neq \emptyset$.*

Proof of Lemma 3.1. By Lemma 3.3, we have to prove that ψ is compact and continuous in $[0, \infty) \times X_1$. For the continuity of ψ, take a sequence $(\lambda^k, f^k) \to (\lambda, f)$ as $k \to \infty$ in the norm topology of $[0, \infty) \times X_1$. Lemma 2.1 implies $\partial_x^k \phi \in L^\infty(0, T; L^2(I))$. The proofs of Lemmas 2.1 and 2.3 imply the strong convergence of $\partial_x^k \phi$ to $\partial_x \phi$ in $L^\infty(0, T; L^2(I))$. Moreover, $\lambda^k \Theta[\phi^k] f^k$ converges strongly to $\lambda \Theta[\phi] f$ in $L^\infty(0, T; X_1)$. For compactness of ψ, note that the term $\lambda^k \Theta[\phi^k] f^k$ is bounded in $L^\infty(0, T; X_1)$ by Lemmas 2.1–2.3. By standard parabolic theory, $L\psi(\lambda^k, f^k) \in L^\infty(0, T; X_1)$ and

$$\partial_t \psi(\lambda^k, f^k) = L e^{Lt} f_0 + L\lambda^k \int_0^t e^{L(t-s)} \Theta[\phi^k] f^k(s)\, ds + \lambda^k \Theta[\phi^k] f^k(t) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where L is defined in (3.1). Using the regularity of L (see [4, 8]),

$$\begin{aligned}
\|\Sigma_1\|_{L^2((0,T)\times\Omega)} &\leq \frac{\beta}{2} \|e^{Lt} f_0\|_{L^2((0,T)\times\Omega)} \leq C\|f_0\|_{L^2(\Omega)},\\
\|\Sigma_2\|_{L^2((0,T)\times\Omega)} &\leq C\lambda^k \|f^k(t)\|^2_{L^2(0,T;X_1)},\\
\|\Sigma_3\|_{L^2((0,T)\times\Omega)} &\leq C\lambda^k \|f^k(t)\|^2_{L^2(0,T;X_1)}.
\end{aligned}$$

Moreover, $\partial_t \psi(\lambda^k, f^k) \in L^2((0, T) \times \Omega)$. Then, [19, Corollary 8] implies that the sequence $\psi(\lambda^k, f^k)$ is relatively compact in the space $C(0, T; X_1)$. The local uniqueness of solutions is obtained with a contraction argument (see also Lemma 2.3). □

Proof of Theorem 1.3. Based on Lemma 3.1, we shall prove that $f \in C([0, t_{\max}), X_1)$ and $t_{\max} = +\infty$. Indeed, $f \in C([0, t_{\max}), X_1)$, from [4, 8]. On the other hand, by computing the time derivative and taking into account (1.9),

$$\frac{1}{2} \frac{d}{dt} \|f\|^2_{X_1} = \langle Lf, f \rangle_{X_1} + \langle \Theta[\phi] f, f \rangle_{X_1}.$$

Using the dissipativity of L and [3, Lemma 2.1], we conclude that

$$\frac{1}{2} \frac{d}{dt} \|f\|^2_{X_1} \leq e^{(\sigma + (1/2)\beta)t} \|f\|^2_{X_1} + \langle \Theta[\phi] f, f \rangle_{X_1}.$$

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Since the operator $\Theta[\phi]f$ is skew-symmetric on $L^2(\Omega)$, by [3, Proposition 2.3] and Lemma 2.1,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f\|^2_{X_1} &\le \Big(\sigma+\frac{1}{2}\beta\Big)\|f\|^2_{X_1} + \langle\Theta[\phi]f, f\rangle_{L^2(\Omega,|v|^2\,dx\,dv)}\\ &\le \Big(\sigma+\frac{1}{2}\beta\Big)\|f\|^2_{X_1} + C\|\partial_x\phi\|_{L^\infty(I)}\|vf\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}\\ &\le \Big(\sigma+\frac{1}{2}\beta\Big)\|f\|^2_{X_1} + C\|f\|^2_{X_1}\|f\|_{L^2(\Omega)}.\end{aligned}$$

On the other hand, as in [3, Lemma 1], it is easy to see that the operator $L-\frac{1}{2}\beta I$ defined on

$$\widetilde{D(L)} = \{f\in L^2(\Omega)\mid vf_x, vf_v, f_{vv}, f_{xv}, f_{xx}\in L^2(\Omega), f(0,v)=f(1,v), f_x(0,v)=f_x(1,v)\}$$

is dissipative in $L^2(\Omega)$. Consequently,

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2(\Omega)} \le \frac{\beta}{2}\|f\|^2_{L^2(\Omega)}, \quad \|f\|^2_{L^2(\Omega)} \le e^{\beta t}\|f_0\|^2_{L^2(\Omega)}$$

and

$$\frac{d}{dt}\|f\|^2_{X_1} \le (2\sigma+\beta+Ce^{\sqrt{\beta t}}\|f_0\|_{L^2(\Omega)})\|f\|^2_{X_1}.$$

Finally, Gronwall's inequality yields $t_{\max}=\infty$. □

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