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Decomposability of von Neumann Algebras and the Mazur Property of Higher Level

Dedicated to Professor Anthony To-Ming Lau on the occasion of his 60th birthday

Zhiguo Hu and Matthias Neufang

Abstract. The decomposability number of a von Neumann algebra $\mathcal{M}$ (denoted by $\operatorname{dec}(\mathcal{M})$) is the greatest cardinality of a family of pairwise orthogonal non-zero projections in $\mathcal{M}$. In this paper, we explore the close connection between $\operatorname{dec}(\mathcal{M})$ and the cardinal level of the Mazur property for the predual $\mathcal{M}_*$ of $\mathcal{M}$, the study of which was initiated by the second author. Here, our main focus is on those von Neumann algebras whose preduals constitute such important Banach algebras on a locally compact group G as the group algebra $L_1(G)$, the Fourier algebra $A(G)$, the measure algebra $M(G)$, the algebra $LUC(G)^*$, etc. We show that for any of these von Neumann algebras, say $\mathcal{M}$, the cardinal number $\operatorname{dec}(\mathcal{M})$ and a certain cardinal level of the Mazur property of $\mathcal{M}_*$ are completely encoded in the underlying group structure. In fact, they can be expressed precisely by two dual cardinal invariants of G: the compact covering number $\kappa(G)$ of G and the least cardinality $\chi(G)$ of an open basis at the identity of G. We also present an application of the Mazur property of higher level to the topological centre problem for the Banach algebra $A(G)^{**}$.

1 Introduction and Preliminaries

Our study of the decomposability of von Neumann algebras was originally motivated by the investigation of a higher level Mazur property for preduals of von Neumann algebras. So, we begin by recalling the definition of the classical Mazur property as well as of property (X), as introduced by Godefroy–Talagrand, see [15, 16]. The formulation of the latter relies on the notion of weakly unconditional Cauchy series (also called weakly unconditionally convergent series), which we will give here for the convenience of the reader.

Definition 1.1 Let X be a Banach space. A series $\sum f_n$ in X is called *weakly unconditionally Cauchy* (wuC) if, for every functional $\Phi \in X^*$, we have $\sum_{n=1}^{\infty} |\langle \Phi, f_n \rangle| < \infty$.

We now come to the central concept.

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Definition 1.2

(i) A Banach space Y is said to have the *Mazur property*, or to satisfy the condition of Mazur, if the following normality criterion is fulfilled: A functional in Y^{**} is normal, *i.e.,* defines an element of Y, if and only if it is w^*-sequentially continuous (on the unit ball of Y^*).

(ii) A Banach space Y is said to have property (X) if the following normality criterion is satisfied: If $f \in Y^{**}$ is a functional such that, for every wuC series $\sum y_n$ in Y^*, the equality

$$\langle f, w^*\text{-}\sum y_n \rangle = \sum \langle f, y_n \rangle$$

holds, then we have $f \in Y$. (Here, the limit $w^*\text{-}\sum y_n$ is taken in the $\sigma(Y^*, Y)$-topology.)

It turns out that in many situations, the above properties are too restrictive, since they require normality to be completely determined only by the *sequential* behavior of the functional. It is thus adequate to refine the concept in such a way that one actually measures the level of w^*-continuity of functionals, which is encoded by the cardinality of nets agains which the functionals must be tested. This leads to the notions of the Mazur property and property (X) of level κ, where κ denotes an arbitrary cardinal number, introduced in [33, Definition 4.1(ii); Definition 4.10]. In order to summarize both notions, we need the following, *cf.* [33, Definition 4.1(i); Definition 4.8]:

Definition 1.3

(i) Let X be a Banach space and $\kappa \geq \aleph_0$ a cardinal number. A functional $f \in X^{**}$ is called w^*- κ-continuous if for all nets $(x_\alpha)_{\alpha\in I} \subseteq \mathrm{Ball}(X^*)$ of cardinality $\aleph_0 \leq |I| \leq \kappa$ such that $x_\alpha \to 0$ in the $\sigma(X^*, X)$-topology, we have $\langle f, x_\alpha \rangle \to 0$.

(ii) Let X be a Banach space. A series $\sum_{\alpha\in I} f_\alpha$ in X is called *weakly unconditionally Cauchy* (wuC for short) if for every functional $\Phi \in X^*$, one has $\sum_{\alpha\in I} |\langle \Phi, f_\alpha \rangle| < \infty$.

We recall here from [33] the generalization of the Mazur property and property (X) for arbitrary cardinality level. It is worth pointing out that in cases when the cardinal level κ is $\aleph_0$, we recover the classical condition of Mazur and property (X), respectively (see Remark 2.4(v) below).

Definition 1.4 Let X be a Banach space and $\kappa \geq \aleph_0$ a cardinal number.

(i) We say that X has the level κ Mazur property if every w^*- κ-continuous functional in X^{**} is actually w^*-continuous, *i.e.,* defines an element of X.

(ii) We say that X has property (X) of level κ if the following normality criterion holds: A functional $\Phi \in X^{**}$ belongs to X if, for every wuC series $\sum_{\alpha\in I} f_\alpha$ in X^* of cardinality $|I| \leq \kappa$, one has

$$\langle \Phi, w^*\text{-}\sum_{\alpha\in I} f_\alpha \rangle = \sum_{\alpha\in I} \langle \Phi, f_\alpha \rangle.$$

(Here, the limit $w^*\text{-}\sum_{\alpha\in I} f_\alpha$ is taken in the $\sigma(X^*, X)$-topology.)

Next we give several basic statements concerning the above properties which will be needed in the sequel.

Remark 1.5 Let $\kappa \geq \aleph_0$ be a cardinal number.

(i) Property (X) of level κ trivially implies the Mazur property of level κ.
(ii) Both the Mazur property and property (X) of level κ are stable under isomorphisms and pass to (closed) linear subspaces, *cf.* [33, Remark 4.3; Remark 4.12].

The particular interest of this more general concept lies in the fact that by providing a *quantitative* version of the above mentioned classical properties, it reveals an intimate link between the w^*-continuity properties of functionals on X^* and certain Banach space properties of the underlying space X. This becomes especially apparent if X is a function space where the domain is, say, a locally compact group G — the standard situation in abstract harmonic analysis. It is interesting to see that in this case, a certain cardinal level of the Mazur property or property (X) in the above sense is intimately related to special cardinal invariants of the group G.

In fact, we find that in the case of the group algebra $L_1(G)$, the corresponding cardinal invariant is just the compact covering number $\kappa(G)$ of G, whereas for the Fourier algebra $A(G)$, the decisive number is given by the least cardinality $\chi(G)$ of an open basis at the identity of G — as one could expect by looking at the situation for a locally compact abelian group G (with dual group $\widehat{G}$), where we can identify $A(G)$ with $L_1(\widehat{G})$ via the inverse Fourier transform and we have $\chi(G) = \kappa(\widehat{G})$. Thus, the above properties show a close connection to the structure of the underlying group, which is not completely reflected in the coarser concept of the classical condition of Mazur or property (X).

We need to stress that, as we shall show, the key link between the higher level analogues of these properties and the cardinal invariants of the group is given by an intrinsic property of the von Neumann algebra whose predual is the Banach space considered. The crucial notion is the one of κ-decomposability, as introduced by Akemann–Anderson in [1, p. 54], where they studied Lyapunov theorems for singular maps on von Neumann algebras. Unaware of [1], the second author re-introduced this concept in [33, Definition 3.6], which we recall here for the sake of completeness. We thank David Sherman for kindly bringing to our attention the book by Akemann–Anderson [1].

Definition 1.6 Let κ be a cardinal number. A von Neumann algebra $\mathcal{M}$ is called *κ-decomposable* if every family of pairwise orthogonal non-zero projections in $\mathcal{M}$ has at most cardinality κ. The least cardinality κ_0 such that $\mathcal{M}$ is κ_0-decomposable is called the *decomposability number* of $\mathcal{M}$, denoted by $\mathrm{dec}(\mathcal{M})$.

This notion naturally extends the well-known concept of countable decomposability (or σ-finiteness) for von Neumann algebras. See [33, Proposition 3.8] for a useful equivalent formulation.

In the following, we shall briefly outline our main motivations to study the decomposability of von Neumann algebras. It turns out that the investigation of the latter is intimately connected with intriguing questions stemming from abstract harmonic analysis and the general theory of topological groups, the theory of Banach and operator algebras, and Banach space theory.

• Since a von Neumann algebra $\mathcal{M}$ is determined by its projections, the number of pairwise orthogonal (non-zero) projections in $\mathcal{M}$ is, of course, a very natural invariant for the algebra. Projection techniques are among the most powerful and fundamental tools in operator algebras, and the decomposability number measures the richness of the algebra in this crucial respect.

• Furthermore, it turns out that the concept of decomposability unifies different cardinal invariants in harmonic analysis from a more abstract operator algebraic viewpoint. More precisely, if G is any locally compact abelian group with dual group $\widehat{G}$, there is a corresponding duality between the compact covering number of G and the local weight of $\widehat{G}$, namely $\kappa(G) = \chi(\widehat{G})$. An immediate question is whether it is possible to extend this duality beyond the context of abelian groups. As our study shows, for any locally compact group G, the above cardinals $\kappa(G)$ and $\chi(G)$ may be interpreted precisely as the decomposability numbers of the von Neumann algebras $L_\infty(G)$ and $VN(G)$, respectively. But the latter are known to be Kac algebras (*cf.* Enock–Schwartz [10]) which are dual to each other, an abstract notion of duality which generalizes the classical Pontryagin duality of locally compact abelian groups. Hence, the concept of decomposability captures this duality in the most general framework.

• As we shall prove in Section 2, the decomposability number provides us with a very handy cardinal level of the Mazur property and property (X). The latter are properties formulated for general Banach spaces which were introduced by Neufang [33]. They have proved extremely useful in various applications in harmonic analysis and Banach algebra theory, such as the solution to a conjecture by Ghahramani–Lau (on topological centres) as well as a conjecture by Hofmeier–Wittstock (on automatic continuity of module homomorphisms on von Neumann algebras), which were recently established by the second author, *cf.* [34, 35].

• Combining our results with techniques of Hu [23], we shall give another application of the concept of decomposability to the topological centre question in Section 8. There we shall be concerned with the problem of determining the centre of the second dual of the Fourier algebra $A(G)$, a question which is still unsolved in general but has been studied in abstract harmonic analysis for more than 20 years. We shall prove that it is enough to solve this question for certain classes of groups, such as second countable ones.

• In connection with topological centres, we would like to mention yet another intriguing question on which the notion of decomposability sheds new light. Namely, as is well known, there are two natural concepts of "maximal" Arens irregularity for general Banach algebras: extreme non-Arens regularity, on the one hand, and strong Arens irregularity, on the other hand, as introduced by Granirer [18], and Dales–Lau [7], respectively. However, up to now, the relation between these two concepts has remained mysterious. But we have discovered that the notion of decomposability may provide an intimate link between the above two properties, for it plays a crucial role in the study of extreme non-Arens regularity undertaken by the first author (*cf.* [22, 23]), as well as in the investigation of strong Arens irregularity by the second author, *cf.* [34, 36]. Moreover, our work on decomposability indicates that there might exist a Banach algebra which is strongly Arens irregular without being extremely non-Arens regular (no such example is known so far!), a line of research that we shall intensely pursue.

• Of course, an obvious application of decomposability is the generalization of results established for σ-finite von Neumann algebras to the general case. In this context, we restrict ourselves to mentioning, as an example, Chu's characterization of the second dual von Neumann algebras among all σ-finite von Neumann algebras, *cf.* [4]. Here, his result on the equivalence of σ-finiteness of the von Neumann algebra and its predual being weakly compactly generated may have a general version involving the decomposability number, which in turn motivates intriguing questions in Banach space theory.

Finally, we would like to point out several intriguing phenomena which occurred to us in our study of the decomposability of von Neumann algebras, and which were unexpected.

• Given a von Neumann algebra $\mathcal{M}$, one may think that the decomposability number of the second dual $\mathcal{M}^{**}$ is completely determined by the cardinal $\operatorname{dec}(\mathcal{M})$, and one may guess that $\operatorname{dec}(\mathcal{M}^{**}) = 2^{2^{\operatorname{dec}(\mathcal{M})}}$. But it turns out that typically, the right-hand side is just a lower bound for $\operatorname{dec}(\mathcal{M}^{**})$. Moreover, quite surprisingly, when determining the latter cardinal exactly, a cardinal different from $\operatorname{dec}(\mathcal{M})$ comes into play. And it is fascinating to remark that, as we shall see in Section 7, in case $\mathcal{M} = L_\infty(G)$ or $\mathcal{M} = VN(G)$, this second cardinal is precisely the decomposability number of the dual Kac algebra!

• In view of the fact that $\operatorname{dec}(L_\infty(G)) = \kappa(G) \cdot \aleph_0$ and $\operatorname{dec}(VN(G)) = \chi(G) \cdot \aleph_0$ for every infinite locally compact group G, it is natural to expect that $\kappa(G)$ plays the dominant role in determining the decomposability numbers of algebras stemming from function spaces such as $C_0(G)^{**}$, $LUC(G)^{**}$ and $L_\infty(G)^{**}$, whereas $\chi(G)$ may predominate the values of $\operatorname{dec}(C^*(G)^{**})$, $\operatorname{dec}(UC(\widehat{G})^{**})$ and $\operatorname{dec}(VN(G)^{**})$. Nevertheless, it turns out that even the algebra $C_0(G)^{**}$ behaves differently, for as we shall prove, we have $\operatorname{dec}(C_0(G)^{**}) = \kappa(G) \cdot 2^{\chi(G)}$ for every infinite locally compact group G. Here, $\chi(G)$ plays a more important role than $\kappa(G)$, which is certainly surprising at first glance. Hence, the decomposability number tells us that there is a hidden structure of $C_0(G)$ which appears when passing to the second dual, namely the Kac algebra structure of $VN(G)$. From the viewpoint of the theory of Kac algebras, the above phenomenon may now be seen as an instance of the fact that the so-called Fourier–Stieltjes algebra of the Kac algebra $VN(G)$ is precisely $M(G) = C_0(G)^*$, and the decomposability number, revealing this link, even allows us to *measure* the extent to which the structure of $C_0(G)^{**}$ is determined by $VN(G)$. (We shall study the dual situation for $B(G)$, *i.e.,* the Fourier–Stieltjes algebra of the Kac algebra $L_\infty(G)$, on a different occasion.)

• It is well known that the C^*-algebra $\mathcal{K}(H)$ of compact operators on a Hilbert space H may be viewed as the non-commutative analogue of $c_0(I)$, whereas the von Neumann algebra $\mathcal{B}(H)$ of bounded operators on H parallels $l_\infty(I)$ (here, $\dim(H) = |I|$). For a general infinite locally compact Hausdorff space Ω, nevertheless, the C^*-

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algebra $C_0(\Omega)$ may of course be very different from $\mathcal{K}(H)$. We shall see that the notion of decomposability permits us not only to illustrate clearly both these analogies and differences, but even to *quantify* them. On the one hand, we shall prove that $\operatorname{dec}(l_\infty(I)^{**}) = 2^{2^{\operatorname{dec}(l_\infty(I))}} = 2^{2^{|I|}}$ and $\operatorname{dec}(\mathcal{B}(H)^{**}) = 2^{2^{\operatorname{dec}(\mathcal{B}(H))}} = 2^{2^{\dim(H)}}$. On the other hand, we have $\operatorname{dec}(c_0(I)^{**}) = \operatorname{dec}(l_\infty(I)) = |I| = \operatorname{dec}(c_0(I))$ and $\operatorname{dec}(\mathcal{K}(H)^{**}) = \operatorname{dec}(\mathcal{B}(H)) = \dim(H) = \operatorname{dec}(\mathcal{K}(H))$, where the decomposability number of a C^*-algebra is defined in exactly the same fashion as we did for a von Neumann algebra. Hence, we see that the decomposability numbers of the two von Neumann algebras, when raised to the second dual level, both increase by two cardinal levels, whereas the C^*-algebras remain stable. Moreover, we shall see that $\operatorname{dec}(C_0(\Omega)^{**}) \geq |\Omega|$ holds for any infinite locally compact Hausdorff space Ω. This tells us exactly how big the "gap" between the discrete case $c_0(I)$ and the continuous case $C_0(\Omega)$ is. Note that if Ω has no compact open subsets, then $C_0(\Omega)$ has no non-trivial projections.

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure λ, $L_\infty(G)$ is the abelian von Neumann algebra of essentially bounded λ-measurable functions on G, and $VN(G)$ is the von Neumann algebra generated by the left regular representation of G. It is well known that both $L_\infty(G)$ and $VN(G)$ are von Neumann subalgebras of $\mathcal{B}(L_2(G))$ (the latter denoting the von Neumann algebra of all bounded linear operators on the Hilbert space $L_2(G)$), and the group algebra $L_1(G)$ (resp., the Fourier algebra $A(G)$) can be identified with the predual of $L_\infty(G)$ (resp., $VN(G)$). Furthermore, $L_\infty(G)$ and $VN(G)$ are Kac algebras which are dual to each other, and together they generate $\mathcal{B}(L_2(G))$ as a von Neumann algebra, *cf.* [10].

The paper is organized as follows. In Section 2, for a general von Neumann algebra $\mathcal{M}$, we reveal the close connection between the level of the Mazur property (resp., property (X)) of the predual $\mathcal{M}_*$ of $\mathcal{M}$ and the decomposability number of $\mathcal{M}$. It is also shown in this section that $\operatorname{dec}(\mathcal{M})$ is attained and determined by the families of pairwise orthogonal normal states on $\mathcal{M}$.

Section 3 is devoted to determining the decomposability number of $L_\infty(G)$ and characterizing the groups G for which the group algebra $L_1(G)$ has the classical Mazur property, resp., property (X). We shall establish the results dual to the ones obtained in Section 3 for the Fourier algebra $A(G)$ in Section 4. The results of Section 3 are also completely extended to the measure algebra $M(G)$ in Section 5; actually, the dual of any commutative C^*-algebra is considered in this section. All the decomposability numbers of von Neumann algebras on a locally compact group G studied in these three sections are expressed precisely by the cardinals $\kappa(G)$ and $\chi(G)$. Among the cardinal invariants of a locally compact group G, we are most interested in $\kappa(G)$ and $\chi(G)$ when we work on the group algebra $L_1(G)$ and the Fourier algebra $A(G)$. A particular reason for this is that these two cardinals are dual to each other in many aspects. For example, if G is abelian, then $\kappa(\hat{G}) = \chi(G)$ and $\chi(\hat{G}) = \kappa(G)$ (*cf.* Hewitt–Ross [20, (24.48)]); for any non-compact amenable group G, $L_\infty(G)$ has exactly $2^{2^{\kappa(G)}}$ many topologically left invariant means, whereas for any non-discrete group G, the size of the set of topologically invariant means on $VN(G)$ is $2^{2^{\chi(G)}}$ (*cf.* Lau–Paterson [32] and Hu [21], respectively). For a general locally compact group G, the dual relation between $\kappa(G)$ and $\chi(G)$ is further exploited in this paper through our study of the decomposability of von Neumann algebras on G (see §3–§7).

Let $LUC(G)$ denote the C^*-algebra of bounded left uniformly continuous functions on G. In Section 6, for *all* infinite locally compact groups G, the precise size of the decomposability number $\operatorname{dec}(LUC(G)^{**})$ is also obtained explicitly in terms of the cardinals $\kappa(G)$ and $\chi(G)$. A local structure theorem for $G^{\mathcal{LUC}}$ (the $\mathcal{LUC}$-compactification of G) by Lau–Medghalchi–Pym [29] plays a key role in this part, which also leads to the determination of the exact cardinality of $G^{\mathcal{LUC}}$. Furthermore, the dual version for the C^*-algebra $UC(\hat{G})$ of uniformly continuous linear functionals on $A(G)$ is proved to be true for a large class of groups G.

The decomposability of the second dual of $L_\infty(G)$ and $VN(G)$ is investigated in Section 7. For $\mathcal{M} = L_\infty(G)$ or $VN(G)$, it turns out that $\operatorname{dec}(\mathcal{M}^{**})$ is completely determined by $\operatorname{dec}(\mathcal{M})$ if $\operatorname{dec}(\mathcal{M})$ is equal to the Hilbert space dimension $\dim(L_2(G))$ of $L_2(G)$. As a consequence, for a large class of locally compact groups G, we are able to precisely evaluate $\operatorname{dec}(L_\infty(G)^{**})$ (resp., $\operatorname{dec}(VN(G)^{**})$) by using the cardinal $\kappa(G)$ (resp., $\chi(G)$). Meanwhile, $L_\infty(G)$ and $VN(G)$ provide us with von Neumann algebras $\mathcal{M}$ for which the cardinal $\operatorname{dec}(\mathcal{M}^{**})$ is far away from $2^{2^{\operatorname{dec}(\mathcal{M})}}$.

In Section 8, we present an application of the Mazur property of higher level to the topological centre problem for the Banach algebra $A(G)^{**}$. It is shown that, for any locally compact group G with a large compact covering number, the topological centre problem for $A(G)^{**}$ can be reduced to the one for the algebras $A(H)^{**}$ of some open subgroups H of G with compact covering number dominated by $\operatorname{dec}(VN(G))$.

2 The General Situation

The main result of Neufang [33] characterizes, for the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$, the classical condition of Mazur in terms of the decomposability of $\mathcal{M}$ and shows at the same time that in this situation, the Mazur property and property (X) are actually equivalent. To state this result, we recall that a cardinal number κ is called *real-valued measurable* if for every set S of cardinality κ, there exists a probability measure μ on the power set of S which vanishes on singletons, *cf.* Gardner–Pfeffer [14, Definition 4.12]; in the following, we will briefly write "measurable" for "real-valued measurable". We note here that the above definition is different from the usual one which requires the measure μ to be κ-additive, *cf.* [14, p. 972].

Theorem 2.1 *For a von Neumann algebra $\mathcal{M}$, the following are equivalent.*

(i) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has the Mazur property.*
(ii) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has property (X) of Godefroy–Talagrand.*
(iii) *The von Neumann algebra $\mathcal{M}$ is κ-decomposable for some non-measurable cardinal number κ.*

Proof See Neufang [33, Theorem 3.17]. ■

Here, we give an analogue of the above result in the setting of the Mazur property and property (X) of higher level. Namely, we shall prove the following:

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Theorem 2.2 *Let $\mathcal{M}$ be a von Neumann algebra and κ a cardinal number. Consider the following statements.*

(i) *The von Neumann algebra $\mathcal{M}$ is κ-decomposable.*
(ii) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has property (X) of level $\kappa \cdot \aleph_0$.*
(iii) *The predual $\mathcal{M}_*$ of $\mathcal{M}$ has the Mazur property of level $\kappa \cdot \aleph_0$.*

Then we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii).

Proof (i)$\Rightarrow$(ii). Since a bounded linear functional on a von Neumann algebra is normal if and only if its restriction to every abelian von Neumann subalgebra is normal (*cf.* [39, Corollary 1]), it is readily seen that we can assume $\mathcal{M}$ to be abelian.

In the sequel, if (Ω, μ) is a finite measure space, we denote by $L_\infty(\Omega, \mu)$ the abelian von Neumann algebra of essentially bounded μ-measurable functions on Ω. A close inspection of the proof of [37, Proposition 1.18.1] shows that for our abelian von Neumann algebra $\mathcal{M}$, we have

$$\mathcal{M} = \bigoplus_{\alpha \in I}^{l_\infty} L_\infty(\Omega_\alpha, \mu_\alpha)$$

as von Neumann algebras, where $(\Omega_\alpha, \mu_\alpha)$ are finite measure spaces. We thus obtain an isometric isomorphism

$$\mathcal{M}_* = \bigoplus_{\alpha \in I}^{l_1} L_1(\Omega_\alpha, \mu_\alpha).$$

Owing to [33, Corollary 3.20], the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ have property (X) of level $\aleph_0$.

First, assume that $|I| \geq \aleph_0$. Then, of course, the spaces $L_1(\Omega_\alpha, \mu_\alpha)$ in particular have property (X) of level $|I|$. Hence, following [33, Theorem 4.17] and using the stability of the latter property under isomorphisms (*cf.* Remark 1.5(ii) above), we deduce that $\mathcal{M}_*$ also enjoys property (X) of level $|I|$. Since we obviously have $|I| \leq \kappa$, it follows that $\mathcal{M}_*$ has property (X) of level κ $(= \kappa \cdot \aleph_0)$.

Finally, in case $|I|$ is finite, $\mathcal{M}_*$ is of course isomorphic to $L_1(\Omega, \mu)$ where Ω is the disjoint union of $\{\Omega\}_{\alpha \in I}$ and $\mu = \sum \bigoplus_{\alpha \in I} \mu_\alpha$. Again, by [33, Corollary 3.20], $\mathcal{M}_*$ has property (X) of level $\aleph_0$.

(ii)$\Rightarrow$(iii). See Remark 1.5(i). ■

Remark 2.3 It is readily seen that for any infinite set S, the abelian von Neumann algebra $l_\infty(S)$ has the decomposability number $|S|$.

Remark 2.4 (i) As we pointed out earlier, the definition of measurable cardinals used in this paper is different from the usual one which is often used by topologists, *cf.* Jech [24, Definition 10.8]. In particular, according to our definition, if κ_1, κ_2 are cardinals with $\kappa_1 \leq \kappa_2$ and κ_1 is measurable, then κ_2 is also measurable. However, the existence question for both concepts of measurable cardinals is the same, *cf.* [24, Corollary 10.7].

(ii) It is known that the existence of measurable cardinals cannot be proved in ZFC (the axioms of Zermelo–Fraenkel and the axiom of choice). Moreover, in ZFC, the existence of measurable cardinals is equiconsistent with the fact that the Lebesgue measure on $\mathbb{R}$ can be extended to a measure on the power set of $\mathbb{R}$, *cf.* Gardner–Pfeffer [14, p. 972]. See Fremlin [12] and Solovay [38] for more information on measurable cardinals.

(iii) The inverse implications in Theorem 2.2 do not hold in general. It is known that the cardinal $\aleph_1$ is non-measurable, *cf.* Jech [24, Lemma 10.13]. Let S be a set such that $|S| = \aleph_1$. Then, by Theorem 2.1 and Remark 2.3, we see that $l_1(S)$ has the Mazur property of level $\aleph_0$ and property (X) of level $\aleph_0$, but $l_\infty(S)$ fails to be countably decomposable.

(iv) If measurable cardinals exist, then there exists a von Neumann algebra $\mathcal{M}$ such that $\mathcal{M}_*$ does not have the classical Mazur property. In fact, Edgar proved that for any set S, $l_1(S)$ has the classical Mazur property if and only if $|S|$ is non-measurable, *cf.* [9, Theorem 5.10].

(v) Let X be Banach space. It is easy to see that $f \in X^{**}$ is w^*-sequentially continuous if and only if f is w^*- $\aleph_0$-continuous. It can also be shown that if $\sum y_n$ is a wuC series in X^*, then w^*- $\lim_F \sum_{n\in F} y_n$ exists in X^*, where F denotes finite subsets of $\mathbb{N}$. Therefore, X has the Mazur property, resp., property (X), of level $\aleph_0$ if and only if X has the classical Mazur property, resp., property (X).

Obviously, $\operatorname{dec}(\mathcal{M}) < \infty$ if $\mathcal{M}$ is a finite dimensional von Neumann algebra. In fact, the converse is also true.

Proposition 2.5 *Let $\mathcal{M}$ be a von Neumann algebra. Then* $\operatorname{dec}(\mathcal{M}) < \infty$ *if and only if $\mathcal{M}$ is finite dimensional.*

Proof Suppose $\operatorname{dec}(\mathcal{M}) = n < \infty$. Then there exist pairwise orthogonal non-zero projections $p_1, \dots, p_n$ in $\mathcal{M}$ such that $\sum_{i=1}^n p_i = \mathrm{id}$. Assume that $\mathcal{M}$ is infinite dimensional. Then $\mathcal{M}$ is not a reflexive von Neumann algebra. In particular, there exists a non-zero singular positive linear functional ϕ on $\mathcal{M}$. By [40, Theorem III.3.8], for each $1 \le i \le n$, there exists a non-zero projection $q_i \le p_i$ in $\mathcal{M}$ such that $\phi(q_i) = 0$. It follows that $\phi(\sum_{i=1}^n q_i) = 0$ and hence $\sum_{i=1}^n q_i \ne \mathrm{id}$. Let $q_{n+1} = \mathrm{id} - \sum_{i=1}^n q_i$. Then $q_1, q_2, \dots, q_{n+1}$ are pairwise orthogonal non-zero projections in $\mathcal{M}$, contradicting the assumption that $\operatorname{dec}(\mathcal{M}) = n$. Therefore, if $\operatorname{dec}(\mathcal{M}) < \infty$, then $\mathcal{M}$ is finite dimensional. ■

Let $\mathcal{M}$ be a von Neumann algebra and φ a normal positive linear functional on $\mathcal{M}$. Then there exists a smallest projection P in $\mathcal{M}$ such that $\varphi(P) = \varphi(1) = \|\varphi\|$, *cf.* Sakai [37]. The projection P is called the *support* of φ and will be denoted by $S(\varphi)$. If φ_1 and φ_2 are normal positive linear functionals on $\mathcal{M}$, then we say that φ_1 and φ_2 are *orthogonal* if $\|\varphi_1 - \varphi_2\| = \|\varphi_1\| + \|\varphi_2\|$, or equivalently, $S(\varphi_1)$ and $S(\varphi_2)$ are orthogonal projections in $\mathcal{M}$, *cf.* [37].

A projection P in $\mathcal{M}$ is said to be *countably decomposable* if every family of pairwise orthogonal non-zero sub-projections of P in $\mathcal{M}$ is countable. Now suppose $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space H. A projection in $\mathcal{M}$ is said to be *cyclic* if its range is the closed linear subspace of H generated by $\mathcal{M}'\xi$ for some vector $\xi \in H$, where $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ in $\mathcal{B}(H)$. The following results on cyclic/countably decomposable projections can be found in Kadison–Ringrose, *cf.* [25, (5.5.9), (5.5.15), (7.6.13) and (7.2.7)], respectively.

(1) Every projection in $\mathcal{M}$ is a sum of pairwise orthogonal cyclic projections in $\mathcal{M}$.
(2) Every cyclic projection in $\mathcal{M}$ is countably decomposable in $\mathcal{M}$.
(3) A projection P in $\mathcal{M}$ is countably decomposable if and only if $P = S(\varphi)$ for some normal state φ on $\mathcal{M}$.
(4) A projection P in $\mathcal{M}$ is cyclic if and only if $P = S(\varphi)$ for some *vector state* φ on $\mathcal{M}$, *i.e.,* there exists a unit vector $\xi \in H$ such that $\varphi(x) = \langle x\xi|\xi\rangle$ for all $x \in \mathcal{M}$.

It is easy to see that $\mathcal{M}$ is $\operatorname{dec}(\mathcal{M})$-decomposable and $\operatorname{dec}(\mathcal{M})$ is the greatest cardinality of a family of pairwise orthogonal non-zero projections in $\mathcal{M}$. The following result shows that the cardinal $\operatorname{dec}(\mathcal{M})$ actually can be attained by the cardinality of such a family of *cyclic* projections in $\mathcal{M}$. Hence, $\operatorname{dec}(\mathcal{M})$ is completely determined by the families of pairwise orthogonal (normal) vector states on $\mathcal{M}$. This fact will be frequently used in the sequel in the study of the decomposability of the second dual $\mathcal{A}^{**}$ of a C^*-algebra $\mathcal{A}$. For convenience, let $\operatorname{ost}(\mathcal{M}_*)$ denote the greatest cardinality of a family of pairwise orthogonal normal states on $\mathcal{M}$.

Theorem 2.6 *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H. Then*

(i) *there exists a family $\mathcal{P}$ of pairwise orthogonal non-zero cyclic projections in $\mathcal{M}$ with $\sum_{P\in\mathcal{P}} P = \mathrm{id}$ such that $|\mathcal{P}| = \operatorname{dec}(\mathcal{M})$. Furthermore, $|\mathcal{P}| = \operatorname{dec}(\mathcal{M})$ holds for any such infinite family of cyclic projections in $\mathcal{M}$.*
(ii) $\operatorname{dec}(\mathcal{M}) = \operatorname{ost}(\mathcal{M}_*) = |\Phi|$ *for some family Φ of pairwise orthogonal vector states on $\mathcal{M}$.*

Proof (i) Suppose $\operatorname{dec}(\mathcal{M}) < \infty$. Then there exists a family $\{P_i\}_{i=1}^n$ of pairwise orthogonal non-zero projections in $\mathcal{M}$ such that $n = \operatorname{dec}(\mathcal{M})$. Clearly, we have $\sum_{i=1}^n P_i = \mathrm{id}$. By Fact (1) above, each P_i ($i \le n$) must be cyclic in $\mathcal{M}$.

In the following, assume that $\operatorname{dec}(\mathcal{M})$ is infinite. First, we show that $\mathcal{M}$ contains an *infinite* family $\mathcal{P}$ of pairwise orthogonal non-zero cyclic projections such that $\sum_{P\in\mathcal{P}} P = \mathrm{id}$. By Proposition 2.5, one can see that $\mathcal{M}$ must contain an infinite sequence $\{E_n\}$ of pairwise orthogonal non-zero projections. We may assume that each E_n is cyclic in $\mathcal{M}$ (*cf.* Fact (1)) and $\sum_{n=1}^\infty E_n \neq \mathrm{id}$. Note that $\mathrm{id} - \sum_{n=1}^\infty E_n$ is a non-zero projection in $\mathcal{M}$. By Fact (1), $\mathrm{id} - \sum_{n=1}^\infty E_n$ is the sum of a family $\mathcal{Q}$ of pairwise orthogonal (non-zero) cyclic sub-projections of $\mathrm{id} - \sum_{n=1}^\infty E_n$ in $\mathcal{M}$. Then $\mathcal{P} = \mathcal{Q} \cup \{E_n\}_{n=1}^\infty$ is the required family of cyclic projections in $\mathcal{M}$.

Next, we prove that $|\mathcal{P}| = \operatorname{dec}(\mathcal{M})$. Obviously, $|\mathcal{P}| \le \operatorname{dec}(\mathcal{M})$. Conversely, let $\mathcal{U}$ be an arbitrary family of pairwise orthogonal non-zero projections in $\mathcal{M}$. According to [25, (6.3.9)], $|\mathcal{U}| \le |\mathcal{P}|$ since $\sum_{U\in\mathcal{U}} U \le \mathrm{id} = \sum_{P\in\mathcal{P}} P$. It follows that $\operatorname{dec}(\mathcal{M}) \le |\mathcal{P}|$ and hence $|\mathcal{P}| = \operatorname{dec}(\mathcal{M})$.

Finally, by [25, (6.3.9)] again, $|\mathcal{U}| = |\mathcal{P}| = \operatorname{dec}(\mathcal{M})$ for any infinite family $\mathcal{U}$ of pairwise orthogonal non-zero cyclic projections in $\mathcal{M}$ with $\sum_{U\in\mathcal{U}} U = \mathrm{id}$.

(ii) Let Φ be the family of vector states on $\mathcal{M}$ corresponding to the family $\mathcal{P}$ of projections in $\mathcal{M}$ as obtained in (i) (*cf.* Fact (4)). Then Φ is pairwise orthogonal and

$|\Phi| = |\mathcal{P}| = \operatorname{dec}(\mathcal{M})$. On the other hand, $|\Phi| \leq \operatorname{ost}(\mathcal{M}_*) \leq \operatorname{dec}(\mathcal{M})$, *cf.* Fact (3). Therefore, $\operatorname{dec}(\mathcal{M}) = \operatorname{ost}(\mathcal{M}_*) = |\Phi|$. ■

Let $\mathcal{A}$ be a C^*-algebra. Then the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$ is a von Neumann algebra. It is known that for any linear functional φ on $\mathcal{A}^{**}$, φ is a normal state on $\mathcal{A}^{**}$ if and only if φ is a state on $\mathcal{A}$, *cf.* Dixmier [8, (12.1.3(iii))]. So, we will use $\operatorname{ost}(\mathcal{A}^*)$ to denote $\operatorname{ost}((\mathcal{A}^{**})_*)$, which is now equal to the greatest cardinality of a family of pairwise orthogonal states on $\mathcal{A}$. For a normed linear space X, we use $\operatorname{dense}(X)$ to denote the *density character* of X, *i.e.,* the least cardinality of a norm dense subset of X. The following corollary is clear.

Corollary 2.7 *Let $\mathcal{A}$ be a C^*-algebra. Then*

$$\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{ost}(\mathcal{A}^*) \leq \operatorname{dense}(\mathcal{A}^*) \leq |\mathcal{A}^*| \leq 2^{\operatorname{dense}(\mathcal{A})}.$$

Remark 2.8 As pointed out earlier, our main interest here is to investigate the intimate connection between the decomposability of von Neumann algebras arising from function spaces and the cardinal invariants of the underlying domain, especially when the domain is a locally compact group. Our investigation of the dual of the measure algebra $M(G)$ of a locally compact group G has led to considering the decomposability of the second dual of an arbitrary *commutative* C^*-algebra; this will be studied in Section 5. We will explore the decomposability of the second dual of a general C^*-algebra elsewhere.

3 The Case of the Group Algebra $L_1(G)$

We begin by determining the decomposability number of the abelian von Neumann algebra $L_\infty(G) = L_1(G)^*$ for all locally compact groups G. Here, we recall that $\kappa(G)$ denotes the compact covering number of G, *i.e.,* the least cardinality of a compact covering of G.

Theorem 3.1 *Let G be a locally compact group. Then*

(i) $\operatorname{dec}(L_\infty(G)) = \kappa(G)$ *if G is non-compact;*
(ii) $\operatorname{dec}(L_\infty(G)) = \aleph_0$ *if G is compact and infinite;*
(iii) $\operatorname{dec}(L_\infty(G)) = |G|$ *if G is finite.*

Proof (i) Suppose that G is non-compact. Then there exists a family with cardinality $\kappa(G)$ of pairwise disjoint non-empty open subsets of G. By taking the characteristic functions of these open sets, we get a family of pairwise orthogonal non-zero projections in $L_\infty(G)$. It follows that $\operatorname{dec}(L_\infty(G)) \geq \kappa(G)$.

Conversely, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of G such that the closure $\overline{U_\alpha}$ of U_α is compact for all $\alpha \in \Lambda$ and $|\Lambda| = \kappa(G)$. Let $\{\Phi_i\}_{i\in I}$ be a family of pairwise orthogonal non-zero projections in $L_\infty(G)$. Then, for each $i \in I$, $\Phi_i = \chi_{E_i}$ for some measurable set E_i such that E_i is not locally λ-null and $E_i \cap E_j$ is locally λ-null if $i \neq j$, where χ_{E_i} denotes the characteristic function of E_i. Note that $\overline{U_\alpha}$ is compact for all $\alpha \in \Lambda$. It follows that for each $\alpha \in \Lambda$, the set $I_\alpha = \{i \in I : \lambda(E_i \cap U_\alpha) > 0\}$ is at

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most countable and $I = \bigcup_{\alpha\in\Lambda} I_\alpha$. Therefore, $|I| \le |\Lambda| \cdot \aleph_0 = |\Lambda| = \kappa(G)$ and hence $\operatorname{dec}(L_\infty(G)) \le \kappa(G)$. Consequently, $\operatorname{dec}(L_\infty(G)) = \kappa(G)$.

(ii) Assume that G is compact and infinite. Then there exists an infinite sequence of pairwise disjoint non-empty open sets in G. Following the same argument as above, we have $\operatorname{dec}(L_\infty(G)) \ge \aleph_0$. On the other hand, let $\{\Phi_i\}_{i\in I}$ be a family of pairwise orthogonal non-zero projections in $L_\infty(G)$. Again, for each $i \in I$, $\Phi_i = \chi_{E_i}$ for some measurable set E_i such that $\lambda(E_i) > 0$ and $\lambda(E_i \cap E_j) = 0$ if $i \ne j$. Thus, I is at most countable since $\lambda(G) < \infty$. Therefore, $\operatorname{dec}(L_\infty(G)) \le \aleph_0$ and hence $\operatorname{dec}(L_\infty(G)) = \aleph_0$.

(iii) If G is finite, by Remark 2.3, $\operatorname{dec}(L_\infty(G)) = \operatorname{dec}(l_\infty(G)) = |G|$. ■

Corollary 3.2 $\operatorname{dec}(L_\infty(G)) = \kappa(G) \cdot \aleph_0$ *for all infinite locally compact groups G.*

A certain cardinal level of the Mazur property, resp., property (X), for the group algebra $L_1(G)$ has been obtained by Neufang [33]. Now we are ready to characterize the locally compact groups G for which the space $L_1(G)$ enjoys the classical Mazur property, resp., property (X).

Corollary 3.3 *Let G be a locally compact group. Then*

(i) $L_1(G)$ *has the Mazur property (resp., property* (X)*) of level* $\kappa(G) \cdot \aleph_0$.
(ii) $L_1(G)$ *has the classical Mazur property, resp., property* (X)*, if and only if* $\kappa(G)$ *is a non-measurable cardinal.*

Proof (i) See [33, Theorem 4.4; Theorem 4.18].

(ii) For the sufficiency, see [33, Corollary 4.15; Theorem 4.18]. The necessity follows immediately from Theorem 2.1 and Corollary 3.2. ■

4 The Case of the Fourier Algebra $A(G)$

Let G be a locally compact group and let $\chi(G)$ denote the least cardinality of an open basis at the identity of G. It is well known that if G is abelian with dual group $\widehat{G}$, then the Fourier algebra $A(G)$ is identified with $L_1(\widehat{G})$ via the inverse Fourier transform and we have $\chi(G) = \kappa(\widehat{G})$. In this section, for all locally compact groups G, we will prove the dual version of the results on $L_1(G)$ presented in Section 3 for the Fourier algebra $A(G)$.

Theorem 4.1 *Let G be a locally compact group. Then*

(i) $\operatorname{dec}(VN(G)) = \chi(G)$ *if G is non-discrete;*
(ii) $\operatorname{dec}(VN(G)) = \aleph_0$ *if G is discrete and infinite;*
(iii) $\operatorname{dec}(VN(G)) < \infty$ *if G is finite.*

Proof (i) Suppose that G is non-discrete. Then $VN(G)$ contains a family with cardinality $\chi(G)$ of pairwise orthogonal non-zero projections (see Chou [3, Theorem 3.2] for the case $\chi(G) = \aleph_0$ and Hu [21, Lemma 5.1] for the case $\chi(G) > \aleph_0$). Therefore, we have $\operatorname{dec}(VN(G)) \ge \chi(G)$.

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To finish the proof, we only need to show that $VN(G)$ is $\chi(G)$-decomposable. We follow an idea as used in the proof of Hu [22, Proposition 5.1]. Let $\mathcal{U}$ be a compact neighborhood system at the identity e of G such that $|\mathcal{U}| = \chi(G)$. For each $U \in \mathcal{U}$, let $h_U = \frac{1}{\lambda(U)}\chi_U \in L_2(G)$, where $\lambda(U)$ is the left Haar measure of U. By [20, Theorem (20.15)], we have $\lim_U \|h_U * f - f\|_2 = 0$ for all $f \in L_2(G)$. If $T \in VN(G)$ and $f \in C_{00}(G)$ (the latter denoting the space of continuous functions on G with compact support), then $T(h_U * f) = T(h_U) * f$ and hence $T(f) = \lim_U(T(h_U) * f)$ in the $\|\cdot\|_2$-norm. Since $C_{00}(G)$ is $\|\cdot\|_2$-dense in $L_2(G)$, each $T \in VN(G)$ is uniquely determined by the net $\{T(h_U)\}_{U\in\mathcal{U}}$ in $L_2(G)$. In particular, if $T \neq 0$, then $T(h_U) \neq 0$ for some $U \in \mathcal{U}$.

Now let $\{P_i\}_{i\in I}$ be an arbitrary family of pairwise orthogonal non-zero projections in $VN(G)$. For each $U \in \mathcal{U}$, we have $\sum_{i\in I} \|P_i(h_U)\|_2^2 \leq \|h_U\|_2^2$ and hence the set $I_U = \{i \in I : P_i(h_U) \neq 0\}$ is at most countable. Let $i \in I$. Since $P_i \neq 0$, $P_i(h_U) \neq 0$ for some $U \in \mathcal{U}$ by the above discussion, *i.e.*, $i \in I_U$. Thus, we have $I = \bigcup_{U\in\mathcal{U}} I_U$. It follows that $|I| \leq |\mathcal{U}| \cdot \aleph_0 = \chi(G) \cdot \aleph_0 = \chi(G)$. Therefore, $VN(G)$ is $\chi(G)$-decomposable.

(ii) Assume that G is discrete and infinite. Let $h = \chi_{\{e\}} \in L_2(G)$. By the same argument as used in (i), we see that each $T \in VN(G)$ is uniquely determined by the function $T(h)$ in $L_2(G)$. Let $\{P_i\}_{i\in I}$ be an arbitrary family of pairwise orthogonal non-zero projections in $VN(G)$. Then $P_i(h) \neq 0$ for all $i \in I$. It follows that I is at most countable since $\sum_{i\in I} \|P_i(h)\|_2^2 \leq \|h\|_2^2$. Hence, $\operatorname{dec}(VN(G)) \leq \aleph_0$. On the other hand, according to Proposition 2.5, $\operatorname{dec}(VN(G)) \geq \aleph_0$. Therefore, $\operatorname{dec}(VN(G)) = \aleph_0$.

(iii) If G is finite, then $\dim(VN(G)) < \infty$ and hence $\operatorname{dec}(VN(G)) < \infty$. ∎

Corollary 4.2 $\operatorname{dec}(VN(G)) = \chi(G) \cdot \aleph_0$ *for all infinite locally compact groups G.*

We have the following analogue of Corollary 3.3.

Corollary 4.3 *Let G be a locally compact group. Then*

(i) *$A(G)$ has the Mazur property (resp., property (X)) of level $\chi(G) \cdot \aleph_0$.*

(ii) *$A(G)$ has the classical Mazur property, resp., property (X), if and only if $\chi(G)$ is a non-measurable cardinal.*

Proof (i) It follows from Corollary 4.2 and Theorem 2.2.

(ii) Owing to Theorem 2.1, we only have to show that $A(G)$ has the Mazur property if and only if $\chi(G)$ is a non-measurable cardinal. Note that $VN(G)$ is $\chi(G) \cdot \aleph_0$-decomposable, *cf.* Corollary 4.2. Also, $\chi(G) \cdot \aleph_0$ is non-measurable if and only if $\chi(G)$ is non-measurable, *cf.* Remark 2.4(i). Therefore, by Theorem 2.1 and Corollary 4.2, $A(G)$ has the classical Mazur property if and only if $\chi(G)$ is non-measurable. ∎

5 The Case of the Space $M(\Omega)$

Let Ω be a locally compact Hausdorff space. Let $C(\Omega)$ be the C^*-algebra of bounded complex-valued continuous functions on Ω with the supremum norm and let $C_0(\Omega)$ be the C^*-algebra of all $f \in C(\Omega)$ such that f vanishes at infinity. It is well known that $C_0(\Omega)^*$ is identified with the space $M(\Omega)$ of bounded regular complex Borel measures on Ω. So, $M(\Omega)^*$ $(= C_0(\Omega)^{**})$ is a von Neumann algebra. In this section, we will study the decomposability of $M(\Omega)^*$ in terms of the space Ω and derive a certain cardinal level of the Mazur property and property (X) for the space $M(\Omega)$. In particular, for all locally compact groups G, we will determine the decomposability number $\mathrm{dec}(M(G)^*)$ and characterize the groups G for which $M(G)$ has the classical Mazur property, resp., property (X), precisely in terms of $|G|$ (and hence in terms of the two group cardinal invariants $\kappa(G)$ and $\chi(G)$).

We note here that μ is a state on $C_0(\Omega)$ if and only if $\mu \in M(\Omega)$ is a probability measure on Ω. Also, recall that for $\mu, \nu \in M(\Omega)$, $\|\mu - \nu\| = \|\mu\| + \|\nu\|$ if and only if μ and ν are mutually singular, *cf.* Ghahramani–McClure [13, Lemma 1]. So, by Corollary 2.7, $\mathrm{dec}(M(\Omega)^*)$ is just the greatest cardinality of a family of mutually singular probability measures in $M(\Omega)$.

For any topological space X, let $w(X)$ denote the weight of X, *i.e.,* the least cardinality of an open basis for X. Here, we introduce the following cardinal for any locally compact Hausdorff space Ω. Let

$$w_c(\Omega) = \sup\{w(U) : U \text{ is a relatively compact open subset of } \Omega\}.$$

Then $w_c(\Omega) \leq w(\Omega)$, $w_c(\Omega) = w(\Omega)$ if Ω is compact, and $w_c(\Omega) = \aleph_0$ if Ω is discrete and infinite. It is readily seen that $w_c(\Omega)$ is an infinite cardinal if Ω is non-discrete. In general, $w_c(\Omega) \neq w(\Omega)$, *e.g.,* for any infinite discrete group G, $w_c(G) = \aleph_0$ but $w(G) = |G|$, see Lemma 5.4.

Also, one can prove that $w(\Omega) = \kappa(\Omega) \cdot w_c(\Omega)$, where $\kappa(\Omega)$ denotes the least cardinality of a compact covering of Ω. In fact, let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open covering of Ω such that each $\overline{U_\alpha}$ is compact and $|\Lambda| = \kappa(\Omega)$. Then $w(\Omega) \leq \sum_{\alpha\in\Lambda} w(U_\alpha) \leq |\Lambda| \cdot w_c(\Omega) = \kappa(\Omega) \cdot w_c(\Omega)$, *i.e.,* $w(\Omega) \leq \kappa(\Omega) \cdot w_c(\Omega)$. Conversely, let $\mathcal{O}$ be an open basis for Ω such that $|\mathcal{O}| = w(\Omega)$. Let $\mathcal{K} = \{\overline{U} : U \in \mathcal{O} \text{ and } \overline{U} \text{ is compact}\}$. Then $\mathcal{K}$ is a compact covering of Ω and thus $\kappa(\Omega) \leq |\mathcal{K}| \leq |\mathcal{O}| = w(\Omega)$, *i.e.,* $\kappa(\Omega) \leq w(\Omega)$. Therefore, $\kappa(\Omega) \cdot w_c(\Omega) \leq w(\Omega)$ since we obviously have $w_c(\Omega) \leq w(\Omega)$.

Let U be an infinite relatively compact open subset of Ω. Then there exists a relatively compact open set $V \subseteq \Omega$ such that $\overline{U} \subseteq V$. Since $C_0(U)$ can be canonically embedded into $C(\overline{U})$, we have that $\mathrm{dense}(C_0(U)) \leq \mathrm{dense}(C(\overline{U}))$. Note that

$$\mathrm{dense}(C(\overline{U})) = w(\overline{U}),$$

cf. Lacey [28, Theorem 13.1]. Hence, we have $\mathrm{dense}(C_0(U)) \leq w(\overline{U}) \leq w(V) \leq w_c(\Omega)$, *i.e.,*

(†) $\mathrm{dense}(C_0(U)) \leq w_c(\Omega)$ for all infinite relatively compact open subsets U of Ω.

This fact will be used in the sequel.

Note that $|\Omega| \leq \kappa(\Omega) \cdot 2^{w_c(\Omega)}$. We shall show that the decomposability number $\mathrm{dec}(M(\Omega)^*)$ is just between these two cardinals.

Lemma 5.1 *Let Ω be an infinite locally compact Hausdorff space. Then*

$$|\Omega| \leq \mathrm{dec}(M(\Omega)^*) \leq \mathrm{dense}(M(\Omega)) \leq \kappa(\Omega) \cdot 2^{w_c(\Omega)}.$$

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Proof Obviously, $\{\delta_x : x \in \Omega\}$ is a family of mutually singular probability measures in $M(\Omega)$, where δ_x denotes the point mass at x. So, we always have $|\Omega| \leq \operatorname{dec}(M(\Omega)^*) \leq \operatorname{dense}(M(\Omega))$, *cf.* Corollary 2.7.

If Ω is discrete, then, by Remark 2.3, $\operatorname{dec}(M(\Omega)^*) = \operatorname{dec}(l_\infty(\Omega)) = \operatorname{dense}(l_1(\Omega)) = |\Omega| = \kappa(\Omega) \leq \kappa(\Omega) \cdot 2^{w_c(\Omega)}$. In the following, we assume that Ω is non-discrete. Then $w_c(\Omega)$ is an infinite cardinal and hence we may assume that $\operatorname{dense}(M(\Omega)) > \aleph_0$.

According to Comfort [5, Lemma 17], there exists a subset $\{\mu_i\}_{i\in I}$ of $M(\Omega)$ such that $|I| = \operatorname{dense}(M(\Omega))$, $\|\mu_i\| < 1$, and $\|\mu_i - \mu_j\| \geq \frac{1}{2}$ for all $i, j \in I$ with $i \neq j$. Let $i \in I$. Choose $\nu_i \in M(\Omega)$ such that ν_i is supported on some compact subset K_i of Ω and $\|\mu_i - \nu_i\| \leq \frac{1}{8}$. Thus, if $i, j \in I$ and $i \neq j$, then $\|\nu_i - \nu_j\| \geq \frac{1}{4}$. In particular, $\nu_i \neq \nu_j$ whenever $i \neq j$.

Let $\mathfrak{U}$ be an open cover of Ω such that $\mathfrak{U}$ is closed under finite union, $|\mathfrak{U}| = \kappa(\Omega)$, and $\overline{U}$ is compact for all $U \in \mathfrak{U}$. For each $U \in \mathfrak{U}$, let $I_U = \{i \in I : K_i \subseteq U\}$. Then $I = \bigcup_{U\in\mathfrak{U}} I_U$ since each K_i is compact and $\mathfrak{U}$ is closed under finite union. So, we only have to show that $|I_U| \leq 2^{w_c(\Omega)}$ for all $U \in \mathfrak{U}$.

Fix $U \in \mathfrak{U}$. Since $\operatorname{dense}(C_0(U)) \leq w_c(\Omega)$ by (†), there exists a subset $\{f_\alpha\}_{\alpha\in\Lambda}$ of $C_0(U)$ such that $|\Lambda| \leq w_c(\Omega)$ and $\{f_\alpha\}_{\alpha\in\Lambda}$ is norm dense in $C_0(U)$. For $i \in I_U$ and $\alpha \in \Lambda$, let $\Gamma(i)(\alpha) = \langle \nu_i, f_\alpha \rangle$, where ν_i is considered as a measure in $M(U)$ since ν_i is supported on $K_i \subseteq U$. Then $\Gamma(i)$ is a complex-valued function defined on Λ for each $i \in I_U$. Clearly, $\Gamma(i) \neq \Gamma(j)$ if $i, j \in I_U$ and $i \neq j$. Hence, Γ is a one-to-one map from I_U to the set of complex-valued functions on Λ. We derive that $|I_U| \leq (2^{\aleph_0})^{|\Lambda|} \leq 2^{w_c(\Omega)}$ since $w_c(\Omega) \geq \aleph_0$ and $w_c(\Omega) \geq |\Lambda|$. ■

Let $\mathcal{A}$ be a *commutative* C^*-algebra with Gelfand space Ω. Then Ω is a locally compact Hausdorff space and $\mathcal{A} \cong C_0(\Omega)$. So, Lemma 5.1 can be reformulated as follows.

Corollary 5.2 *Let $\mathcal{A}$ be an infinite dimensional commutative C^*-algebra with Gelfand space Ω. Then*

$$|\Omega| \leq \operatorname{dec}(\mathcal{A}^{**}) \leq \operatorname{dense}(\mathcal{A}^*) \leq \kappa(\Omega) \cdot 2^{w_c(\Omega)}.$$

In view of the equality $w(\Omega) = \kappa(\Omega) \cdot w_c(\Omega)$ (*cf.* the fourth paragraph of this section), it can be seen that if Ω is non-compact, *i.e.*, $\mathcal{A}$ is non-unital, then the estimate on $\operatorname{dec}(\mathcal{A}^{**})$ obtained in Corollary 5.2 is, in general, sharper than the inequality $\operatorname{dec}(\mathcal{A}^{**}) \leq 2^{\operatorname{dense}(\mathcal{A})} (= 2^{w(\Omega)})$ and may even be sharper than $\operatorname{dec}(\mathcal{A}^{**}) \leq |\mathcal{A}^*|$, *cf.* Corollary 2.7, see also Theorem 5.5(ii).

Combining Corollary 5.2 and Theorem 2.2, we have

Corollary 5.3 *Let $\mathcal{A}$ be an infinite dimensional commutative C^*-algebra with Gelfand space Ω. Then $\mathcal{A}^*$ has the Mazur property, resp., property (X), of level $\kappa(\Omega)\cdot 2^{w_c(\Omega)}$.*

To express the decomposability number $\operatorname{dec}(M(G)^*)$ explicitly in terms of $\kappa(G)$ and $\chi(G)$ for all locally compact groups G, we need the following lemma, which is essentially contained in [6, §3].

Lemma 5.4 *Let G be an infinite locally compact group. Then*

(i) $w_c(G) = \chi(G) \cdot \aleph_0$.
(ii) $w(G) = \max\{\kappa(G), \chi(G)\}$.
(iii) $|G| = \kappa(G) \cdot 2^{\chi(G)}$.

Proof Suppose that G is discrete. Then $\chi(G) = 1$, $\kappa(G) = w(G) = |G|$, and $w_c(G) = \aleph_0$. So, (i)–(iii) hold.

In the following, we assume that G is non-discrete. Let U be any non-empty relatively compact open subset of G and G_0 a σ-compact open subgroup of G containing U. By [6, Theorem 3.9(i)], $w(G_0) = \chi(G_0)$. So, we have

$$\chi(G) = \chi(G_0) \leq w(U) \leq w(G_0) = \chi(G_0) = \chi(G),$$

i.e., $w(U) = \chi(G)$. By the definition of $w_c(G)$, we have $w_c(G) = \chi(G) = \chi(G) \cdot \aleph_0$ since $\chi(G) \geq \aleph_0$. Therefore, (i) is true. Conclusions (ii) and (iii) are Theorem 3.5(iii) and Theorem 3.12(iii) in [6], respectively. ■

Now we are ready to present the main result of this section.

Theorem 5.5

(i) *Let $\mathcal{A}$ be an infinite dimensional commutative C^*-algebra with Gelfand space Ω satisfying $|\Omega| = \kappa(\Omega) \cdot 2^{w_c(\Omega)}$. Then*

$$\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{dense}(\mathcal{A}^*) = \kappa(\Omega) \cdot 2^{w_c(\Omega)}.$$

In addition, if $\mathcal{A}$ is unital, then $\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{dense}(\mathcal{A}^*) = |\mathcal{A}^*| = 2^{w(\Omega)}$.

(ii) *Let G be an infinite locally compact group. Then*

$$\operatorname{dec}(M(G)^*) = \operatorname{dense}(M(G)) = \kappa(G) \cdot 2^{\chi(G)} = |G|.$$

Proof (i) The first half of the conclusion follows directly from Corollary 5.2. Furthermore, suppose that $\mathcal{A}$ is unital. Then Ω is compact and hence $w_c(\Omega) = w(\Omega) \geq \aleph_0$. We recall the Kruse–Schmidt–Stone theorem, which states that for any Banach space X, we have $|X| = [\operatorname{dense}(X)]^{\aleph_0}$. Hence, $|\mathcal{A}^*| = [\operatorname{dense}(\mathcal{A}^*)]^{\aleph_0} = 2^{w(\Omega)\cdot\aleph_0} = 2^{w(\Omega)} = \operatorname{dec}(\mathcal{A}^{**})$. Therefore, we have $\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{dense}(\mathcal{A}^*) = 2^{w(\Omega)} = |\mathcal{A}^*|$.

(ii) Let G be an infinite locally compact group. If G is discrete, then $M(G) = l_1(G)$ and, by Remark 2.3 and Lemma 5.4(iii), $\operatorname{dec}(M(G)^*) = \operatorname{dec}(l_\infty(G)) = \operatorname{dense}(M(G))$ $= |G| = \kappa(G) \cdot 2^{\chi(G)}$. The equality for the case that G is non-discrete can be derived from the first conclusion of this theorem and Lemma 5.4(i) and (iii). ■

We now obtain an analogue of Corollary 3.3.

Corollary 5.6 *Let G be a locally compact group. Then*

(i) *$M(G)$ has the Mazur property (resp., property (X)) of level $|G| \cdot \aleph_0$.*

(ii) *$M(G)$ has the classical Mazur property, resp., property (X), if and only if $|G|$ is a non-measurable cardinal, which holds if and only if both $\kappa(G)$ and $2^{\chi(G)}$ are non-measurable cardinals.*

Proof The statement is clear if G is finite.

Assume that G is infinite. Combining Theorem 5.5(ii) with Theorem 2.1 and Theorem 2.2 and following the proof of Corollary 4.3, one can see that (i) and the first half of (ii) hold true. We note here that, if κ_1 and κ_2 are cardinal numbers with $\kappa_1 \leq \kappa_2$ and if κ_2 is non-measurable, then κ_1 is also non-measurable, see Remark 2.4(i). By Lemma 5.4(iii), it is readily seen that $|G|$ is non-measurable if and only if both $\kappa(G)$ and $2^{\chi(G)}$ are non-measurable. ∎

Remark 5.7

(i) When G is discrete, Corollary 3.3(ii) and Corollary 5.6(ii) are included in Edgar [9, Theorem 5.10], *cf.* Remark 2.4(iv).

(ii) Corollary 5.6(ii) has been used by the second author in the proof of the following results: if G is non-compact and $|G|$ is non-measurable, then (a) the (left and right) topological centres of $M(G)^{**}$ are exactly $M(G)$; (b) every (left or right) $M(G)^{**}$-module homomorphism on $M(G)^*$ is automatically bounded and w^*-continuous, see [34, Theorem 3.4]. In fact, by Lemma 5.4(iii) and from the proof of [34, Theorem 3.4], it is easily seen that the above (a) and (b) are also true for all locally compact groups G satisfying $\kappa(G) \geq 2^{\chi(G)}$.

(iii) It will be interesting to explore the "dual" setting of Theorem 5.5(ii). A natural question is: when $\mathcal{B} = B(G)$ or $B_\rho(G)$ (where $B(G)$, resp., $B_\rho(G)$, denotes the (resp., reduced) Fourier–Stieltjes algebra of G), for which non-compact and non-abelian groups G do we have $\text{dec}(\mathcal{B}^*) = \chi(G) \cdot 2^{\kappa(G)}$? Furthermore, is there a C^*-algebra $\mathcal{A}$ naturally associated with G such that $\text{dec}(\mathcal{A}^{**}) = \chi(G) \cdot 2^{\kappa(G)}$ holds for all locally compact groups G? Here, we would like to point out that for Fell's group G (*cf.* [27, p. 328]), we have $B(G) = B_\rho(G)$, $\kappa(G) = \chi(G) = \aleph_0$, and $\text{dec}(B(G)^*) = \text{dense}(B(G)) = \aleph_0$. So, one may have $\text{dec}(B(G)^*) < \chi(G) \cdot 2^{\kappa(G)}$ even when G is second countable and amenable.

6 The Decomposability of $LUC(G)^{**}$ and $UC(\hat{G})^{**}$

Let G be a locally compact group and $LUC(G)$ the C^*-algebra of bounded left uniformly continuous functions on G, *i.e.*, all $f \in C(G)$ such that the map $x \mapsto {}_xf$ from G into $C(G)$ is continuous when $C(G)$ has the norm topology, where ${}_xf(t) = f(xt)$ $(t \in G)$. Let $UC(\hat{G})$ be the C^*-algebra generated by operators in $VN(G)$ with compact support as introduced by Granirer [17]. We recall here that the support of an element T of $VN(G)$ is defined to be the closed subset $\text{supp}\, T$ of G such that $x \in \text{supp}\, T$ if and only if for all $u \in A(G)$, $u \cdot T = 0$ implies $u(x) = 0$, where $u \cdot T \in VN(G)$ is defined by $\langle u \cdot T, v\rangle = \langle T, uv\rangle$ $(v \in A(G))$. It is known that when G is abelian with dual group Γ, $UC(\hat{G})$ is identified with $LUC(\Gamma)$. In this section, we will explore the decomposability of the von Neumann algebras $LUC(G)^{**}$ and $UC(\hat{G})^{**}$. It turns out that $LUC(G)^{**}$ and $UC(\hat{G})^{**}$ have plenty of projections even though the C^*-algebras $LUC(G)$ and $UC(\hat{G})$ may rarely have non-trivial projections.

Let $G^{\mathcal{LUC}}$ be the Gelfand space of the commutative C^*-algebra $LUC(G)$. Then $G^{\mathcal{LUC}}$ can be regarded as the $\mathcal{LUC}$-compactification of G and $C(G^{\mathcal{LUC}})$ is $*$-isomorphic to $LUC(G)$. See Berglund–Junghenn–Milnes [2] for more information on the $\mathcal{LUC}$-compactifications. If G is compact, then $G^{\mathcal{LUC}}$ is just G. When G is discrete, $LUC(G) = l_\infty(G)$ so that $G^{\mathcal{LUC}}$ is equal to the Stone–Čech compactification βG of G. For a general locally compact group G, Lau–Medghalchi–Pym [29] proved a "local structure theorem" for $G^{\mathcal{LUC}}$, which plays a key role in our determination of the decomposability number $\operatorname{dec}(LUC(G)^{**})$. We cite this local topological structure theorem here for the convenience of the reader.

Theorem 6.1 ([29, Theorem 2.10]) *Let G be an infinite locally compact group. Let V be any compact symmetric neighborhood of the identity e of G and let $\{z_\alpha\}_{\alpha\in A}$ be a maximal subset of G such that the family $\{Vz_\alpha\}_{\alpha\in A}$ of subsets of G is disjoint. Give A the discrete topology and let $q\colon V^3 \times A \to G$ be the natural map defined by $q(v,\alpha) = vz_\alpha$. Then q extends to a continuous map $q\colon V^3 \times \beta A \to G^{\mathcal{LUC}}$ for which*

(i) *$q(V^2 \times \beta A) = G^{\mathcal{LUC}}$;*
(ii) *if W is open in G, $\operatorname{cl}_G W \subseteq \operatorname{int} V$ and $v \in V^2$, then $q(vW \times \beta A)$ is open in $G^{\mathcal{LUC}}$ and q is a homeomorphism from $vW \times \beta A$ onto $q(vW \times \beta A)$;*
(iii) *if $M \subseteq A$, then $q(vW \times \operatorname{cl}_{\beta A} M) \cap G = q(vW \times M) = \bigcup_{\alpha\in M} vWz_\alpha$.*

From now on, we shall fix V and $\{z_\alpha\}_{\alpha\in A}$ such that the conclusions of Theorem 6.1 hold. We note that $|A|$ is less than or equal to the cellularity of G. So, by [6, Theorem 3.12(v)], $|A| \le \kappa(G)\cdot\aleph_0$. In fact, we have

Lemma 6.2 *If G is non-compact, then $|A| = \kappa(G)$.*

Proof The maximality of $\{z_\alpha\}_{\alpha\in A}$ ensures that $\{V^2 z_\alpha\}_{\alpha\in A}$ covers G, see the proof of [29, Lemma 2.1]. So, $\kappa(G) \le |A|$. Combining this with the inequality $|A| \le \kappa(G)\cdot\aleph_0$, we have $|A| = \kappa(G)$ if G is non-compact. ∎

It is known that for any infinite compact Hausdorff space X, $\operatorname{dense}(C(X)) = w(X)$, *cf.* Lacey [28, Theorem 13.1]. Let S be an infinite set. Then $C(\beta S) \cong l_\infty(S)$ and hence $w(\beta S) = \operatorname{dense}(C(\beta S)) = \operatorname{dense}(l_\infty(S)) = 2^{|S|}$, *i.e.*, $w(\beta S) = 2^{|S|}$. This fact will be used in the proof of the following result on the $\mathcal{LUC}$-compactification $G^{\mathcal{LUC}}$ of G.

Lemma 6.3 *Let G be an infinite locally compact group. Then*

$$w(G^{\mathcal{LUC}}) = \chi(G)\cdot 2^{\kappa(G)} \quad \text{and} \quad |G^{\mathcal{LUC}}| = 2^{w(G^{\mathcal{LUC}})} = 2^{\chi(G)}\cdot 2^{2^{\kappa(G)}}.$$

Proof By Lemma 5.4(ii) and (iii), the assertion holds when G is compact. If G is discrete, then $G^{\mathcal{LUC}} = \beta G$, $w(G^{\mathcal{LUC}}) = 2^{|G|} = 2^{\kappa(G)}$, and $|G^{\mathcal{LUC}}| = 2^{2^{|G|}} = 2^{2^{\kappa(G)}}$, and hence the conclusion is also true.

In the following, we assume that G is non-compact and non-discrete. Let W be an open neighborhood of the identity of G as in Theorem 6.1(ii). Let $x \in G^{\mathcal{LUC}}$. By Theorem 6.1(i), $x = q(v,\alpha)$ for some $v \in V^2$ and $\alpha \in \beta A$. Let $W(x) = q(vW \times \beta A)$.

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Then $W(x)$ is an open neighborhood of x in $G^{\mathcal{LUC}}$ and $W(x)$ is homeomorphic to $vW \times \beta A$.

Now vW is a non-empty relatively compact open subset of G. The proof of Lemma 5.4(i) shows that $w(vW) = \chi(G)$. Note that $w(\beta A) = 2^{|A|}$ and $w(X \times Y) = w(X) \cdot w(Y)$ for all topological spaces X and Y. It follows that

$$\begin{aligned} w(W(x)) &= w(vW \times \beta A) = w(vW) \cdot w(\beta A) = \chi(G) \cdot 2^{|A|} \\ &= \chi(G) \cdot 2^{\kappa(G)} \qquad \text{(Lemma 6.2)}, \end{aligned}$$

i.e., $w(W(x)) = \chi(G) \cdot 2^{\kappa(G)}$ for all $x \in G^{\mathcal{LUC}}$. Thus, $w(G^{\mathcal{LUC}}) \geq \chi(G) \cdot 2^{\kappa(G)}$. On the other hand, since $G^{\mathcal{LUC}}$ is compact, $G^{\mathcal{LUC}}$ is a finite union of open sets of the form $W(x)$. So, $w(G^{\mathcal{LUC}}) \leq \aleph_0 \cdot \chi(G) \cdot 2^{\kappa(G)} = \chi(G) \cdot 2^{\kappa(G)}$. Therefore, $w(G^{\mathcal{LUC}}) = \chi(G) \cdot 2^{\kappa(G)}$.

Note that $G^{\mathcal{LUC}}$ is not extremally disconnected in general, *cf.* Theorem 6.1(ii). So, we need the following proof to conclude that $|G^{\mathcal{LUC}}| = 2^{w(G^{\mathcal{LUC}})}$.

Clearly, $|G^{\mathcal{LUC}}| \leq 2^{w(G^{\mathcal{LUC}})} = 2^{\chi(G)} \cdot 2^{2^{\kappa(G)}}$. Let G_0 be a σ-compact open subgroup of G containing W. Then G_0 can be covered by countably many left translates of W. By Lemma 5.4(iii), $2^{\chi(G)} = 2^{\chi(G_0)} = |G_0| \leq \aleph_0 \cdot |W| = |W|$. Also, $|\beta A| = 2^{2^{|A|}} = 2^{2^{\kappa(G)}}$ (Lemma 6.2). Thus,

$$|G^{\mathcal{LUC}}| \geq |W(x)| = |vW \times \beta A| = |vW| \cdot |\beta A| = |W| \cdot |\beta A| \geq 2^{\chi(G)} \cdot 2^{2^{\kappa(G)}},$$

i.e., $|G^{\mathcal{LUC}}| \geq 2^{\chi(G)} \cdot 2^{2^{\kappa(G)}}$. Therefore, $|G^{\mathcal{LUC}}| = 2^{w(G^{\mathcal{LUC}})} = 2^{\chi(G)} \cdot 2^{2^{\kappa(G)}}$. ∎

Recall that $G^{\mathcal{LUC}}$ is the Gelfand space of $LUC(G)$ and note that $G^{\mathcal{LUC}}$ is compact. Taking $\mathcal{A} = LUC(G)$ in Theorem 5.5(i) and applying Lemma 6.3, we derive the main result of this section.

Theorem 6.4 *Let G be an infinite locally compact group. Then*

$$\operatorname{dec}(LUC(G)^{**}) = \operatorname{dense}(LUC(G)^{*}) = |\,LUC(G)^{*}| = 2^{\chi(G)} \cdot 2^{2^{\kappa(G)}}.$$

Now, let us turn to the decomposability number $\operatorname{dec}(UC(\hat{G})^{**})$. Suppose that G is non-discrete. Let $TIM(\hat{G})$ be the set of topologically invariant means on $VN(G)$, *i.e.*, all $m \in VN(G)^{*}$ such that $\|m\| = m(1) = 1$ and $m(u \cdot T) = u(e)m(T)$ for all $u \in A(G)$ and $T \in VN(G)$. By Hu [21, Proposition 6.1], there exists a subset E of $TIM(\hat{G})$ such that $|E| = 2^{2^{\chi(G)}}$ and $\|m_1 - m_2\| = 2$ for all $m_1, m_2 \in E$ with $m_1 \neq m_2$. Note that for each $m \in TIM(\hat{G})$, $m|_{UC(\hat{G})}$ is a state on $UC(\hat{G})$ and $\|(m_1 - m_2)|_{UC(\hat{G})}\| = \|m_1 - m_2\|$ for all $m_1, m_2 \in TIM(\hat{G})$. Therefore,

$$\operatorname{dec}(UC(\hat{G})^{**}) = \operatorname{ost}(UC(\hat{G})^{*}) \geq |E| = 2^{2^{\chi(G)}},$$

i.e., $\operatorname{dec}(UC(\hat{G})^{**}) \geq 2^{2^{\chi(G)}}$. An upper bound in terms of $\kappa(G)$ and $\chi(G)$ for the cardinal $\operatorname{dec}(UC(\hat{G})^{**})$ is obtained in the following

Lemma 6.5 *Let G be an infinite locally compact group. Then*

$$2^{2^{\chi(G)}} \leq \mathrm{dec}(UC(\hat{G})^{**}) \leq \mathrm{dense}(UC(\hat{G})^{*}) \leq |\,UC(\hat{G})^{*}| \leq 2^{\kappa(G)} \cdot 2^{2^{\chi(G)}}.$$

Proof Suppose that G is discrete. Then $l_1(G)$ is $\|\cdot\|_{UC(\hat{G})}$-dense in $UC(\hat{G})$ and hence $\mathrm{dense}(UC(\hat{G})) = |G| = \kappa(G)$. So, by Corollary 2.7, $\mathrm{dec}(UC(\hat{G})^{**}) \leq \mathrm{dense}(UC(\hat{G})^{*}) \leq |\,UC(\hat{G})^{*}| \leq 2^{\mathrm{dense}(UC(\hat{G}))} = 2^{\kappa(G)}$. Therefore, the conclusion holds when G is discrete.

In the following, we assume that G is non-discrete. Let $\mathcal{T}$ be the set of all operators in $VN(G)$ with compact support. Then $\mathcal{T}$ is norm dense in $UC(\hat{G})$. Owing to Corollary 2.7 and the inequality $\mathrm{dec}(UC(\hat{G})^{**}) \geq 2^{2^{\chi(G)}}$ preceding this lemma, we only need to prove that $(\mathrm{dense}(UC(\hat{G})) \leq)\ |\mathcal{T}| \leq \kappa(G) \cdot 2^{\chi(G)}$. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of G such that each $\overline{U_i}$ is compact, $\mathcal{U}$ is closed under finite union and $|I| = \kappa(G)$. For each $i \in I$, let $\mathcal{T}_i = \{T \in \mathcal{T} : \mathrm{supp}\, T \subseteq U_i\}$ and let G_i be a σ-compact open subgroup of G containing $\overline{U_i}$. Then $\mathcal{T}_i \subseteq VN_{G_i}(G) \cong VN(G_i)$ (*cf.* Eymard [11, Proposition 3.21]), where $VN_{G_i}(G) = \{T \in VN(G) : \mathrm{supp}\, T \subseteq G_i\}$. Note that $\mathrm{dense}(A(G_i)) \leq \mathrm{dense}(L_2(G_i)) = \chi(G_i) = \chi(G)$, *cf.* Lemma 7.6. Thus, $|VN(G_i)| \leq 2^{\mathrm{dense}(A(G_i))} \leq 2^{\chi(G)}$. It follows that $|\mathcal{T}_i| \leq |VN_{G_i}(G)| = |VN(G_i)| \leq 2^{\chi(G)}$ for all $i \in I$. Therefore, $|\mathcal{T}| = |\bigcup_{i\in I} \mathcal{T}_i| \leq |I| \cdot 2^{\chi(G)} = \kappa(G) \cdot 2^{\chi(G)}$. ∎

Theorem 6.6 *Let G be an infinite locally compact group. Then*

$$\mathrm{dec}(UC(\hat{G})^{**}) = \mathrm{dense}(UC(\hat{G})^{*}) = |\,UC(\hat{G})^{*}| = 2^{\kappa(G)} \cdot 2^{2^{\chi(G)}}$$

if either of the following two conditions is satisfied:

(i) $2^{\kappa(G)} \leq 2^{2^{\chi(G)}}$;
(ii) $2^{\kappa(G)} > 2^{2^{\chi(G)}}$ *and G contains an abelian subgroup H such that* $\kappa(H) = \kappa(G)$.

Proof The conclusion for case (i) follows immediately from Lemma 6.5.

In order to prove the assertion for case (ii), suppose that $2^{\kappa(G)} > 2^{2^{\chi(G)}}$ and let H be an abelian subgroup of G such that $\kappa(H) = \kappa(G)$. By Lemma 5.4(iii), we may assume that H is a closed abelian subgroup of G. By Kaniuth–Lau [26, Lemma 3.2(ii)], $UC(\hat{H})$ is identified with a C^*-subalgebra of $UC(\hat{G})$ and hence $UC(\hat{H})^{**}$ is a von Neumann subalgebra of $UC(\hat{G})^{**}$. Let Γ be the dual group of H. Then $\chi(\Gamma) = \kappa(H)$, $\kappa(\Gamma) = \chi(H)$, and $UC(\hat{H}) \cong LUC(\Gamma)$. So, we have

$$\begin{aligned}
\mathrm{dec}(UC(\hat{G})^{**}) &\geq \mathrm{dec}(UC(\hat{H})^{**})\\
&= \mathrm{dec}(LUC(\Gamma)^{**}) && \text{(by Theorem 6.4)}\\
&= 2^{\chi(\Gamma)} \cdot 2^{2^{\kappa(\Gamma)}}\\
&= 2^{\kappa(H)} \cdot 2^{2^{\chi(H)}} && (\text{since } 2^{\kappa(H)} = 2^{\kappa(G)} > 2^{2^{\chi(G)}} \geq 2^{2^{\chi(H)}})\\
&= 2^{\kappa(H)} = 2^{\kappa(G)},
\end{aligned}$$

i.e., $\mathrm{dec}(UC(\hat{G})^{**}) \geq 2^{\kappa(G)}$. Owing to Lemma 6.5, we have $\mathrm{dec}(UC(\hat{G})^{**}) = 2^{\kappa(G)} \cdot 2^{2^{\chi(G)}}$. Therefore, the conclusion holds for case (ii). ∎

Remark 6.7 (i) We are not able to prove the full dual version of Theorem 6.4, *i.e.,* $\operatorname{dec}(UC(\hat{G})^{**}) = 2^{\kappa(G)} \cdot 2^{2^{\chi(G)}}$ for all infinite locally compact groups G, since we do not know whether $\operatorname{dec}(UC(\hat{G})^{**}) \geq 2^{\kappa(G)}$ holds in general. It is interesting to note that for Fell's group G, $\operatorname{dec}(UC(\hat{G})^{**}) = 2^{\kappa(G)} \cdot 2^{2^{\chi(G)}}$ holds even though $\operatorname{dec}(B(G)^{*}) < \chi(G) \cdot 2^{\kappa(G)}$, *cf.* Remark 5.7(iii).

(ii) For a C^*-algebra $\mathcal{A}$, if $\operatorname{dec}(\mathcal{A}^{**}) = |\mathcal{A}^*|$, then we say that $\operatorname{dec}(\mathcal{A}^{**})$ is *maximal*, *cf.* Corollary 2.7. Theorem 6.4 shows that for all infinite locally compact groups G, $\operatorname{dec}(LUC(G)^{**})$ is maximal. And so is $\operatorname{dec}(UC(\hat{G})^{**})$ for a large class of groups G by Theorem 6.6. We shall see in the next section that $\operatorname{dec}(l_\infty(I)^{**})$ and $\operatorname{dec}(\mathcal{B}(H)^{**})$ are also maximal for all infinite sets I and infinite dimensional Hilbert spaces H. It is not clear to us whether this is true for all infinite dimensional unital commutative C^*-algebras $\mathcal{A}$, *cf.* Theorem 5.5(i).

(iii) A related question is: for which infinite dimensional C^*-algebras $\mathcal{A}$ do we have $\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{dense}(\mathcal{A}^*)$, *cf.* Corollary 2.7? Obviously, this equality holds if $\mathcal{A}^*$ is separable (*cf.* Proposition 2.5), which is equivalent to $\mathcal{A}$ being weakly compactly generated, *cf.* Kaniuth–Lau–Schlichting [27, Theorem 2.3]. We note that the equality $\operatorname{dec}(\mathcal{M}) = \operatorname{dense}(\mathcal{M}_*)$ does not hold for a general von Neumann algebra $\mathcal{M}$, *e.g.,* for any infinite compact group G, $\operatorname{dec}(L_\infty(G)) = \aleph_0$ but $L_1(G)$ can be non-separable. Besides the C^*-algebras $LUC(G)$, $UC(\hat{G})$, $l_\infty(I)$ and $\mathcal{B}(H)$ as mentioned in (ii), $\operatorname{dec}(\mathcal{A}^{**}) = \operatorname{dense}(\mathcal{A}^*)$ if $\mathcal{A} = C_0(G)$ and G is any infinite locally compact group, *cf.* Theorem 5.5(ii). The equality also holds for $\mathcal{A} = c_0(I)$ with $|I| \geq \aleph_0$ and $\mathcal{A} = \mathcal{K}(H)$, the C^*-algebra of compact operators on an infinite dimensional Hilbert space H, *cf.* Remark 2.3 and Theorem 7.3, respectively.

7 The Decomposability of $\mathcal{M}^{**}$

In this section, we shall investigate the decomposability of the second dual of the von Neumann algebras $L_\infty(G)$ and $VN(G)$. We begin with the following simple consequence of Theorem 5.5(i) on the cardinal $\operatorname{dec}(l_\infty(I)^{**})$ for any infinite set I. Note that $\operatorname{dec}(l_\infty(I)) = |I|$, *cf.* Remark 2.3.

Lemma 7.1 *Let I be an infinite set. Then* $\operatorname{dec}(l_\infty(I)^{**}) = 2^{2^{|I|}}$.

Proof Let $\mathcal{A} = l_\infty(I)$. Then $\mathcal{A}$ is an infinite dimensional unital commutative C^*-algebra with Gelfand space βI (the Stone–Čech compactification of I). Note that $|\beta I| = 2^{w(\beta I)} = 2^{2^{|I|}}$. By Theorem 5.5(i), we have $\operatorname{dec}(l_\infty(I)^{**}) = 2^{w(\beta I)} = 2^{2^{|I|}}$. ■

Lemma 7.2 *Let $\mathcal{M}$ be an infinite dimensional von Neumann algebra. Then*

$$\operatorname{dec}(\mathcal{M}^{**}) \geq 2^{2^{\operatorname{dec}(\mathcal{M})}}.$$

Proof Let $\{P_i\}_{i\in I}$ be a family of pairwise orthogonal non-zero projections in $\mathcal{M}$ such that $|I| = \operatorname{dec}(\mathcal{M})$, *cf.* Theorem 2.6(i). Since $\mathcal{M}$ is infinite dimensional, by Proposition 2.5, $|I| = \operatorname{dec}(\mathcal{M})$ is infinite. Let $\mathcal{R}$ be the von Neumann algebra generated by $\{P_i\}_{i\in I}$. Then $\mathcal{R}$ is $*$-isomorphic to $l_\infty(I)$, *i.e.,* $\mathcal{R} = \{\sum_{i\in I} f_i P_i : (f_i) \in$

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$l_\infty(I)\}$ and $\|\sum_{i\in I} f_i P_i\|_{\mathcal{R}} = \|(f_i)\|_{l_\infty(I)}$, *cf.* the proof of [33, Theorem 3.12]. So, by Lemma 7.1, $\operatorname{dec}(\mathcal{R}^{**}) = \operatorname{dec}(l_\infty(I)^{**}) = 2^{2^{|I|}} = 2^{2^{\operatorname{dec}(\mathcal{M})}}$, *i.e.*, $\operatorname{dec}(\mathcal{R}^{**}) = 2^{2^{\operatorname{dec}(\mathcal{M})}}$. Also, we note that $\mathcal{R}^{**}$ is a $*$-subalgebra of the von Neumann algebra $\mathcal{M}^{**}$, since $\mathcal{R}$ is a von Neumann subalgebra of $\mathcal{M}$. Therefore, $\operatorname{dec}(\mathcal{M}^{**}) \geq \operatorname{dec}(\mathcal{R}^{**}) = 2^{2^{\operatorname{dec}(\mathcal{M})}}$. ■

For any Hilbert space H, let $\mathcal{B}(H)$ be the von Neumann algebra of all bounded linear operators on H. We can express both $\operatorname{dec}(\mathcal{B}(H))$ and $\operatorname{dec}(\mathcal{B}(H)^{**})$ precisely in terms of the Hilbert space dimension $\dim(H)$ of H as follows.

Theorem 7.3 *Let H be an infinite dimensional Hilbert space. Then*

$$\operatorname{dec}(\mathcal{B}(H)) = \dim(H) \quad \textit{and} \quad \operatorname{dec}(\mathcal{B}(H)^{**}) = 2^{2^{\dim(H)}}.$$

Proof The equality $\operatorname{dec}(\mathcal{B}(H)) = \dim(H)$ follows from Neufang [33, Corollary 3.9].

By Lemma 7.2 and the above equality, we have $\operatorname{dec}(\mathcal{B}(H)^{**}) \geq 2^{2^{\dim(H)}}$. On the other hand, let $\{e_i\}_{i\in I}$ be an orthonormal basis for H. Then $|I| = \dim(H)$. Note that each T in $\mathcal{B}(H)$ is uniquely determined by the infinite matrix $(\langle Te_i \mid e_j\rangle)$, where $\langle\cdot\mid\cdot\rangle$ denotes the inner product on H. It follows that $|\mathcal{B}(H)| \leq (2^{\aleph_0})^{|I\times I|} = 2^{|I|}$. Hence, by Corollary 2.7, we have

$$\operatorname{dec}(\mathcal{B}(H)^{**}) \leq 2^{\operatorname{dense}(\mathcal{B}(H))} \leq 2^{|\mathcal{B}(H)|} \leq 2^{2^{|I|}},$$

i.e., $\operatorname{dec}(\mathcal{B}(H)^{**}) \leq 2^{2^{\dim(H)}}$. Therefore, $\operatorname{dec}(\mathcal{B}(H)^{**}) = 2^{2^{\dim(H)}}$. ■

Let $\mathcal{M}$ be an infinite dimensional von Neumann algebra acting on a Hilbert space H. Then $\operatorname{dense}(\mathcal{M}) \leq \operatorname{dense}(\mathcal{B}(H)) \leq 2^{\dim(H)}$, see the proof of Theorem 7.3. By Corollary 2.7, we have $\operatorname{dec}(\mathcal{M}^{**}) \leq 2^{2^{\dim(H)}}$. Combining this inequality with Lemma 7.2, we have

Corollary 7.4 *Let $\mathcal{M}$ be an infinite dimensional von Neumann algebra acting on a Hilbert space H. Then*

$$2^{2^{\operatorname{dec}(\mathcal{M})}} \leq \operatorname{dec}(\mathcal{M}^{**}) \leq 2^{2^{\dim(H)}}.$$

In particular, $\operatorname{dec}(\mathcal{M}^{**}) = 2^{2^{\operatorname{dec}(\mathcal{M})}}$ *if* $\operatorname{dec}(\mathcal{M}) = \dim(H)$.

In order to evaluate $\operatorname{dec}(L_\infty(G)^{**})$ and $\operatorname{dec}(VN(G)^{**})$, we need Lemma 7.6 below. We are unable to find any reference to it except for Hewitt–Ross [19, Theorem 28.2], where Lemma 7.6 is proved to be true for all infinite compact groups G. For the sake of completeness, we include the proof here. First, we have

Lemma 7.5 *Let G be an infinite locally compact group and $1 \leq p < \infty$. Then* $\operatorname{dense}(L_p(G)) \leq \max\{\kappa(G), \chi(G)\}$.

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Proof Let $\{U_i\}_{i\in I}$ be an open covering of G which is closed under finite union such that $|I| = \kappa(G)$ and U_i is relatively compact for all $i \in I$. For each $i \in I$, let

$$C_i(G) = \{f \in C_{00}(G) : \operatorname{supp} f \subseteq \overline{U_i}\}.$$

Fix $i \in I$. Note that $C_i(G) \subseteq C(\overline{U_i})$ and the density character of $C(\overline{U_i})$, where $C(\overline{U_i})$ is equipped with the supremum norm, is $w(\overline{U_i})$ (*cf.* [28, Theorem 13.1]), which is less than or equal to $w(G) = \max\{\kappa(G), \chi(G)\}$ (Lemma 5.4(ii)). Also, if $A \subseteq C_i(G)$ and A is dense in $C_i(G)$ with respect to the supremum norm, then A is also $\|\cdot\|_p$-dense in $C_i(G)$. Consequently, $\operatorname{dense}((C_i(G), \|\cdot\|_p)) \leq \max\{\kappa(G), \chi(G)\}$ for all $i \in I$. Since $C_{00}(G) = \bigcup_{i\in I} C_i(G)$ with $|I| = \kappa(G)$ and $C_{00}(G)$ is $\|\cdot\|_p$-dense in $L_p(G)$, it follows that $\operatorname{dense}(L_p(G)) \leq \max\{\kappa(G), \chi(G)\}$. ■

Using the decomposability numbers of the two von Neumann algebras $L_\infty(G)$ and $VN(G)$, we are able to provide a shorter proof of an even more general version of Hewitt–Ross [19, Theorem 28.2] as follows.

Lemma 7.6 *Let G be an infinite locally compact group. Then*

$$\dim(L_2(G)) = \operatorname{dense}(L_2(G)) = \max\{\kappa(G), \chi(G)\}.$$

Proof Following Theorem 7.3, Corollary 3.2 and Corollary 4.2, we have

$$\begin{aligned} \dim(L_2(G)) &= \operatorname{dec}(\mathcal{B}(L_2(G))) \\ &\geq \max\{\operatorname{dec}(L_\infty(G)), \operatorname{dec}(VN(G))\} \\ &= \max\{\kappa(G)\cdot\aleph_0, \chi(G)\cdot\aleph_0\} \;=\; \max\{\kappa(G), \chi(G)\}. \end{aligned}$$

On the other hand, for any infinite dimensional Hilbert space H, $\dim(H) = \operatorname{dense}(H)$. Hence, by Lemma 7.5,

$$\dim(L_2(G)) = \operatorname{dense}(L_2(G)) \leq \max\{\kappa(G), \chi(G)\}.$$

Therefore, $\dim(L_2(G)) = \operatorname{dense}(L_2(G)) = \max\{\kappa(G), \chi(G)\}$. ■

Combining Lemma 7.2 with Theorem 7.3 and Lemma 7.6, we have

Corollary 7.7 *Let G be an infinite locally compact group. Let* $\mathcal{M} = L_\infty(G)$ *or* $VN(G)$. *Then* $2^{2^{\operatorname{dec}(\mathcal{M})}} \leq \operatorname{dec}(\mathcal{M}^{**}) \leq 2^{2^{\max\{\kappa(G),\chi(G)\}}}$.

Now, for the von Neumann algebra $\mathcal{M} = L_\infty(G)$, resp., $VN(G)$, satisfying $\operatorname{dec}(\mathcal{M}) = \dim(L_2(G))$, we are ready to show that $\operatorname{dec}(\mathcal{M}^{**})$ can be expressed precisely by the *single* group cardinal invariant $\kappa(G)$, resp., $\chi(G)$, as follows.

Theorem 7.8 *Let G be an infinite locally compact group. Then*

(i) $\operatorname{dec}(L_\infty(G)^{**}) = 2^{2^{\operatorname{dec}(L_\infty(G))}} = 2^{2^{\kappa(G)\cdot\aleph_0}}$ *if* $\kappa(G)\cdot\aleph_0 \geq \chi(G)$, *for example, if G is metrizable.*

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(ii) $\mathrm{dec}(VN(G)^{**}) = 2^{2^{\mathrm{dec}(VN(G))}} = 2^{2^{\chi(G)\cdot\aleph_0}}$ *if* $\chi(G)\cdot\aleph_0 \geq \kappa(G)$, *for example, if* G *is* σ*-compact.*

(iii) $\max\{\mathrm{dec}(L_\infty(G)^{**}), \mathrm{dec}(VN(G)^{**})\} = 2^{2^{\dim(L_2(G))}} = 2^{2^{\max\{\kappa(G),\chi(G)\}}}$.

Proof This follows immediately from Corollary 7.7, Corollary 3.2 and Corollary 4.2. ∎

Let $\mathcal{M} = L_\infty(G)$ or $VN(G)$, $\widehat{\mathcal{M}}$ denote the dual Kac algebra of $\mathcal{M}$, and $H = L_2(G)$. It is well known that $L_\infty(G)$ and $VN(G)$ are dual to each other as Kac algebras and $\mathcal{B}(H)$ is generated by $\mathcal{M}$ and $\widehat{\mathcal{M}}$, *cf.* Enock–Schwartz [10]. Thus it is interesting to reformulate Lemma 7.6 and Theorem 7.8 as follows:

(i) $\mathrm{dec}(\mathcal{M}^{**}) = 2^{2^{\mathrm{dec}(\mathcal{M})}}$ if $\mathrm{dec}(\mathcal{M}) \geq \mathrm{dec}(\widehat{\mathcal{M}})$;

(ii) $\mathrm{dec}(\mathcal{B}(H)) = \max\{\mathrm{dec}(\mathcal{M}), \mathrm{dec}(\widehat{\mathcal{M}})\}$;

(iii) $\mathrm{dec}(\mathcal{B}(H)^{**}) = \max\{\mathrm{dec}(\mathcal{M}^{**}), \mathrm{dec}(\widehat{\mathcal{M}}^{**})\} = 2^{2^{\max\{\mathrm{dec}(\mathcal{M}),\mathrm{dec}(\widehat{\mathcal{M}})\}}}$.

When $\mathrm{dec}(\mathcal{M}) < \mathrm{dec}(\widehat{\mathcal{M}})$, it is natural to ask whether we still have $\mathrm{dec}(\mathcal{M}^{**}) = 2^{2^{\mathrm{dec}(\mathcal{M})}}$ and how far away from $2^{2^{\mathrm{dec}(\mathcal{M})}}$ the cardinal $\mathrm{dec}(\mathcal{M}^{**})$ can be if the equality does not hold. One particular reason for this consideration lies in the fact that a certain cardinal level of the Mazur property, resp., property (X), of the space $\mathcal{M}^*$ can be traced back to that of $\mathcal{M}_*$ if $\mathrm{dec}(\mathcal{M}^{**}) = 2^{2^{\mathrm{dec}(\mathcal{M})}}$, *cf.* Theorem 2.2. We will further investigate the problem mentioned above on another occasion. At this point, we only note that $\mathrm{dec}(L_\infty(G)^{**}) \geq \mathrm{dec}(C_0(G)^{**}) \geq 2^{\chi(G)}$ for all infinite locally compact groups G, *cf.* Theorem 5.5, and $\mathrm{dec}(VN(G)^{**}) \geq \mathrm{dec}(UC(\hat{G})^{**}) \geq 2^{\kappa(G)}$ for groups G as in Theorem 6.6. This fact shows that the cardinal number $\mathrm{dec}(\mathcal{M}^{**})$ can be arbitrarily far away from $2^{2^{\mathrm{dec}(\mathcal{M})}}$. It also reveals that $\mathrm{dec}(\widehat{\mathcal{M}})$ will join with $\mathrm{dec}(\mathcal{M})$ to determine $\mathrm{dec}(\mathcal{M}^{**})$ when $\mathrm{dec}(\mathcal{M}) < \mathrm{dec}(\widehat{\mathcal{M}})$. We do have the evidence that $\mathrm{dec}(\mathcal{M}^{**}) = 2^{\mathrm{dec}(\widehat{\mathcal{M}})^{\mathrm{dec}(\mathcal{M})}}$ when $\mathcal{M} = L_\infty(G)$ or $VN(G)$ for a large class of locally compact groups G.

Remark 7.9 Let α be an arbitrary cardinal. Then there exists a compact abelian group G such that $\chi(G) = \alpha$. Note that $\mathrm{dec}(L_\infty(G)) = \aleph_0$ and $\mathrm{dec}(L_\infty(G)^{**}) \geq 2^{\chi(G)} = 2^\alpha$, *cf.* Corollary 3.2 and Theorem 5.5. Hence, by Theorem 2.1, $L_1(G)$ has the classical Mazur property and property (X). However, if a measurable cardinal λ exists, then $L_1(G)^{**}$ will fail to have the above two properties whenever $2^\alpha \geq \lambda$, *cf.* Remark 2.4 and Theorem 2.1.

8 An Application to the Topological Centre Problem

Let G be a locally compact group. It is known that the topological centre $Z_t(L_1(G)^{**})$ of the Banach algebra $L_1(G)^{**}$ is precisely $L_1(G)$, see Lau–Losert [30]. It is also known that for a large class of locally compact groups G, including amenable discrete groups, the motion group, the "$ax + b$"-group and the Heisenberg group, the topological centre $Z_t(A(G)^{**})$ of the Banach algebra $A(G)^{**}$ is $A(G)$, *cf.* Lau–Losert [31]. In this section, we present an application of the Mazur property of higher level to the topological centre problem for $A(G)^{**}$. It turns out that if G has a large compact covering number, then the topological centre problem for the algebra $A(G)^{**}$ can be reduced to the one for the algebras $A(H)^{**}$ of some open subgroups H of G with compact covering number dominated by $\operatorname{dec}(VN(G))$. We note here that the spaces $A(H)^{**}$ may behave better than $A(G)^{**}$ in the sense that the equality $\operatorname{dec}(VN(H)^{**}) = 2^{2^{\operatorname{dec}(VN(H))}}$ holds, which ensures that a certain cardinal level of the Mazur property of $A(H)^{**}$ can be traced back to that of $A(H)$, *cf.* the last paragraph of Section 7.

As is well known, the multiplication on $A(G)$ gives rise to two Banach algebra multiplications on $A(G)^{**}$ (known as the first and the second Arens multiplications on $A(G)^{**}$) which extend the multiplication on $A(G)$. In the following, the space $A(G)^{**}$ will always be considered to be equipped with the first Arens multiplication. That is, for $\varphi, \psi \in A(G)$, $T \in A(G)^*$, and $m, n \in A(G)^{**}$, the products $T \cdot \varphi$, $n \cdot T \in A(G)^*$ and $m \cdot n \in A(G)^{**}$ are defined by

$$\langle T \cdot \varphi, \psi \rangle = \langle T, \varphi\psi \rangle, \quad \langle n \cdot T, \varphi \rangle = \langle n, T \cdot \varphi \rangle, \quad \text{and} \quad \langle m \cdot n, T \rangle = \langle m, n \cdot T \rangle.$$

Since $A(G)$ is commutative, $Z_t(A(G)^{**})$ is just the algebraic centre of $A(G)^{**}$, *i.e.*, $Z_t(A(G)^{**}) = \{m \in A(G)^{**} : m \cdot n = n \cdot m \text{ for all } n \in A(G)^{**}\}$.

Let H be an open subgroup of G. Let $r\colon A(G) \to A(H)$ be the restriction map and $t\colon A(H) \to A(G)$ the trivial extension map, *i.e.*, $(tu)(x) = 0$ for $x \in G - H$. Obviously, $r \circ t = 1$. Even though $t \circ r \neq id$ when $H \neq G$, we still have

$$(*) \qquad t(r(\varphi)u) = \varphi\, t(u) \ \text{ for all } \varphi \in A(G) \text{ and } u \in A(H).$$

This fact will be used later. The adjoint r^* of r is a $*$-isomorphism of the von Neumann algebra $VN(H)$ $(= A(H)^*)$ onto the von Neumann subalgebra $VN_H(G)$ of $VN(G)$, *cf.* Eymard [11, Proposition 3.21]. Also, it is easy to see that r^{**} is an algebraic homomorphism of $A(G)^{**}$ onto $A(H)^{**}$ and t^{**} is an algebraic isomorphism of $A(H)^{**}$ into $A(G)^{**}$. To get the main result of this section, we need the following elementary results on the images of the topological centres under the maps t^{**} and r^{**}, respectively. We include a proof here for completeness.

Lemma 8.1

(i) $t^{**}[Z_t(A(H)^{**})] \subseteq Z_t(A(G)^{**})$.
(ii) $r^{**}[Z_t(A(G)^{**})] = Z_t(A(H)^{**})$.

Proof (i) Owing to $(*)$, it is readily seen that for all $n \in A(G)^{**}$, $T \in A(G)^*$ and $p \in A(H)^{**}$, we have $t^*(n \cdot T) = r^{**}(n) \cdot t^*(T)$ and $r^*(p \cdot t^*(T)) = t^{**}(p) \cdot T$.

Let $p \in Z_t(A(H)^{**})$. Then, for all $n \in A(G)^{**}$ and $T \in A(G)^*$, we have

$$\begin{aligned}\langle t^{**}(p) \cdot n, T \rangle &= \langle t^{**}(p), n \cdot T \rangle = \langle p, t^*(n \cdot T) \rangle \\ &= \langle p, r^{**}(n) \cdot t^*(T) \rangle = \langle r^{**}(n), p \cdot t^*(T) \rangle \ \ (\text{since } p \in Z_t(A(H)^{**})) \\ &= \langle n, r^*(p \cdot t^*(T)) \rangle = \langle n, t^{**}(p) \cdot T \rangle = \langle n \cdot t^{**}(p), T \rangle.\end{aligned}$$

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So, $t^{**}(p) \cdot n = n \cdot t^{**}(p)$ for all $n \in A(G)^{**}$, *i.e.*, $t^{**}(p) \in Z_t(A(G)^{**})$. Therefore, (i) holds.

(ii) The inclusion $Z_t(A(H)^{**}) \subseteq r^{**}[Z_t(A(G)^{**})]$ follows from (i) and the identity $r^{**} \circ t^{**} = \mathrm{id}$. Conversely, let $m \in Z_t(A(G)^{**})$. To get $r^{**}(m) \in Z_t(A(H)^{**})$, let $p \in A(H)^{**}$. Since $A(H)^{**} = r^{**}(A(G)^{**})$, there exists an $n \in A(G)^{**}$ such that $p = r^{**}(n)$. Note that r^{**} is an algebraic homomorphism. Thus,

$$\begin{aligned} r^{**}(m) \cdot p &= r^{**}(m \cdot n) \quad (\text{since } m \in Z_t(A(G)^{**}) \\ &= r^{**}(n \cdot m) = r^{**}(n) \cdot r^{**}(m) = p \cdot r^{**}(m), \end{aligned}$$

i.e., $r^{**}(m) \cdot p = p \cdot r^{**}(m)$ for all $p \in A(H)^{**}$. So, $r^{**}(m) \in Z_t(A(H)^{**})$. Therefore, $r^{**}[Z_t(A(G)^{**})] \subseteq Z_t(A(H)^{**})$ and hence $r^{**}[Z_t(A(G)^{**})] = Z_t(A(H)^{**})$. ∎

For convenience, for any open subgroup H of G and $m \in A(G)^{**}$, let m_H denote $r^{**}(m) \in A(H)^{**}$, where $r\colon A(G) \to A(H)$ is the restriction map.

Proposition 8.2 *Let G be an infinite locally compact group and $\mathcal{H}$ the collection of open subgroups H of G satisfying $\kappa(H) \leq \mathrm{dec}(VN(G))$. Let $m \in A(G)^{**}$. If $m_H \in A(H)$ for all $H \in \mathcal{H}$, then $m \in A(G)$.*

Proof Since $\mathrm{dec}(VN(G)) = \chi(G) \cdot \aleph_0$ (*cf.* Corollary 4.2), we may assume that $\kappa(G) > \chi(G) \cdot \aleph_0$. Suppose that $m \in A(G)^{**}$ and $m_H \in A(H)$ for all $H \in \mathcal{H}$.

Note that $A(G)$ has the Mazur property of level $\chi(G) \cdot \aleph_0$ (*cf.* Corollary 4.3(i)). To get $m \in A(G)$, we let $(T_i)_{i\in I}$ be a net in the unit ball of $VN(G)$ $(= A(G)^*)$ such that $|I| \leq \chi(G) \cdot \aleph_0$ and $T_i \to 0$ in the $\sigma(VN(G), A(G))$-topology. Now we only have to prove that $\langle m, T_i \rangle \to 0$.

Since $|I| \leq \chi(G) \cdot \aleph_0$, by [23, Corollary 4.4; Theorem 5.3], there exists an $H \in \mathcal{H}$ such that $T_i \in VN_H(G) = r^*(VN(H))$ for *all* $i \in I$. Thus, for each $i \in I$, there is an $S_i \in VN(H)$ such that $T_i = r^*(S_i)$. Since $r\colon A(G) \to A(H)$ is a surjection, we have that $S_i \to 0$ in the $\sigma(VN(H), A(H))$-topology. Note that $m_H \in A(H)$. It follows that

$$\langle m, T_i \rangle = \langle m, r^*(S_i) \rangle = \langle r^{**}(m), S_i \rangle = \langle m_H, S_i \rangle \to 0,$$

i.e., $\langle m, T_i \rangle \to 0$. ∎

Let $\mathcal{H}_0$ be the family of all σ-compact open subgroups of G. In [31, Lemma 6.3], Lau–Losert proved that if $Z_t(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$, then for any $m \in Z_t(A(G)^{**})$, there exists an $H \in \mathcal{H}_0$ such that $m - m_H$ vanishes on $B_\rho(G) \cdot VN(G)$. Note that if G is amenable, then $B_\rho(G) \cdot VN(G) = VN(G)$. Therefore, if G is amenable and $Z_t(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}_0$, then $Z_t(A(G)^{**}) = A(G)$. The following result shows that without the assumption of amenability, the above assertion remains true if $\mathcal{H}_0$ is replaced by the family $\mathcal{H}$.

Theorem 8.3 *Let G and $\mathcal{H}$ be the same as in Proposition 8.2. If $Z_t(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}$, then $Z_t(A(G)^{**}) = A(G)$.*

In particular, if G is a metrizable (non-amenable) locally compact group such that $Z_t(A(H)^{**}) = A(H)$ *for all σ-compact open subgroups H of G, then* $Z_t(A(G)^{**}) = A(G)$.

Proof Assume that $Z_t(A(H)^{**}) = A(H)$ for all $H \in \mathcal{H}$. To get the nontrivial inclusion $Z_t(A(G)^{**}) \subseteq A(G)$, let $m \in Z_t(A(G)^{**})$. Then, by Lemma 8.1(ii), $m_H \in Z_t(A(H)^{**})$ for all $H \in \mathcal{H}$. By the assumption, $m_H \in A(H)$ for all $H \in \mathcal{H}$. It follows from Proposition 8.2 that $m \in A(G)$. ■

Remark 8.4 (i) The algebra $A(H)^{**}$ can be identified with a closed subalgebra of $A(G)^{**}$ via the algebraic isomorphism t^{**}. By Lemma 8.1(i), it is readily seen that if $Z_t(A(G)^{**}) = A(G)$, then $Z_t(A(H)^{**}) = A(H)$ for all open subgroups H of G. So, the converse of Theorem 8.3 is obviously true.

(ii) Let $\mathcal{H}$ be the same family of open subgroups of G as in Proposition 8.2. Then we have $\bigcup_{H\in\mathcal{H}} A(H) = A(G)$ and, by Lemma 8.1(i), $\bigcup_{H\in\mathcal{H}} Z_t(A(H)^{**}) \subseteq Z_t(A(G)^{**})$ under the algebraic isomorphisms t^{**}. Therefore, Theorem 8.3 would be trivial if one could prove that $\bigcup_{H\in\mathcal{H}} Z_t(A(H)^{**}) = Z_t(A(G)^{**})$. However, it is unknown whether this equality holds in general.

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Department of Mathematics and Statistics
University of Windsor
Windsor, ON
N9B 3P4
e-mail: zhiguohu@uwindsor.ca

School of Mathematics and Statistics
Carleton University
Ottawa, ON
K1S 5B6
e-mail: mneufang@math.carleton.ca