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# SATURATED FORMATIONS AND SYLOW NORMALISERS

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Sufficient conditions are provided in order that some classes of finite soluble groups, defined by properties of the Sylow normalisers, are saturated formations.

## 0. INTRODUCTION

Let $h : \mathbb{P} \to \{\text{group classes}\}$ be a function which associates with each $p$ a (possibly empty) class of groups $h(p)$, contained in some universe $\mathcal{B}$ of finite groups. The operation N on the functions $\mathbb{P} \to \{\text{group classes}\}$ is defined as follows:

$$\mathrm{N}h := \big(G \in \mathcal{B} \mid \mathrm{N}_G(G_p) \in h(p), \quad \text{for every prime } p \ \text{ which divides } |G|\big)$$

where $G_p \in \mathrm{Syl}_p(G)$.

In this paper we provide sufficient conditions in order that $\mathrm{N}h$ is a saturated formation, we suggest a way to construct a wide class of such saturated formations and a local definition for them.

It is easy to observe that, if $h(p)$ is Q-closed (closed under epimorphic images), for every prime $p$, then $\mathrm{N}h$ is Q-closed, whereas nothing analogous occurs for other frequently used closure operations. For instance, if $h(p)$ is the class $\mathcal{T}$ of finite groups with ordered Sylow tower (for every prime $p$), $\mathcal{T}$ is an S-closed saturated Fitting formation, instead $\mathrm{N}\mathcal{T}=\mathrm{N}h$ is neither a formation nor a Fitting class ($\mathrm{N}\mathcal{T}$ is closed under none of the operations $\mathrm{R}_o$, $\mathrm{N}_0$, $\mathrm{S}_n$). The classes $\mathrm{N}h$ can have some interesting properties, though they do not inherit the closure properties of the classes $h(p)$, an example is provided by the class $\mathrm{N}\mathcal{U}$, where $\mathcal{U}$ is the formation of supersoluble groups. The class $\mathrm{N}\mathcal{U}$ has been studied in 1988 by Fedri and Serena [**5**] and in 1991 by the same authors with Bryce [**3**].

The operation N was introduced in 1970 by Glaubermann [**6**], who proved that, if $x$ is the formation function defined for each $p \in \mathbb{P}$ by $x(p) = \mathcal{S}_p$ =class of $p$-groups, then $\mathrm{N}x = \bigcup_{p\in\mathbb{P}} \mathcal{S}_p$.

In 1986 Bianchi, Gillio and Hauck [**2**], generalising the cited result of Glaubermann, proved that $\mathrm{N}\mathcal{N} = \mathcal{N}$, where $\mathcal{N}$ is the formation of nilpotent groups. A further generalisation was obtained in 1999 by Ballester-Bolinches and Shemetkov [**1**], who proved

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that, if $y$ is the formation function which associates with each prime $p$ the formation $y(p) = \mathcal{F}_{p'}\mathcal{S}_p$ of $p$-nilpotent groups, then $\mathrm{N}y = \mathcal{N}$

As a concrete example we shall consider a saturated formation function introduced by Huppert in [7], and as corollaries we get the results of Fedri and Serena on the class $\mathrm{N}\mathcal{U}$ and, for soluble groups only, the result of Ballester-Bolinches and Shemetkov [1]. An interesting corollary is also the following one:

*The class of soluble groups, in which normalisers of Sylow $p$-subgroups are $p$-supersoluble, is a saturated formation* (see Theorem B).

Most of our notation is standard and can be found in [4]. "Group" will stand for "finite soluble group".

## 1. The main result

In this section we prove the following theorem:

**Theorem 1.1.** *Let $g$ be a formation function and $\pi$ a set of primes such that $\mathcal{S}_{p'} \cap \mathcal{S}_{\pi'} \subseteq g(p) \subseteq \mathcal{S}_{\pi'}$, for all primes $p$. Then $\mathrm{N}g$ is a formation. Moreover, if $g$ is a saturated formation function, then $\mathrm{N}g$ is a saturated formation.*

Proof: We have already observed that $\mathrm{N}g$ is a homomorph. Now we prove that $\mathrm{N}g$ is $\mathrm{R}_0$-closed. On the assumption that it is not let $G$ be a group in $\mathrm{R}_0(\mathrm{N}g) \setminus \mathrm{N}g$. Since $G$ is in $\mathrm{R}_0(\mathrm{N}g)$, it has two normal subgroups $K_1$ and $K_2$ such that $G/K_i \in \mathrm{N}g$ $(i = 1, 2)$, and $K_1 \cap K_2 = 1$.

On the other hand, since $G \notin \mathrm{N}g$, there exists a prime $p$ dividing $|G|$ such that $N_G(G_p) \notin g(p)$. If $p$ divides $G/K_i$ (for $i = 1, 2$), we obtain

$$N_G(G_p)/K_i \cap N_G(G_p) \in g(p),$$

from which the contradiction $N_G(G_p) \in g(p)$ follows, because $g(p)$ is $\mathrm{R}_0$-closed. Suppose, without loss of generality, $G/K_1 \in \mathcal{S}_{p'}$. Since $K_1 \cap K_2 = 1$, we get $N_G(G_p)/K_2 \cap N_G(G_p) \in g(p)$. If $G/K_1 \in \mathcal{S}_{\pi'}$, we have $N_G(G_p)/K_1 \cap N_G(G_p) \in \mathcal{S}_{p'} \cap \mathcal{S}_{\pi'} \subseteq g(p)$ and so once more we have the contradiction $N_G(G_p) \in g(p)$. Let $q \in \pi$ be a prime dividing $|G/K_1|$. Since $G/K_1 \in \mathrm{N}g$, we have $N_G(G_q)/K_1 \cap N_G(G_q) \in g(q) \subseteq \mathcal{S}_{\pi'}$; it follows, as $q \notin \pi'$, that $q$ does not divide $|N_G(G_q)/K_1 \cap N_G(G_q)|$ and therefore $G_q \subseteq K_1$, obtaining the contradiction that $q$ does not divide $|G/K_1|$. Hence the assumption that $\mathrm{N}g$ is not $\mathrm{R}_0$-closed is false.

Now we are going to prove that, if $g(p)$ is saturated for every prime $p$, then $\mathrm{N}g$ is saturated. If not then let $G$ be a group of minimal order in $E_\Phi(\mathrm{N}g) \setminus \mathrm{N}g$. A routine argument shows that $G$ is monolithic and, if $N$ is the socle of $G$, we have, for some prime $q$, $N \subseteq \Phi(G) \subset O_q(G) = \mathrm{Fit}(G)$ and $G/N \in \mathrm{N}g$. On the other hand, since $G \notin \mathrm{N}g$, there exists a prime $p$ dividing $|G|$ such that $N_G(G_p) \notin g(p)$. Now, since

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$G/N \in \mathrm{N}g$ and $q$ divides $|G/N|$ (because $N \subseteq \Phi(G)$), we have $N_G(G_q)/N \in g(q) \subseteq \mathcal{S}_{\pi'}$ and $N_G(G_p)/N \cap N_G(G_p) \in g(p) \subseteq \mathcal{S}_{\pi'}$; consequentely $p, q \notin \pi$ and therefore $N_G(G_p)/G_p \in \mathcal{S}_{p'} \cap \mathcal{S}_{\pi'} \subseteq g(p)$.

If $q \neq p$, we get $N \cap G_p = 1$ and obtain the contradiction $N_G(G_p) \in g(p)$; therefore $q = p$. Then, since $O_{p'}(G) = 1$ and $N \subseteq \Phi(G)$, we have $O_{p'}\bigl(N_G(G_p)/N\bigr) = 1$, which implies that $O_{p',p}\bigl(N_G(G_p)/N\bigr) = G_p/N$; consequently $G_p/N$ is the intersection of the centralisers of all chief $p$-factors of $N_G(G_p)/N$.

Now a well-known theorem of Lubeseder (see [**4**, IV, (4.6) Theorem]) shows that $g(p)$ is locally defined by some formation function $\mathcal{F}$, because, by hypothesis, $g(p)$ is a saturated formation. Then, setting $N_G(G_p)/N = \Gamma$, we have $\mathrm{Aut}_\Gamma(H/K) \in \mathcal{F}(p)$ for all chief $p$-factors $H/K$ of $\Gamma$, that is $\Gamma/C_\Gamma(H/K) \in \mathcal{F}(p)$; it follows $\Gamma/O_{p',p}(\Gamma) = \Gamma/\Gamma_p \cong N_G(G_p)/G_p \in \mathcal{F}(p)$, from which we obtain that $\mathrm{Aut}_{N_G(G_p)}(H/K) \in \mathcal{F}(p)$, for all chief $p$-factors $H/K$ of $N_G(G_p)$ such that $N \subseteq K$. On the other hand, if $H/K$ is a chief $p$-factor of $N_G(G_p)$ such that $H \subseteq N$, we have $G_p \subseteq C_{N_G(G_p)}(H/K)$ and so $\mathrm{Aut}_{N_G(G_p)}(H/K) \in \mathcal{F}(p)$, because it is a homomorphic image of $N_G(G_p)/G_p \in \mathcal{F}(p)$; thus $\mathrm{Aut}_{N_G(G_p)}(H/K) \in \mathcal{F}(p)$, for all chief $p$-factors $H/K$ of $N_G(G_p)$. Now, since $N_G(G_p)/N \in g(p) = LF(\mathcal{F})$, it is obvious that, for all primes $q$ different from $p$ the group of automorphisms induced by $N_G(G_p)$ on a chief $q$-factor belongs to $\mathcal{F}(q)$. Thus we obtain the contradiction $N_G(G_p) \in LF(\mathcal{F}) = g(p)$. ∎

## 2. The saturated formations $\mathrm{N}\widetilde{f_\pi}$

A function $f : \mathbb{P} \to \{\text{group classes}\}$ is called a [saturated] formation function if $f(p)$ is a [saturated] formation for all $p \in \mathbb{P}$. If $f$ is a formation function, it is well known that the class

$$LF(f) = \bigl(G \in \mathcal{S} \mid \mathrm{Aut}_G(H/K) \in f(p),\ \text{for all chief } p\text{-factors } H/K \text{ of } G\bigr).$$

is a saturated formation, called locally defined by $f$. Moreover every saturated formation can be locally defined.

If $\pi$ is a subset of the set $\mathbb{P}$ of all primes, the class $\mathcal{S}_\pi$ is the class of $\pi$-groups. If $G$ is a group, $G_\pi$ denotes a Hall $\pi$-subgroup of $G$; in particular $G_p$ is a Sylow $p$-subgroup of $G$ (if $p$ does not divide $|G|$, $G_p = 1$).

DEFINITION 2.1: Let $f$ be a function $\mathbb{P} \to \{\text{group classes}\}$. The function $f^*$ is defined as follows

$$f^*(p) = \bigl(G \in \mathcal{S} \mid \mathrm{N}_G(G_p)/G_p \in f(p)\bigr) \quad (p \in \mathbb{P}).$$

**LEMMA 2.2.** *If $f : \mathbb{P} \to \{\text{group classes}\}$ is a formation function, then $f^*$ is a formation function.*

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PROOF: It is easy to observe that $f^*(p)$ is a homomorph. It remains to prove that $f^*(p)$ is $\mathrm{R}_0$-closed. Let $G$ be a group with two normal subgroups $N_1$ and $N_2$ such that $G/N_i \in f^*(p)$ $(i = 1,2)$ and $N_1 \cap N_2 = 1$. If $G_p \in \mathrm{Syl}_p(G)$ we have

$$\frac{N_G(G_p)}{G_p(N_i \cap N_G(G_p))} \cong \frac{N_{G/N_i}((G/N_i)_p)}{(G/N_i)_p} \in f(p) \quad (i = 1,2).$$

It follows, since $f(p)$ is $\mathrm{R}_0$-closed, that $N_G(G_p)/G_p \in f(p)$, observing that

$$G_p(N_1 \cap N_G(G_p)) \cap G_p(N_2 \cap N_G(G_p)) = G_p.$$

Thus $G \in f^*(p)$. □

DEFINITIONS 2.3: Let $f$ be a formation function and $\pi$ be a (possibly empty) set of primes. The formation function $f/\pi$ is defined as follows:

$$(f/\pi)(p) := \begin{cases} \emptyset & \text{if } p \in \pi \\ f(p) & \text{if } p \notin \pi \end{cases} \qquad (p \in \mathbb{P}).$$

For every prime $p$ the formation function $(f/\pi, p)$ is defined as follows:

$$(f/\pi, p)(q) := \begin{cases} \emptyset & \text{if } q \in \pi \\ f(p) & \text{if } q = p \notin \pi \\ \mathcal{S} & \text{if } q \neq p \text{ and } q \notin \pi \end{cases} \qquad (q \in \mathbb{P}).$$

The saturated formation $LF((f/\pi, p))$ locally defined by the formation function $(f/\pi, p)$ will be denoted by $\widetilde{f}_\pi(p)$. Thus two formation functions are defined:

$$\widetilde{f}_\pi : p \in \mathbb{P} \to \widetilde{f}_\pi(p) \quad \text{and} \quad f^*_\pi : p \in \mathbb{P} \to (f/\pi)^*(p).$$

If $\pi = \emptyset$, we shall use the following notation:

$$(f,p) := (f/\emptyset, p) \text{ and } \widetilde{f} := \widetilde{f}_\emptyset.$$

The main result is the following theorem.

**THEOREM A.** *Let $f$ be a formation function and $\pi$ a set of primes. Then:*

(i) $\mathrm{N}\widetilde{f}_\pi$ *is a saturated formation;*
(ii) $\mathrm{N}\widetilde{f}_\pi$ *is locally defined by the formation function* $f^*_\pi$.

Before we proceed with the proof, let us state some easy consequences of the definitions.

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**PROPOSITION 2.4.** *Let $f$ be a formation function and $\pi$ a set of primes. Then, for every prime $p$, we have*

$$\mathcal{S}_{p'} \cap \mathcal{S}_{\pi'} \subseteq \widetilde{f}_\pi(p) \subseteq \mathcal{S}_{\pi'}$$

*and therefore* $\mathrm{N}\widetilde{f}_\pi \subseteq \mathcal{S}_{\pi'}$. *In particular:*

(i) $\widetilde{f}_\pi(p) = \mathcal{S}_{\pi'}$, *for every* $p \in \pi$;

(ii) *if* $\pi = \emptyset$, $\mathcal{S}_{p'} \subseteq \widetilde{f}(p)$ *for every prime* $p$.

PROOF: Let $G \in \widetilde{f}_\pi(p) = LF\big((f/\pi, p)\big)$ and let $q$ be a prime dividing $|G|$. If $H/K$ is a chief $q$-factor of $G$, we have $\mathrm{Aut}_G(H/K) \neq \emptyset$ and so $(f/\pi, p)(q) \neq \emptyset$; therefore, by definition, $q \notin \pi$. Hence $G$ is a $\pi'$-group. Now let $G \in \mathcal{S}_{p'} \cap \mathcal{S}_{\pi'}$. If $H/K$ is a chief $q$-factor of $G$, then $q \neq p$ and $q \notin \pi$; therefore, by definition, $(f/\pi, p)(q) = \mathcal{S}$ and so $\mathrm{Aut}_G(H/K) \in (f/\pi, p)(q)$. Hence $G \in \widetilde{f}_\pi(p)$. ∎

The inclusion $\mathrm{N}\widetilde{f}_\pi \subseteq \mathcal{S}_{\pi'}$ is an obvious consequence of the definitions and of the inclusion $\widetilde{f}_\pi(p) \subseteq \mathcal{S}_{\pi'}$.

**PROPOSITION 2.5.** *Let $f$ be a formation function and $\pi$ a set of primes. Then:*

$$f^*_\pi(p) = \begin{cases} \emptyset & \text{if } p \in \pi \\ f^*(p) & \text{if } p \notin \pi \end{cases} \qquad (p \in \mathbb{P}).$$

*In particular, if $\pi = \emptyset$, $f^*_\emptyset = f^*$.*

PROOF: It follows easily from definitions. ∎

**PROPOSITION 2.6.** *Let $f$ be a formation function and $\pi$ a set of primes. Then, for every prime $p$, we have:*

$$\widetilde{f}_\pi(p) = \mathcal{S}_{\pi'} \cap \widetilde{f}(p).$$

PROOF: It follows easily from the definitions. ∎

**PROPOSITION 2.7.** *Let $f$ be a formation function and $\pi$ a set of primes. Then $LF(f^*_\pi) = \mathcal{S}_{\pi'} \cap LF(f^*)$.*

PROOF: Let $G \in LF(f^*_\pi)$ and let $p$ be a prime dividing $|G|$. If $H/K$ is a chief $p$-factor of $G$ we have $\mathrm{Aut}_G(H/K) \in f^*_\pi(p)$, that is $G/C \in f^*_\pi$ where $C = C_G(H/K)$. Therefore, by definition of $f^*_\pi = (f/\pi)^*$, we obtain that

$$N_G(G_p)/G_p\big(C \cap N_G(G_p)\big) \in (f/\pi)(p);$$

it follows $(f/\pi)(p) \neq \emptyset$ and so $p \notin \pi$ and $\mathrm{Aut}_G(H/K) \in f^*(p)$. Thus

$$G \in \mathcal{S}_{\pi'} \cap LF(f^*).$$

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The inclusion $\mathcal{S}_{\pi'} \cap LF(f^*) \subseteq LF(f^*_\pi)$ is an easy consequence of the definitions. ∎

**PROPOSITION 2.8.** *Let $f$ be a formation function and $\pi$ a set of primes. Then:*

$$\mathrm{N}\widetilde{f}_\pi = \mathcal{S}_{\pi'} \cap \mathrm{N}\widetilde{f}.$$

PROOF: It follows easily from the definitions. ∎

**THEOREM 2.9.** (A. 1st part.) *Let $f$ be a formation function and $\pi$ a set of primes. Then $\mathrm{N}\widetilde{f}_\pi$ is a saturated formation.*

PROOF: It follows immediately from Theorem 1.1, by recalling Proposition 2.4. ∎

**THEOREM 2.10.** *Let $f$ be a formation function. Then $\mathrm{N}\widetilde{f}$ is a saturated formation and is locally defined by the formation function $f^*$.*

PROOF: First we prove the inclusion $\mathrm{N}\widetilde{f} \subseteq LF(f^*)$. Let $G$ be a group of minimal order in $\mathrm{N}\widetilde{f} \setminus LF(f^*)$. Since $LF(f^*)$ is a saturated formation, $G$ belongs to the Q-boundary of $LF(f^*)$ and so $G$ is primitive. Then we have $G = KN$, where $N = \mathrm{Soc}(G) = O_p(G)$ (for some prime $p$) and $K \in LF(f^*)$. Since $G \notin LF(f^*)$ we have

$$\mathrm{Aut}_G(N) \cong G/N \cong K \notin f^*(p),$$

that is $N_K(K_p)/K_p \notin f(p)$, where $K_p = K \cap G_p$. On the other hand, since $G \in \mathrm{N}\widetilde{f}$, we have

$$N_G(G_p) = N N_K(K_p) \in \widetilde{f}(p) = LF\big((f,p)\big),$$

that is $N_G(G_p)/C_{N_G(G_p)}(A/B) \in f(p)$, for all chief $p$-factors $A/B$ of $N_G(G_p)$; it follows that $N_G(G_p)/G_p \in f(p)$, because $G_p = O_{p',p}\big(N_G(G_p)\big)$; then, since $N_G(G_p)/G_p \cong N_K(K_p)/K_p$, we obtain the contradiction $N_K(K_p)/K_p \in f(p)$.

Now we are going to prove the inclusion $LF(f^*) \subseteq \mathrm{N}\widetilde{f}$.

Let $G$ be a group of minimal order in $LF(f^*) \setminus \mathrm{N}\widetilde{f}$. Since $\mathrm{N}\widetilde{f}$ is a saturated formation (Theorem 2.9), $G$ belongs to the Q-boundary of $\mathrm{N}\widetilde{f}$ and so is primitive. Then, as above, $G = KN$, where $N = \mathrm{Soc}(G) = O_p(G)$ is a minimal normal subgroup of $G$ and $K \in \mathrm{N}\widetilde{f}$. Since $G \notin \mathrm{N}\widetilde{f}$, there exists a prime $q$ dividing $|G|$ such that $N_G(G_q) \notin \widetilde{f}(q) = LF\big((f,q)\big)$. If $q \neq p$, we may suppose $G_q = K_q$ $(G_q \subseteq K)$ and therefore $N_G(G_q) = C_N(K_q)N_K(K_q) \notin \widetilde{f}(q)$, so there exists a chief $q$-factor $A/B$ of $N_K(K_q)$ such that $N_K(K_q)/C_{N_K(K_q)}(A/B) \notin f(q)$ and this contradicts $K \in \mathrm{N}\widetilde{f}$. Thus $q = p$. Therefore we have

$$N_G(G_p) = N N_K(K_p) \notin \widetilde{f}(p) = LF\big((f,p)\big)$$

(where $K_p = K \cap G_p$) and so $N_G(G_p)/O_{p,p'}\big(N_G(G_p)\big) \notin f(p)$. On the other hand, since $G/N \in \mathrm{N}\widetilde{f}$ and $O_{p',p}\big(N_G(G_p)\big) = G_p$, we get

$$N_G(G_p)/O_{p',p}\big(N_G(G_p)\big) = N_G(G_p)/G_p \in f^*(p),$$

from which we obtain the contradiction $N_G(G_p) \in \widetilde{f}(p)$. $\square$

**THEOREM 2.11.** (A. 2nd part.) *Let $f$ be a formation function and $\pi$ a set of primes. Then* $\mathrm{N}\,\widetilde{f}_\pi = LF(f_\pi^*)$.

PROOF: It follows immediately from Theorem 2.10, by recalling Propositions 2.7 and 2.8. $\square$

## 3. Some applications of Theorem A

In this section, by choosing particular formation functions and set of primes in Theorem A, we obtain some interesting examples of saturated formations, that can be defined by the operation N and for which therefore it can also be obtained a local definition.

Moreover we obtain, as corollaries, a result of Fedri and Serena on the class $\mathrm{N}\,\mathcal{U}$ [**5**] and, for soluble groups only, the result of Ballester-Bolinches and Shemetkov cited in the introduction [**1**].

A class of meaningful examples is obtained if we consider the following well-known saturated formation functions $\widetilde{a}_n$ ($n$ a positive integer), which have been introduced by B. Huppert in [**7**] (see [**8**, VI, 8]).

Let $n$ be a positive integer. The formation function $a_n$ is defined as follows:

$$a_n : p \in \mathbb{P} \to \mathcal{A}_{p^n-1}$$

where $\mathcal{A}_{p^n-1}$ is the formation of Abelian groups whose exponent divides $p^n - 1$.

Denote by $\pi(n)$ the set of primes which divide $n$. According to our definitions we have, for every prime $p$:

$$\big(a_n/\pi(n),p\big) : q \in \mathbb{P} \to \begin{cases} \emptyset & \text{if} \quad q \mid n \\ \mathcal{A}_{p^n-1} & \text{if} \quad q = p \nmid n \\ \mathcal{S} & \text{if} \quad q \neq p \quad \text{and} \quad q \nmid n \end{cases}$$

Let $\widetilde{a}_n := (\widetilde{a_n})_{\pi(n)}$, that is $\widetilde{a}_n(p) = LF\Big(\big(a_n/\pi(n),p\big)\Big)$, $(p \in \mathbb{P})$. It is well known that

$$\widetilde{a}_n(p) = \Big(G \in \mathcal{S} \mid \big(|G|,n\big) = 1 \text{ and } \overline{r}_p(G) \text{ either divides } n \text{ or is } 0\Big)$$

where $\overline{r}_p(G)$ is the arithmetic $p$-rank of $G$ (see [**8**, VI, 8.3 Hilfsatz]).

We deduce immediately the following result.

**THEOREM B.** *Let $n$ be a positive integer. Then the class $\mathrm{N}\widetilde{a}_n$ is a saturated formation and it is locally defined by the formation function*

$$a_n^* : p \in \mathbb{P} \to \begin{cases} \emptyset & \textit{if} \quad p \mid n \\ (G \in \mathcal{S} \mid N_G(G_p)/G_p \in \mathcal{A}_{p^n-1}) & \textit{if} \quad p \nmid n \end{cases}.$$

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*In particular (for $n = 1$): the class $\mathrm{N}\widetilde{a}_1$ of groups, in which normalisers of Sylow $p$-subgroups are $p$-supersoluble, is a saturated formation and is locally defined by the formation function*

$$a_1^* : p \in \mathbb{P} \to (G \in \mathcal{S} \mid N_G(G_p)/G_p \in \mathcal{A}_{p-1}).$$

**THEOREM.** (Fedri-Serena, [**5**, Proposition 1.2].) *Let $p$ and $q$ be primes. Then $\mathcal{S}_{\{p,q\}} \cap \mathrm{N}\mathcal{U}$ is a saturated formation and is locally defined by the formation function*

$$a_{\{p,q\}}^* : t \in \mathbb{P} \to \begin{cases} \emptyset & \text{if } t \neq p, q \\ (G \in \mathcal{S} \mid N_G(G_p)/G_p \in \mathcal{A}_{t-1}) & \text{if } t = p \text{ or } q. \end{cases}$$

PROOF: It is enough to observe that $\mathcal{S}_{\{p,q\}} \cap \mathrm{N}\mathcal{U} = \mathcal{S}_{\{p,q\}} \cap \mathrm{N}\widetilde{a}_1$, therefore the statement follows immediately from Theorem B. ∎

The following result is well known.

**PROPOSITION 3.3.** *The (saturated) formation locally defined by the formation function*

$$c : p \in \mathbb{P} \to \big(G \in \mathcal{S} \mid \mathrm{Syl}_p(G) = \mathrm{Carter}(G)\big)$$

*coincides with the formation $\mathcal{N}$ of nilpotent groups.*

**THEOREM.** (Ballester-Bolinches and Shemetkov, [**1**, Corollary 3].) *If the normalisers of Sylow $p$-subgroups of a group $G$ are $p$-nilpotent for every prime $p$, then $G$ is nilpotent.*

PROOF: Let $e$ be the formation function defined by $e(p) = 1$, for all primes $p$. In our notation we have

$$(e,p)(q) = \begin{cases} (1) & \text{if } q = p \\ \mathcal{S} & \text{if } q \neq p \end{cases} \qquad (q \in \mathbb{P})$$

and so $\widetilde{e}(p) = \mathcal{S}_{p'}\mathcal{S}_p$ is the formation of $p$-nilpotent groups. Then from Theorem 2.3 we deduce that $\mathrm{N}\widetilde{e}$ is locally defined by the formation function

$$e^* : p \in \mathbb{P} \to e^*(p) = \big(G \in \mathcal{S} \mid N_G(G_p)/G_p = 1\big) = \big(G \in \mathcal{S} \mid \mathrm{Syl}_p(G) = \mathrm{Carter}(G)\big).$$

It follows, by Proposition 3.3, that $\mathrm{N}\widetilde{e} = \mathcal{N}$ is the formation of nilpotent groups. ∎

## REFERENCES

[1] A. Ballester-Bolinches and L.A. Shemetkov, 'On Normalizers of Sylow Subgroups in Finite Groups', *Siberian Math. J.* **40** (1999), 1–2.

[2] M.G. Bianchi, A. Gillio Berta Mauri and P. Hauck, 'On finite groups with nilpotent Sylow normalizers', *Arch. Math.* **47** (1986), 193–197.

[3] R.A. Bryce, V. Fedri and L. Serena, 'Bounds on the Fitting length of finite soluble groups with supersoluble Sylow normalizers', *Bull. Austral. Math. Soc.* **44** (1991), 19–31.

[4] K. Doerk and T. Hawkes, *Finite soluble groups*, de Gruyter Expositions in Maths. **4** (W. de Gruyter, Berlin, 1992).

[5] V. Fedri and L. Serena, 'Finite soluble groups with supersoluble Sylow normalizers', *Arch. Math.* **50** (1988), 11–18.

[6] G. Glaubermann, 'Prime-power factor groups of finite groups II.', *Math. Z.* **117** (1970), 46–56.

[7] B. Huppert, 'Zur Gaschützschen theorie der formationen', *Math. Ann.* **164** (1966), 133–141.

[8] B. Huppert, *Endliche Gruppen I* (Springer Verlag, Berlin, New York, 1967).

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