# TWO INTEGRALS INVOLVING MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

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§ 1. *Introductory.* In § 2 a product of two modified Bessel Functions of the Second Kind is expressed as an integral with a function of the same type as a factor of the integrand. In § 3 an integral involving a product of these functions, regarded as functions of their orders, is evaluated in terms of another function of this kind. These results were suggested by a study of Mellin's inversion formula.

§ 2. *Product of two modified Bessel Functions.* The formula to be proved is

$$K_m(a)K_n(b) = \int_{-\infty}^{\infty} e^{-u(m-n)} \left(\frac{ae^u+be^{-u}}{ae^{-u}+be^u}\right)^{\frac{1}{2}(m+n)} K_{m+n}[\surd\{(ae^u+be^{-u})(ae^{-u}+be^u)\}]\,du, \quad \text{.........(1)}$$

where $R(a)\geqslant 0$, $R(b)\geqslant 0$, $R(a+b)>0$.

This formula is a generalisation of a formula of Nicholson's (**1**), which can be deduced by putting $b=a$.

The proof is based on the formulae (**2, 3**)

$$K_n(z)=\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-z\cosh t \pm nt}\,dt, \quad \text{.....................................(2)}$$

where $R(z)>0$, and

$$K_n(z)=\tfrac{1}{2}z^n\int_0^{\infty} e^{-\frac{1}{2}(t+z^2/t)}\,t^{-n-1}\,dt, \quad \text{.................................(3)}$$

where $R(z^2)>0$.

From (2), if $R(a)>0$, $R(b)>0$,

$$K_m(a)K_n(b)=\tfrac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a\cosh s - b\cosh t - ms + nt}\,ds\,dt.$$

Here make the transformation

$$s=u+v,\ t=u-v$$

and get

$$\tfrac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a\cosh(u+v)-b\cosh(u-v)-u(m-n)-v(m+n)}\,du\,dv$$

$$=\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-u(m-n)}\,du\int_{-\infty}^{\infty} e^{-a\cosh(u+v)-b\cosh(u-v)-v(m+n)}\,dv.$$

Now in the inner integral put $w=(ae^u+be^{-u})e^v$ and it becomes

$$(ae^u+be^{-u})^{m+n}\int_0^{\infty} e^{-\frac{1}{2}(w+R^2/w)}\,w^{-m-n-1}\,dw,$$

where
$$R^2=a^2+b^2+2ab\cosh 2u=(ae^u+be^{-u})(ae^{-u}+be^u).$$

Thus, by (3), the inner integral is equal to

$$2\{(ae^u+be^{-u})/R\}^{m+n}K_{m+n}(R),$$

so giving formula (1).

§ 3. *Integral of a Product of two modified Bessel Functions.* The formula to be proved is

$$\frac{1}{2\pi i}\int k^{-2\zeta}\frac{K_{\mu+\zeta}(a)}{a^{\mu+\zeta}}\frac{K_{\nu-\zeta}(b)}{b^{\nu-\zeta}}\,d\zeta$$

$$=\tfrac{1}{2}k^{\mu-\nu}\left(\frac{k+k^{-1}}{a^2k+b^2k^{-1}}\right)^{\frac{1}{2}(\mu+\nu)}K_{\mu+\nu}[\sqrt{\{(k+k^{-1})(a^2k+b^2k^{-1})\}}], \quad\ldots\ldots(4)$$

where $a\neq 0$, $b\neq 0$ and the integral is taken up the entire length of the imaginary axis. The proof is based on the two following formulae (**4, 5**) :

$$(\tfrac{1}{2}z)^{-n}K_n(z)=\frac{1}{8\pi i}\int\Gamma(\tfrac{1}{2}\zeta)\Gamma(\tfrac{1}{2}\zeta-n)(\tfrac{1}{2}z)^{-\zeta}\,d\zeta, \quad\ldots\ldots(5)$$

where the integral is taken up the imaginary axis with loops, if necessary, to ensure that the poles of the integrand lie to the left of the contour ; and

$$\frac{1}{2\pi i}\int\Gamma(a+\zeta)\Gamma(b-\zeta)z^{-\zeta}\,d\zeta=z^a\Gamma(a+b)(1+z)^{-a-b}, \quad\ldots\ldots(6)$$

where the integral is taken up the imaginary axis with loops, if necessary, to ensure that the poles of $\Gamma(a+\zeta)$ lie to the left and those of $\Gamma(b-\zeta)$ to the right of the contour. On replacing $\zeta$ by $\zeta-a$ the integral reduces to an $E$-function.

On substituting from (5) on the left of (4) and changing the order of integration it becomes

$$\frac{2^{-\mu-\nu-4}}{(2\pi i)^3}\int\Gamma(\tfrac{1}{2}s)\left(\frac{a}{2}\right)^{-s}ds\int\Gamma(\tfrac{1}{2}t)\left(\frac{b}{2}\right)^{-t}dt\int k^{-2\zeta}\Gamma(\tfrac{1}{2}s-\mu-\zeta)\Gamma(\tfrac{1}{2}t-\nu+\zeta)\,d\zeta.$$

From (6) it follows that the inmost integral is equal to

$$2\pi i\,k^{t-2\nu}\,\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}t-\mu-\nu)(1+k^2)^{-\frac{1}{2}s-\frac{1}{2}t+\mu+\nu}.$$

Thus, on replacing $s$ and $t$ by $u+v$ and $u-v$ the expression reduces to

$$\frac{2^{-\mu-\nu-3}}{(2\pi i)^2}\frac{(1+k^2)^{\mu+\nu}}{k^{2\nu}}\int\int\left(\frac{a}{2}\right)^{-u-v}\left(\frac{b}{2}\right)^{-u+v}\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{u-v}{2}\right)\Gamma(u-\mu-\nu)(1+k^2)^{-u}k^{u-v}\,du\,dv.$$

Now the last line may be written

$$\int\Gamma(u-\mu-\nu)\left\{\frac{ab(1+k^2)}{4k}\right\}^{-u}du\int\Gamma\left(\frac{u+v}{2}\right)\Gamma\left(\frac{u-v}{2}\right)\left(\frac{ak}{b}\right)^{-v}dv, \quad\ldots\ldots(A)$$

and, from (6), on replacing $v$ by $2v$, it is seen that the inner integral is equal to

$$2\pi i\times 2\left(\frac{ak}{b}\right)^{u}\Gamma(u)\left(1+\frac{a^2k^2}{b^2}\right)^{-u}.$$

Hence, on replacing $u$ by $\frac{1}{2}u$ and applying (5) the expression (A) becomes

$$(2\pi i)^2\,4\,(\tfrac{1}{2}R)^{-\mu-\nu}K_{\mu+\nu}(R),$$

where $$R=\sqrt{\{(k+k^{-1})(a^2k+b^2k^{-1})\}},$$

and from this formula (4) follows.

## REFERENCES

(**1**) Nicholson, J. W., *Quart. J. Math.*, **42** (1911), 220–223.

(**2**), (**3**) Watson, G. N., *Bessel Functions*, pp. 182, 183.

(**4**) Titchmarsh, E. C., *Fourier Integrals*, p. 197.

(**5**) MacRobert, T. M., *Complex Variable*, 4th ed., p. 374.

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