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# SOME EXTREMAL PROPERTIES OF BIPARTITE SUBGRAPHS

C. S. EDWARDS

1. $G_p = (V, X)$ is a graph on $p$ vertices, with vertex set $V$ and edge set $X$. Occasionally, to avoid ambiguity, $V$ and $X$ will be written $V(G_p)$ and $X(G_p)$, respectively.

2. $H(S + 1)$ shall denote any bipartite graph with $S + 1$ edges. Consistent with a standard notation, see Harary [**1**, Chapter 2, p. 18], $\text{ex}(p, H(S + 1))$ denotes the maximum number of edges in any graph $G_p = (V, X)$, subject only to the constraints that $|V| = p$ and that $G_p$ contains no subgraph isomorphic to any $H(S + 1)$, i.e. that $G_p$ has no bipartite subgraph with $S + 1$ edges.

Evidently, for $p$ sufficiently large, any such graph $G_p$ with a maximum number of edges will contain a subgraph isomorphic to some $H(S)$: otherwise at least one edge could be added to $X$ without breaking the constraints above.

3. The principal result of this paper is as follows: for all $p$,

$$\text{ex}(p, H(S + 1)) \leqq [2(S + \tfrac{1}{4}) - (S + \tfrac{1}{4})^{\frac{1}{2}}], \text{ for all } S \geqq 0. \tag{3.1}$$

As usual, $[x]$ denotes the largest integer not exceeding $x$. Much of the remainder of this paper is directed towards establishing the result of (3.1).

4. The elements of $V$ can be ordered in $p!$ different ways, not all of which may be distinguishable within the graph $G_p$. Let $I$ denote one such ordering of the $p$ different vertices of $G_p$. Any ordering will append each of the integers 1 to $p$, inclusive, to precisely one of the vertices of $G_p$, every such vertex having one appended integer: thus $v_r^I$ is defined as the particular vertex of $G_p$ to which is appended the integer $r$ by the ordering $I$, $1 \leqq r \leqq p$.

5. For a particular ordering $I$, the sequence $(v_p^I, v_{p-1}^I, \ldots, v_1^I)$ will represent the corresponding ordering of the vertices of $G_p$. Let $G_{p-1}^I = G_p - v_p^I$, $G_{p-2}^I = G_{p-1}^I - v_{p-1}^I = G_p^I - v_p^I - v_{p-1}^I$, and in general,

$$G_{p-r}^I = G_p - \bigcup_{i=0}^{r-1} v_{p-i}^I,$$

for all $r$, $0 \leqq r < p$, and all $I$. Thus $G_{p-r}^I$ is the induced subgraph of $G_p$ generated by the vertex set

$$V(G_p) - \{v_p^I, v_{p-1}^I, \ldots, v_{p-r+1}^I\} = \{v_{p-r}^I, v_{p-r-1}^I, \ldots, v_1^I\}.$$

As a natural extension of the above notation, let $G_p^I = G_p$, for all $I$.

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Define $\deg_r(v_k^I)$ to be the degree of the vertex of $G_p$, denoted as $v_k^I$ by the ordering $I$, within the induced subgraph $G_r^I$ of $G_p$ defined by the vertex set $\{v_r^I, v_{r-1}^I, \ldots, v_1^I\}$, $1 \leqq k \leqq r \leqq p$.

Noting that $G_{r-1}^I$ is formed from $G_r^I$ by removing $v_r^I$, and all edges incident to $v_r^I$, from $G_r^I$, for all $r$, $1 < r \leqq p$, then it follows that

$$|X(G_r^I)| = \sum_{i=2}^{r} \deg_i(v_i^I), \text{ for all } r, I, \ 1 \leqq r \leqq p.$$

6. $G_r^I$, as defined in paragraph 5, is uniquely determined by $G_p$ and the particular set of $(p - r)$ vertices of $G_p$ which are the first $(p - r)$ vertices within the ordering $I = (v_p^I, \ldots, v_{r+1}^I, v_r^I, \ldots, v_1^I)$. Evidently, any change of the relative order in which the vertices $v_p^I, v_{p-1}^I, \ldots, v_{r+1}^I$, are successively removed from $G_p$ has no effect on the induced subgraph $G_r^I$ obtained. Further, the relative ordering of the last $r$ vertices within the ordering $I$, i.e., $v_r^I, v_{r-1}^I, \ldots, v_1^I$, is not determined by $G_r^I$ in any respect: thus the ordering of the $r$ vertices of $G_r^I$ can be freely chosen, as if $G_r^I$ were a graph in its own right: indeed, hereafter, $G_r^I$ will often be treated as a graph, sometimes with no reference to any particular $G_p$ from which it may have been derived and to the ordering $I$ used to derive $G_r^I$ from $G_p$.

It can be seen that any given induced subgraph of $G_p$ can be obtained from $G_p$ by some ordering $I$. For suppose the induced subgraph has $r$ vertices, $1 \leqq r \leqq p$. We take these $r$ vertices and any ordering $I$ with these $r$ vertices as last $r$ vertices in the ordering. $G_r^I$ is then the given induced subgraph. Evidently there will be $r!(p - r)!$ orderings $I$ by which a given induced subgraph $G_r^I$ may be obtained from $G_p$.

In spite of the multiplicity of orderings $I$ which may be used to derive a given induced subgraph $G_r^I$ of $G_p$, the notation described in paragraph 5 will be convenient for the purposes of this paper because of its compactness.

7. A simple function defined upon the vertices of a graph is now introduced. This function will be used extensively during the remainder of this paper, together with some of its properties: the next few paragraphs will be devoted to deriving the properties required.

For all $r$, $I$, and $1 \leqq r \leqq p$, define

$$(7.1) \qquad t(v_r^I) = \begin{cases} 1, & \text{if } \deg_r(v_r^I) \text{ is odd,} \\ 0, & \text{otherwise,} \end{cases}$$

for any particular graph $G_p$.

Thus for each ordering $I$, the $t$-function assigns to each vertex of $G_p$ a unique value which is either 0 or 1.

In paragraph 6 it has been pointed out that the ordering, say $J$, of the vertices of $G_r^I$ can be chosen independently of any ordering $I$ of the vertices of $G_p$ by which $G_r^I$ may have been derived from $G_p$. Thus, without inconsistency, we may introduce the following functions of $G_r^I$ and any ordering $J$

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of its vertices, for all $r$, $1 \leqq r \leqq p$, and all $I$:

(7.2) $$T^J(G_r^I) = \sum_{i=2}^{r} t(v_i^J), \qquad T^*(G_r^I) = \max_J T^J(G_r^I).$$

Putting $r = p$, and recalling that $G_p^I = G_p$, for all $I$, we have:

(7.3) $$T^J(G_p) = \sum_{i=2}^{p} t(v_i^J), \qquad T^*(G_p) = \max_J T^J(G_p).$$

8. From this point up to the end of paragraph 16 a series of theorems are stated and proved, each theorem involving the functions $t$ and $T$.

9. THEOREM 1. *$T^*(G_r^I) = 0$, if and only if $X(G_r^I) = \emptyset$, for all subgraphs $G_r^I$, and all $r$.*

*Proof.* $X(G_r^I) = \emptyset$ implies that $\deg_r(v_r^J), \deg_{r-1}(v_{r-1}^J), \ldots, \deg_2(v_2^J)$ are all even, for all $J$, and so $t(v_r^J) = t(v_{r-1}^J) = \ldots = t(v_2^J) = 0$, implying $T^*(G_r^I) = 0$. This completes the proof in this direction.

Conversely, $X(G_r^I) \neq \emptyset \Rightarrow$ there exist a pair of distinct vertices of $G_r^I$, which can be named $v_r^J, v_{r-1}^J$, under some ordering $J$ of the vertices of $G_r^I$, such that $(v_r^J, v_{r-1}^J) \in X(G_r^I)$. Thus we have the following disjoint alternatives:

(i) at least one of $\deg_r(v_r^J)$, $\deg_r(v_{r-1}^J)$, is odd;

(ii) both $\deg_r(v_r^J)$, $\deg_r(v_{r-1}^J)$, are even integers $\geqq 2$.

Suppose that at least one of $\deg_r(v_r^J)$, $\deg_r(v_{r-1}^J)$, is odd: in this case we can always choose $J$ such that $\deg_r(v_r^J)$ is odd. Then by definition $t(v_r^J) = 1$.

Now suppose that both $\deg_r(v_r^J)$, $\deg_r(v_{r-1}^J)$, are even integers $\geqq 2$. Then $t(v_r^J) = 0$ and $\deg_{r-1}(v_{r-1}^J)$ is odd; thus $t(v_r^J) + t(v_{r-1}^J) = 1$. It follows that if $X(G_r^I) \neq \emptyset$ then $T^J(G_r^I) \geqq 1$. Hence $T^*(G_r^I) \geqq 1$, which completes the proof.

10. THEOREM 2. *$T^*(G_r^I) \geqq t(v_r^I) + T^*(G_{r-1}^I)$, for all $r$, $1 < r \leqq p$, and all $G_p$, $p$, $I$.*

*Proof.* If the theorem is not true, then there exist $r$, $p$, where $1 < r \leqq p$, a graph $G_p$ and an ordering $I$, such that $T^*(G_r^I) < t(v_r^I) + T^*(G_{r-1}^I)$.

(7.2) $\Rightarrow$ there exists an ordering $J = (v_{r-1}^J, \ldots, v_1^J)$ of the $(r-1)$ vertices of $G_{r-1}^I$, such that $T^*(G_{r-1}^I) = T^J(G_{r-1}^I)$. It follows that it is possible to construct the ordering $K = (v_r^I, v_{r-1}^J, v_{r-2}^J, \ldots, v_1^J)$ of the vertices of $G_r^I$. Then

$$T^*(G_r^I) < t(v_r^I) + T^*(G_{r-1}^I) = T^K(G_r^I),$$

which implies that $T^*(G_r^I) < T^K(G_r^I)$, contrary to the definition (7.2). This completes the proof.

11. THEOREM 3. *$T^I(G_p) = T^*(G_p) \Rightarrow T^I(G_r^I) = T^*(G_r^I)$, for all $r$, $1 \leqq r \leqq p$.*

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*Proof*. If the theorem is not true, then (7.2) implies there exists $r$, $1 \leqq r \leqq p - 1$, such that $T^*(G_r^I) > T^I(G_r^I)$. Now there exists an ordering $J = (v_r^J, v_{r-1}^J, \ldots, v_1^J)$ of the vertices of $G_r^I$ such that $T^*(G_r^I) = T^J(G_r^I)$. Therefore $T^J(G_r^I) > T^I(G_r^I)$.

Also by above it is possible to construct an ordering

$$K = (v_p^I, v_{p-1}^I, \ldots, v_{r+1}^I, v_r^J, v_{r-1}^J, \ldots, v_1^J)$$

of the vertices of $G_p$. Then

$$T^K(G_p) = \sum_{i=r+1}^{p} t(v_i^I) + T^J(G_r^I), \quad \text{and} \quad T^I(G_p) = \sum_{i=r+1}^{p} t(v_i^I) + T^I(G_r^I).$$

It follows that $T^K(G_p) > T^I(G_p) = T^*(G_p)$, contrary to definition. This completes the proof.

12. THEOREM 4. *If* $T^*(G_p) = T^I(G_p) = K$, *then there exists a strictly increasing positive integer single-valued function* $r(k)$, *for* $k = 0, 1, 2, \ldots, K$, *where* $r(k) \leqq p$, *such that* $T^*(G_{r(k)}^I) = k$.

*Proof*. By assumption, $T^*(G_p) = T^I(G_p) = K$. Then by Theorem 3, $T^*(G_r^I) = T^I(G_r^I)$, for all $r$ such that $1 \leqq r \leqq p$. Thus

$$\begin{aligned} (12.1) \qquad T^*(G_r^I) &= t(v_r^I) + T^I(G_{r-1}^I) = t(v_r^I) + T^*(G_{r-1}^I), \\ &\Rightarrow T^*(G_r^I) - T^*(G_{r-1}^I) = t(v_r^I), \\ &\Rightarrow 0 \leqq T^*(G_r^I) - T^*(G_{r-1}^I) \leqq 1, \end{aligned}$$

for all $r$ such that $1 < r \leqq p$.

(12.1) $\Rightarrow T^*(G_1^I), T^*(G_2^I), \ldots, T^*(G_{p-1}^I), T^*(G_p)$ is a non-decreasing sequence of non-negative integers in which consecutive terms differ by at most unity, and in which $T^*(G_1^I) = 0$ and $T^*(G_p) = K$. Thus every integer from 0 to $K$, inclusive, appears at least once within the sequence.

Then for given $G_p$ and given $k$ such that $0 \leqq k \leqq K$, there exists a smallest integer $r(k)$ such that $T^*(G_{r(k)}^I) = k$. Evidently $r(k)$ is a strictly increasing single-valued function of $k$, for all $k$ such that $0 \leqq k \leqq K$. This completes the proof.

13. We define

$$M(G_p) = \max_i \deg_p(v_i), \qquad v_i \in V(G_p).$$

Thus $M(G_p)$ is the maximum vertex degree which occurs in the graph $G_p$.

THEOREM 5. $T^*(G_p) \geqq [(M(G_p) + 1)/2]$ *for all* $G_p$, *and* $p$.

*Proof*. The proof proceeds by induction on $T^*(G_p)$. Suppose that the theorem is true for all $G_p$ and $p$, such that $T^*(G_p) \leqq K$, $K \geqq 0$.

Let $G_p'$ be some graph on $p$ vertices, for some $p$, such that $T^*(G_p') = K + 1$. It may be assumed that $G_p'$ exists: otherwise, Theorem 4 $\Rightarrow T^*(G_p) \leqq K$, for all graphs $G_p$, and all $p$ (it is trivial to show that there exist $p$, $G_p$, such that

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$T^*(G_p) > K$, for all $K$: for example, consider the graph formed of a single Hamiltonian chain on not less than $K + 2$ vertices).

Let $v'$ be some vertex of $G_p'$ of maximum degree. $V(G_p') - \{v'\}$ either contains a vertex of odd degree in $G_p'$, or $V(G_p') - \{v'\}$ only contains vertices of even degree in $G_p'$.

First, suppose that $V(G_p') - \{v'\}$ contains a vertex of odd degree in $G_p'$: this implies that it is possible to find an ordering $I$ of the $p$ vertices of $G_p'$ such that $\deg_p(v_p^I)$ is odd, $v_p^I \neq v'$. Then $t(v_p^I) = 1$, and by Theorem 2, $T^*(G_p') \geqq 1 + T^*(G_p' - v_p^I)$, and so $T^*(G_p' - v_p^I) \leqq K$. Then

$$[(M(G_p' - v_p^I) + 1)/2] \leqq K.$$

However, since $v_p^I \neq v'$, it follows that $M(G_p' - v_p^I) + 1 \geqq M(G_p')$. Thus $[M(G_p')/2] \leqq K$, which implies

$$(13.1) \qquad [(M(G_p') + 1)/2] \leqq K + 1 = T^*(G_p').$$

Secondly, suppose $V(G_p') - v'$ contains only vertices of even degree in $G_p'$. $T^*(G_p') = K + 1 \Rightarrow X(G_p') \neq \emptyset$, by Theorem 1. Then there exist in $V(G_p') - \{v'\}$, a pair of vertices adjacent in $G_p'$: for otherwise all vertices in $V(G_p') - \{v'\}$ have degree, in $G_p'$, at most 1, $\Rightarrow$ all vertices in $V(G_p') - \{v'\}$ have degree 0 in $G_p'$, $\Rightarrow X(G_p') = \emptyset$, contrary to the above. Thus we can choose an ordering $I$ of the vertices of $G_p'$ such that $v_p^I, v_{p-1}^I \neq v'$ where $(v_p^I, v_{p-1}^I) \in X(G_p')$, and such that $\deg_p(v_p^I)$, $\deg_p(v_{p-1}^I)$, are both even integers $\geqq 2$. Then $\deg_p(v_p^I)$ is even and $\deg_{p-1}(v_{p-1}^I)$ is odd; hence $t(v_p^I) + t(v_{p-1}^I) = 1$. By Theorem 2,

$$T^*(G_p') \geqq t(v_p^I) + T^*(G_{p-1}'^I) \geqq t(v_p^I) + t(v_{p-1}^I) + T^*(G_{p-2}'^I), \text{ for all } I.$$

Thus, $T^*(G_p') \geqq 1 + T^*(G_{p-2}'^I)$, and so $T^*(G_{p-2}'^I) \leqq K$, from which we conclude that

$$(13.2) \qquad [(M(G_{p-2}'^I) + 1)/2] \leqq K.$$

By our choice of ordering,

$$(13.3) \quad M(G_{p-2}'^I) + 2 \geqq M(G_p'), \Rightarrow M(G_p') - 1 \leqq M(G_{p-2}'^I) + 1.$$

(13.2), (13.3), imply

$$(13.4) \quad [(M(G_p') - 1)/2] \leqq K, \Rightarrow [(M(G_p') + 1)/2] \leqq K + 1 = T^*(G_p').$$

If $T^*(G_p) = 0$, then by Theorem 1 $X(G_p) = \emptyset$, $\Rightarrow M(G_p) = 0$. Thus the induction hypothesis holds for $K = 0$. Then by (13.1) and (13.4), the induction hypothesis extends to all non-negative integer values of $T^*(G_p)$ for all $G_p$, and all $p$. This completes the proof of Theorem 5.

14. THEOREM 6. *$T^*(G_p) = R \Rightarrow |X(G_p)| \leqq \binom{2R+1}{2}$, for all $G_p$, and all $p$.*

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*Proof.* The proof proceeds by induction on $T^*(G_p)$. Suppose the result holds whenever $T^*(G_p) \leqq K$, $K \geqq 0$.

Let $G_p'$, for some $p$, be a graph such that $T^*(G_p') = K + 1$ (there exists such a graph by the initial remarks in the proof of Theorem 5). Then by Theorem 5, $[(M(G_p') + 1)/2] \leqq K + 1$. Thus, if $M(G_p')$ is odd, then $M(G_p') \leqq 2K + 1$, while if $M(G_p')$ is even, then $M(G_p') \leqq 2K + 2$.

$G_p'$ either contains a vertex of odd degree, or all vertices of $G_p'$ are of even degree.

First assume $G_p'$ contains a vertex of odd degree. Then we can find an ordering $I$ such that $\deg_p(v_p^I)$ is odd. By Theorem 2

$$T^*(G_p') \geqq 1 + T^*(G_p' - v_p^I),$$

and so $T^*(G_p' - v_p^I) \leqq K$. Thus, by the induction hypothesis

$$|X(G_p' - v_p^I)| \leqq \binom{2K+1}{2}.$$

Now, since $\deg_p(v_p^I)$ is odd and does not exceed $M(G_p')$, it follows that

$$\deg_p(v_p^I) \leqq 2K + 1, \Rightarrow |X(G_p')| \leqq 2K + 1 + |X(G_p' - v_p^I)|.$$

It now follows that

$$(14.1) \quad |X(G_p')| \leqq \binom{2K+1}{1} + \binom{2K+1}{2} = \binom{2K+2}{2},$$

$$\Rightarrow |X(G_p')| < \binom{2(K+1)+1}{2}.$$

Now we consider the case where $G_p'$ contains only vertices of even degree. By Theorem 1, $T^*(G_p') = K + 1, \Rightarrow X(G_p') \neq \emptyset$. Thus $G_p'$ necessarily contains a pair of adjacent vertices of even degree: moreover, we can find an ordering $I$ such that $v_p^I, v_{p-1}^I$, are adjacent in $G_p'$ and of even degree: also, $\deg_{p-1}(v_{p-1}^I)$ is odd, and so by Theorem 2, $T^*(G_p') \geqq 1 + T^*(G_{p-2}^I)$. Then $T^*(G_{p-2}'^I) \leqq K$, and by induction $|X(G_{p-2}'^I)| \leqq \binom{2K+1}{2}$. Now, $\deg_p(v_p^I)$, $1 + \deg_{p-1}(v_{p-1}^I)$, are both even and neither exceeds $M(G_p')$; thus, $|X(G_p')| \leqq (2K + 2) + (2K + 1) + |X(G_{p-2}'^I)|$, and so

$$(14.2) \qquad |X(G_p')| \leqq \binom{2K+2}{1} + \binom{2K+1}{1} + \binom{2K+1}{2},$$

$$\Rightarrow |X(G_p')| \leqq \binom{2(K+1)+1}{2}.$$

If the given induction hypothesis holds whenever $T^*(G_p) \leqq K$, then (14.1), (14.2), imply the induction hypothesis holds whenever $T^*(G_p) \leqq K + 1$, for all $G_p$, and all $p$. Moreover, Theorem 1 implies the induction hypothesis holds for $K = 0$. It follows that the induction hypothesis extends to all non-negative integer values of $K$. This completes the proof of Theorem 6.

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15. Let $H^*(G_r^I)$ be a bipartite graph with edge set of maximum cardinality, subject to the constraint that $H^*(G_r^I)$ is isomorphic to some subgraph of $G_r^I$. We define

$$b(G_r^I) = |X(H^*(G_r^I))|,$$

where $X(H^*(G_r^I))$ is the edge set of $H^*(G_r^I)$. Thus $b(G_r^I)$ is the number of edges in a bipartite subgraph of $G_r^I$ which has most edges. Evidently $b(G_1^I) = 0$, for all orderings $I$.

THEOREM 7. $2b(G_p) - |X(G_p)| \geqq T^*(G_p) \geqq [(M(G_p) + 1)/2]$ *for all* $G_p$, *and all* $p$.

*Proof.* The above paragraph implies there exists a partition of $V(G_{r-1}^I)$, the vertex set of $G_{r-1}^I$, into $V'$ and $V''$, such that each edge of the particular $H^*(G_{r-1}^I)$ is incident to a vertex in $V'$ and to a vertex in $V''$. Now $v_r^I \notin V(G_{r-1}^I)$ but, by definition, is adjacent to $\deg_r(v_r^I)$ vertices in $G_{r-1}^I$. Thus $v_r^I$ is adjacent to not less than $\frac{1}{2}(\deg_r(v_r^I) + t(v_r^I))$ vertices in $V'$ and/or to not less than $\frac{1}{2}(\deg_r(v_r^I) + t(v_r^I))$ vertices in $V''$. Thus $b(G_r^I) \geqq b(G_{r-1}^I) + \frac{1}{2}(\deg_r(v_r^I) + t(v_r^I))$, for all $r$, $2 \leqq r \leqq p$, and all $G_r^I, I$ (since a bipartite subgraph of $G_r^I$, with $v_r^I$ as one vertex, can always be formed to include any bipartite subgraph of $G_{r-1}^I$).

Putting $r = p$ initially, $(p - 1)$ successive applications of the above to $G_p, G_{p-1}^I, \ldots, G_2^I$, respectively, yields

$$b(G_p) \geqq \frac{1}{2}\left(\sum_{i=2}^{p} (\deg_i(v_i^I) + t(v_i^I))\right),$$

for all $I$, $G_p$, and $p$. Then, by the remark at the end of paragraph 5,

$$2b(G_p) \geqq |X(G_p)| + T^I(G_p), \text{ for all } I, \Rightarrow 2b(G_p) - |X(G_p)| \geqq T^*(G_p),$$

for all $G_p$, and all $p$. The result now follows by Theorem 5; this completes the proof.

16. THEOREM 8. $|X(G_p)| \leqq [2(b(G_p) + \frac{1}{4}) - (b(G_p) + \frac{1}{4})^{\frac{1}{2}}]$, *for all* $G_p$, *and all* $p$.

*Proof.* Define

$$y(G_p) = 2b(G_p) - |X(G_p)| \tag{16.1}$$

Then by Theorem 7, $T^*(G_p) \leqq y(G_p)$, and so by Theorem 6,

$$|X(G_p)| \leqq \binom{2y(G_p) + 1}{2}. \tag{16.2}$$

$$\begin{aligned}
(16.1),\ (16.2), &\Rightarrow 2b(G_p) - y(G_p) \leqq y(G_p)(2y(G_p) + 1),\\
&\Rightarrow b(G_p) \leqq y(G_p)(y(G_p) + 1),\\
&\Rightarrow y(G_p) \geqq -\tfrac{1}{2} + (b(G_p) + \tfrac{1}{4})^{\frac{1}{2}},
\end{aligned}$$

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since $y(G_p) \geqq 0$ by Theorem 7. Then (16.1) implies $|X(G_p)| \leqq 2(b(G_p) + \frac{1}{4}) - (b(G_p) + \frac{1}{4})^{\frac{1}{2}}$. This completes the proof since $|X(G_p)|$ is an integer.

17. Let $H(S + 1)$, $\text{ex}(p, H(S + 1))$, be defined as in paragraph 2.

THEOREM 9. $\text{ex}(p, H(S + 1)) \leqq [2(S + \frac{1}{4}) - (S + \frac{1}{4})^{\frac{1}{2}}]$, *for all* $p$, *and all* $S \geqq 0$.

*Proof.* If the theorem is true for all $p$ sufficiently large, then the theorem will be true for all $p$: for, $\text{ex}(p, H(S + 1))$ is a non-decreasing function of $p$, for fixed $S$, since we may always choose $p$ vertices from any vertex set with not less than $p$ vertices.

By paragraph 2, for all $p$ sufficiently large, there exists a graph $G_p$ such that $|X(G_p)| = \text{ex}(p, H(S + 1))$ and $b(G_p) = S$. Then by Theorem 8,

$$\text{ex}(p, H(S + 1)) \leqq [2(S + \tfrac{1}{4}) - (S + \tfrac{1}{4})^{\frac{1}{2}}],$$

for $p$ sufficiently large, and all $S \geqq 0$. By our previous remarks, the proof is now complete.

18. COROLLARY. $\text{ex}(p, H([N^2/4] + 1)) \leqq \binom{N}{2}$, *for all* $p$, *and all* $N \geqq 0$.

*Proof.* If $S = [N^2/4]$, where $N$ is some non-negative integer, then by Theorem 9,

$$\text{(18.1)} \quad \text{ex}(p, H([N^2/4] + 1)) \leqq [2([N^2/4] + \tfrac{1}{4}) - ([N^2/4] + \tfrac{1}{4})^{\frac{1}{2}}].$$

If $N$ is odd, $[N^2/4] + \frac{1}{4} = N^2/4$. It follows that

$$\text{(18.2)} \quad [2([N^2/4] + \tfrac{1}{4}) - ([N^2/4] + \tfrac{1}{4})^{\frac{1}{2}}] = [N^2/2 - N/2] = [N(N - 1)/2] = N(N - 1)/2.$$

If $N$ is even, then $[N^2/4] = N^2/4$. $N \geqq 0$ implies $0 < (N^2/4 + \frac{1}{4})^{\frac{1}{2}} - N/2 \leqq \frac{1}{2}, \Rightarrow N/2 < (N^2/4 + \frac{1}{4})^{\frac{1}{2}} \leqq \frac{1}{2} + N/2$. Thus, if $N$ is non-negative and even, then

$$\text{(18.3)} \quad [2([N^2/4] + \tfrac{1}{4}) - ([N^2/4] + \tfrac{1}{4})^{\frac{1}{2}}] = [N^2/2 + \tfrac{1}{2} - (N^2/4 + \tfrac{1}{4})^{\frac{1}{2}}] = N^2/2 - N/2 = N(N - 1)/2.$$

(18.1), (18.2), (18.3), $\Rightarrow \text{ex}(p, H([N^2/4] + 1)) \leqq \binom{N}{2}$, for all $p$, and all $N \geqq 0$.

19. $K_N$, the complete graph on $N$ vertices, has a number of interesting properties in the context of this paper. These properties will be considered now. The Corollary to Theorem 9 will be relevant in this context: in particular it is possible to replace the inequality sign by the equality sign in this corollary to give the following theorem.

THEOREM 10. $\text{ex}(p, H([N^2/4] + 1)) = \binom{R}{2}$, *where* $R = \min(p, N)$, *for all* $N, p$.

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*Proof.* Since a bipartite graph contains no triangle, Turan's theorem [**2**] $\Rightarrow$ $b(K_N) \leqq [N^2/4]$, where $b(K_N)$ = number of edges in a bipartite subgraph of $K_N$ with greatest number of edges. (This definition is consistent with paragraph 15.)

The vertex set of $K_N$ may be partitioned into two sets, one set containing [N/2] vertices and the other set containing $[(N+1)/2]$ vertices. It follows that $K_N$ has a bipartite subgraph with $[N/2][(N+1)/2] = [N^2/4]$ edges. Thus, it follows that

$$b(K_N) = [N^2/4]. \tag{19.1}$$

Then (19.1) and the corollary to Theorem 9 imply $\text{ex}(p, H([N^2/4]+1)) = \binom{N}{2}$, for all $p \geqq N$, since $K_N$ has $\binom{N}{2}$ edges. Evidently, if $p \leqq N$, then $\text{ex}(p, H([N^2/4]+1)) = \binom{p}{2}$. This completes the proof.

20. It is possible to prove Theorem 10 directly without recourse to Theorem 8, or to Theorem 9 or its corollary. The proof uses the properties $b(K_N) = [N^2/4]$, $|X(K_N)| = \binom{N}{2}$, and is as follows:

Suppose there exists a graph $G_p = (V, X)$ such that $b(G_p) = [N^2/4]$, $|X| > \binom{N}{2}$. Then by Theorem 7,

$$2[N^2/4] - |X| \geqq T^*(G_p), \Rightarrow 2[N^2/4] - \binom{N}{2} > T^*(G_p). \tag{20.1}$$

However, $2[N^2/4] - \binom{N}{2} = [N/2]$, for all $N \geqq 0$. Thus by (20.1), $T^*(G_p) < [N/2], \Rightarrow T^*(G_p) \leqq [(N-2)/2]$. By Theorem 6, it now follows that

$$|X| \leqq \binom{2[(N-2)/2]+1}{2} = \binom{2[N/2]-1}{2},$$

which implies $|X| \leqq \binom{N-1}{2}$, contrary to our assumption. Thus we must have $|X| \leqq \binom{N}{2}$ if $b(G_p) = [N^2/4]$, for all $G_p$, and all $p$. Now $K_N$ is a graph such that $b(K_N) = [N^2/4]$, $|X(K_N)| = \binom{N}{2}$, for all $N > 0$. So

$$\text{ex}(p, H([N^2/4]+1)) = \binom{N}{2}, \text{ for all } p \geqq N > 0.$$

Evidently, $p \leqq N \Rightarrow \text{ex}(p, H([N^2/4]+1)) = \binom{p}{2}$. This completes the proof.

*Note.* Theorem 10 was first conjectured by A. J. Maal who passed the con-

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jecture to the author in personal communication and whom the author wishes to thank for stimulating his interest in this area.

21. In this paragraph it is seen how certain further properties of $K_N$ illustrate earlier results within this paper.

(i) The symmetry of $K_N \Rightarrow T^*(K_N) = T^I(K_N)$, for all $I$, and all $N > 0$.

(ii) For $K_N$, by paragraph 5, $t(v_{2k}{}^I) = 1$, for all $k$, where $1 \leqq k \leqq [N/2]$, and also $t(v_{2k+1}{}^I) = 0$, for $k$, $3 \leqq 2k + 1 \leqq N$, for all $I$, and $N > 0$.

(iii) (i) and (ii) $\Rightarrow T^*(K_N) = T^I(K_N) = [N/2]$, for all $I$, and all $N > 0$. Evidently, $M(K_N) = N - 1$, for $N > 0$.

(iv) (iii) $\Rightarrow$ Theorem 5 is satisfied for $K_N$, for $N > 0$.

(v) $|X(K_N)| = \binom{N}{2}$, $\Rightarrow |X(K_N)| \leqq \binom{2[N/2]+1}{2}$, for $N > 0$.

(vi) (iii) and (v) $\Rightarrow$ Theorem 6 is satisfied for $K_N$, for $N > 0$.

(vii) From paragraph 20, $2[N^2/4] - \binom{N}{2} = [N/2]$, for $N \geqq 0$; also $b(K_N) = [N^2/4]$.

(viii) From (iii) and (vii) it follows that Theorem 7 is satisfied for $K_N$, for all $N > 0$.

(ix) By paragraph 18 it follows that

$$[2([N^2/4] + \tfrac{1}{4}) - ([N^2/4] + \tfrac{1}{4})^{\frac{1}{2}}] = \binom{N}{2},$$

for all $N > 0$: since $|X(K_N)| = \binom{N}{2}$, it follows that Theorem 8 is satisfied for $K_N$.

From the above it is easily seen that, for all $T^*(G_p)$, or $M(G_p)$, and all $p$ sufficiently large, there exists a graph $G_p$, e.g. with unique component $K_N$ where $N$ is appropriately chosen, such that the inequalities in Theorems 5, 6 and 7 become equalities: in this sense the results of Theorems 5, 6 and 7 are best-possible. Further, we have seen that, if we choose $G_p$ to have unique component $K_N$, then equality holds between the two sides in the result of Theorem 8, for all $N \leqq p$.

22. Theorem 7 provides two lower bounds for $b(G_p)$, namely

$$\{\tfrac{1}{2}(|X(G_p)| + [\tfrac{1}{2}(M(G_p) + 1)])\} \quad \text{and} \quad \{\tfrac{1}{2}(|X(G_p)| + T^*(G_p))\},$$

where $\{x\}$ denotes the smallest integer $\geqq x$.

For any given graph $G_p$, the determination of $b(G_p)$ is not necessarily trivial: in this context it may be useful to have lower bounds for $b(G_p)$ such as those above. However, in general, $T^*(G_p)$ is not explicitly known even though (7.2), (7.3), Theorem 2, Theorem 3, imply that an algorithm of dynamic programming type can be established to find $T^*(G_p)$ for any particular $G_p$. Further, it may not be considered appropriate to bound the unknown value of $b(G_p)$ in terms of the solution $T^*(G_p)$ of a more or less complex individually

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applied algorithm. Evidently, any lower bound to $T^*(G_p)$ can be substituted for $T^*(G_p)$ in the above lower bound to give an alternative lower bound for $b(G_p)$. In the next theorem a lower bound for $T^*(G_p)$, often distinct from $[\frac{1}{2}(M(G_p) + 1)]$, is established: in the context of this paper this result has some intrinsic interest.

THEOREM 11. $T^*(G_p) \geqq \{\frac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1)\}$, *for all* $G_p$, *and all* $p$.

*Proof.* We define $x$ and $T$ by $x = |X(G_p)|$, $T = T^*(G_p)$. Then by Theorem 6 $0 \leqq x \leqq T(2T + 1)$, for all $G_p$, and all $p$. Thus, $0 < 2x + \frac{1}{4} \leqq (2T + \frac{1}{2})^2$. It follows that

$$(2T + \tfrac{1}{2} + (2x + \tfrac{1}{4})^{\frac{1}{2}})(2T + \tfrac{1}{2} - (2x + \tfrac{1}{4})^{\frac{1}{2}}) \geqq 0. \tag{22.1}$$

If $2T + \frac{1}{2} < (2x + \frac{1}{4})^{\frac{1}{2}}$, then $(22.1) \Rightarrow 2T + \frac{1}{2} \leqq -(2x + \frac{1}{4})^{\frac{1}{2}} \Rightarrow T \leqq -\frac{1}{2}$, contrary to (7.3). Thus $2T + \frac{1}{2} \geqq (2x + \frac{1}{4})^{\frac{1}{2}}$, from which it follows that $T \geqq \{\frac{1}{4}((8x + 1)^{\frac{1}{2}} - 1)\}$, for all $G_p$, and all $p$. This completes the proof.

23. THEOREM 12. $b(G_p) \geqq \{\frac{1}{2}(|X(G_p)| + \{\frac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1)\})\}$.

*Proof.* This follows directly from Theorem 11, and the initial remarks in paragraph 22.

24. It is possible to derive another bound for $b(G_p)$ from Theorem 8: this bound is:

$$b(G_p) \geqq \{\tfrac{1}{2}(|X(G_p)| + \tfrac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1))\}$$

for all $G_p$, and all $p$. It is evident that this bound is never superior to that given by Theorem 12, since

$$\{\tfrac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1)\} \geqq \tfrac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1).$$

It is now possible to formulate the last result of this paper: this gives a lower bound for $b(G_p)$ in terms of $|X(G_p)|$ and $M(G_p)$, both known or observable numbers for any given graph $G_p$:

THEOREM 13.
$b(G_p) \geqq \{\frac{1}{2}(|X(G_p)| + \max(\{\frac{1}{4}((8|X(G_p)| + 1)^{\frac{1}{2}} - 1)\}, [\frac{1}{2}(M(G_p) + 1)]))\}$, *for all* $G_p$, *and all* $p$.

*Proof.* This follows directly from Theorem 12 and, once again, the initial remarks of paragraph 22.

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*University of Birmingham,*
*Birmingham, England*

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