BULL. AUSTRAL. MATH. SOC. 18C15, 18A40, 18D10
VOL. 26 (1982), 221-237.

TWO ADDENDA TO THE AUTHOR'S 'TRANSFINITE CONSTRUCTIONS'

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Since the author's article "A unified treatment of transfinite constructions ...", in Volume 22 (1980) of this *Bulletin*, had an encyclopaedic goal, he now takes the opportunity to answer two further questions raised since that article was submitted. The lesser of these asks whether the *only* pointed endofunctors for which every action is an isomorphism are the well-pointed ones, at least when the endofunctor is cocontinuous; a counter-example provides a negative answer. The more important question concerns the reflexion from the comma-category T/A into the category of algebras for the pointed endofunctor T of A, and the algebra-reflexion sequence which converges to this reflexion; and asks for simplified descriptions in the special case where T is *cocontinuous*. We give closed formulas in this case, both for the reflexion and for the sequence which converges to it. The reader may wonder why we care about the approximating sequence when we have a closed formula for the reflexion; the answer is that, in certain applications, we need to separate the roles of finite colimits and filtered ones.

1. Introduction

Recall from the author's article [1] that a *pointed endofunctor* (T, τ) on a category A is an endofunctor T of A along with a natural

Received 19 April 1982. The author gratefully acknowledges the assistance of the Australian Research Grants Committee.

https://doi.org/10.1017/S0004972700005724 Published online by Cambridge University Press

transformation $\tau : 1 \to T$, and that (T, τ) is called *well-pointed* if $\tau T = T\tau : T \to T^2$. A *T-algebra* is an object A together with an *action* $a : TA \to A$, the latter being a map satisfying the *unit axiom* $a.\tau A = 1$. If T has more structure, such as that of a monad, there are more axioms for the action a to satisfy; but we never suppose T to have less structure. When we want to consider the algebras for a mere, unpointed, endofunctor H of $\mathbb{A}$, given by actions $HA \to A$ subject to no axioms, we treat them as the algebras for the pointed endofunctor $T = 1 + H$.

In the case of a *well-pointed* (T, τ) , it is shown in Proposition 5.2 of [1] that every action $a : TA \to A$ is an isomorphism. Thus here A admits an action only when $\tau A : A \to TA$ is invertible, and then admits the unique action $(\tau A)^{-1}$; in consequence, the forgetful functor $U : T\text{-Alg} \to \mathbb{A}$ is fully faithful.

The goal of [1] was to give a comprehensive account of old and new transfinite-sequence existence proofs in categorical algebra, unified by the technique of embedding the T-algebras as a full subcategory in the comma-category $T/\mathbb{A}$, and then exhibiting them as the algebras for a *well-pointed* endofunctor on $T/\mathbb{A}$. This reduces the reflectivity of T-Alg in $T/\mathbb{A}$ to the question of the reflectivity, in the case of a *well-pointed* T , of T-Alg in $\mathbb{A}$. Transfinite-sequence arguments of various degrees of delicacy were now used to give sufficient conditions for this last; in the simpler cases the sequence $A \to TA \to T^2A \to \ldots$, continued transfinitely, ultimately stops and gives the desired reflexion. Then, translating back to the case of a general pointed T , we get an *algebra-reflexion sequence* for each object of $T/\mathbb{A}$, giving its reflexion into T-Alg .

In line with the encyclopaedic aim of [1], the author would now like to record the answers to two further questions raised since that manuscript was written.

In a letter to the author of 14 December 1979, Michael Barr asked whether the pointed (T, τ) is *necessarily* well-pointed if every action is an isomorphism - at least in the case where T is cocontinuous. As Barr pointed out, the question is not a pressing one, but a positive answer might save an author from giving a complicated argument where a simple one

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would suffice; he had a particular instance in mind. We show by an example that the answer is in fact negative.

Recent work of the author with Anders Kock on synthetic differential geometry has shown that we need at our disposal a simplified closed form for the algebra-reflexion sequence in the special case when T is *cocontinuous* - as for instance when T is $D \otimes -$ for a pointed object D in a monoidal biclosed category A. Of course the sequence stops for a cocontinuous T at its ωth term, and gives the reflexion. However no closed formula was given in [1] except the classical expression $\sum_{n \in \mathbb{N}} E^{(n)}$ for the free monoid on an *unpointed* E in a biclosed monoidal A. We therefore treat below the three cases of a pointed endofunctor T, an unpointed endofunctor H, and a monad $\mathbf{T}$, under the hypothesis of cocontinuity.

2. The counter-example for Barr's question

Let R be the associative $(\mathbb{Z}/2\mathbb{Z})$-algebra of dimension 3 with vector-space basis $\{1, e, f\}$ and with multiplication given by $e^2 = e$, $fe = f$, $ef = f^2 = 0$. Take for A the category of R-bimodules, and write $\otimes$ for $\otimes_R$. The principal 2-sided ideals I and J of R generated by e and f respectively satisfy $IJ = 0$ and $JI = J$. Let $k : R \to R/I = K$ be the canonical map. In each of the exact sequences

$$J \otimes I \to J \otimes R \xrightarrow[J \otimes k]{} J \otimes K \to 0 , \quad I \otimes J \to R \otimes J \xrightarrow[k \otimes J]{} K \otimes J \to 0 ,$$

the middle object is isomorphic to J; the image of $J \otimes I \to J$ is J while that of $I \otimes J \to J$ is 0; consequently $J \otimes K = 0$, while $k \otimes J$ is an isomorphism between non-zero objects.

Set $D = K \oplus J$, making it a pointed object via $d = \binom{k}{0} : R \to K \oplus J$. If an action $a : D \otimes A \to A$ is given by $(b, c) : (K \otimes A) \oplus (J \otimes A) \to A$, the unit axiom becomes $b(k \otimes A) = 1$; forcing $b : K \otimes A \to A$ to be an isomorphism since $k \otimes A$ is an epimorphism. We now have $J \otimes A \cong J \otimes K \otimes A = 0$ since $J \otimes K = 0$; whence $a : D \otimes A \to A$ is itself an isomorphism.

Yet $D \otimes d$, $d \otimes D : D \to D \otimes D$ do not coincide; for as maps

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$K \oplus J \to (K \otimes K) \oplus (J \otimes K) \oplus (K \otimes J) \oplus (J \otimes J)$ they differ in the component $J \to K \otimes J$, which is 0 for $D \otimes d$ but is the non-zero $k \otimes J$ for $d \otimes D$.

3. The free-algebra sequence for a cocontinuous pointed endofunctor

Let $\tau : 1 \to T : \mathsf{A} \to \mathsf{A}$ where A is cocomplete and T is cocontinuous. Write $\sigma_n : T^n \to T_n$ for the joint coequalizer of the maps $T^i \tau T^{n-i-1} : T^{n-1} \to T^n$ where $0 \le i \le n-1$; and note that σ_0 is $1 : 1 \to 1$ while σ_1 is $1 : T \to T$, the first non-trivial case being the coequalizer $\sigma_2 : T^2 \to T_2$ of $\tau T,\ T\tau : T \to T^2$. Observe that T_n is cocontinuous since each T^m is so.

PROPOSITION 1. *There is a unique* ϕ_{mn} *rendering commutative*

$$(1)\qquad \begin{array}{ccc} T^m T^n & \xrightarrow{\ \sigma_m \sigma_n\ } & T_m T_n \\ & {}_{\sigma_{m+n}}\searrow & \downarrow{\scriptstyle \phi_{mn}} \\ & & T_{m+n} \end{array} \;;$$

clearly ϕ_{mn} *is epimorphic.*

Proof. From the fact that $\sigma_m T^n$ and $T_m \sigma_n$ are certain joint coequalizers, from the naturality of $T_m T^i \tau T^{n-1-i}$, and from the fact that $\sigma_m T^{n-1}$ is epimorphic, it easily follows that $\sigma_m \sigma_n$, as $(T_m \sigma_n) . (\sigma_m T^n)$, is the universal map which jointly coequalizes $\tau T^{m-1} T^n, \ldots, T^{m-1} \tau T^n$ and jointly coequalizes $T^m \tau T^{n-1}, \ldots, T^m T^{n-1} \tau$. The result follows. □

PROPOSITION 2. *We have* $\phi_{0n} = \phi_{n0} = 1 : T_n \to T_n$. *The diagram*

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$$(2)\qquad \begin{array}{ccc} T_mT_nT_k & \xrightarrow{T_m\phi_{nk}} & T_mT_{n+k} \\ {\scriptstyle \phi_{mn}T_k}\downarrow & & \downarrow{\scriptstyle \phi_{m,n+k}} \\ T_{m+n}T_k & \xrightarrow[\phi_{m+n,k}]{} & T_{m+n+k} \end{array}$$

commutes, and it is a pushout for $n > 0$.

Proof. The first statement is clear because $\sigma_0 = 1$. As for the second, it suffices to consider the composite of (2) with the epimorphism $\sigma_m\sigma_n\sigma_k$, which by (1) is

$$(3)\qquad \begin{array}{ccc} T^{m+n+k} & \xrightarrow{\sigma_m\sigma_{n+k}} & T_mT_{n+k} \\ {\scriptstyle \sigma_{m+n}\sigma_k}\downarrow & & \downarrow{\scriptstyle \phi_{m,n+k}} \\ T_{m+n}T_k & \xrightarrow[\phi_{m+n,k}]{} & T_{m+n+k} \end{array}$$

This commutes by (1), each leg being σ_{m+n+k} . Moreover, by the proof of Proposition 1, the pushout of $\sigma_m\sigma_{n+k}$ and $\sigma_{m+n}\sigma_k$ is the universal map which jointly coequalizes the four sets $\tau T^{m-1}T^{n+k}, \ldots, T^{m-1}\tau T^{n+k}$; $T^m\tau T^{n+k-1}, \ldots, T^mT^{n+k-1}\tau$; $\tau T^{m+n-1}T^k, \ldots, T^{m+n-1}\tau T^k$; and $T^{m+n}\tau T^{k-1}, \ldots, T^{m+n}T^{k-1}\tau$. If $n > 0$ we have the overlap that makes this equal to σ_{m+n+k} . □

We now make $n \mapsto T_n$ into a functor $\omega \to A$ from the ordered set $\omega = \{0, 1, 2, \ldots\}$, defining the transition-map $T_n^{n+1} : T_n \to T_{n+1}$ by

$$(4)\qquad \begin{array}{ccc} T_n & \xrightarrow{\tau T_n} & TT_n \\ & {\scriptstyle T_n^{n+1}}\searrow & \downarrow{\scriptstyle \phi_{1n}} \\ & & T_{n+1} \end{array}$$

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and the transition-maps $T_n^m : T_n \to T_m$ for $m \geq n$ by composition. (We use this notation to agree with that of [1]; the reader is unlikely to confuse T_n^m with the power $(T_n)^m$, which will never occur.) Note that (4) is equivalent to its composite with the epimorphism σ_n , which by (1) and naturality is

(5)
$$\begin{array}{ccc} T^n & \xrightarrow{\tau T^n} & T^{n+1} \\ {\scriptstyle \sigma_n}\downarrow & & \downarrow{\scriptstyle \sigma_{n+1}} \\ T_n & \xrightarrow[T_n^{n+1}]{} & T_{n+1} \end{array}$$

by the definition of σ_{n+1} , we could replace τT^n here by $T^i \tau T^{n-i}$ for any i with $0 \leq i \leq n$.

PROPOSITION 3. *For* $m' \geq m$ *and* $n' \geq n$ *we have commutativity in*

(6)
$$\begin{array}{ccc} T_m T_n & \xrightarrow{T_m^{m'} T_n^{n'}} & T_{m'} T_{n'} \\ {\scriptstyle \phi_{mn}}\downarrow & & \downarrow{\scriptstyle \phi_{m'n'}} \\ T_{m+n} & \xrightarrow[T_{m+n}^{m'+n'}]{} & T_{m'+n'} \end{array} .$$

Proof. It suffices to consider the cases $m' = m + 1$, $n' = n$ and $m' = m$, $n' = n + 1$. In these cases we have only to compose (6) with the epimorphism $\sigma_m \sigma_n$, and use (1) and (5). □

REMARK 4. If we regard ω as a monoidal category with tensor product $m + n$, we can see Propositions 2 and 3 as asserting that $\left(T_n, T_n^m\right)$ and ϕ_{mn} constitute a monoidal functor $\omega \to \mathrm{End}\, A$.

We now define T_∞ as the colimit of this functor $n \mapsto T_n$ from ω to A , with colimit-cone $T_n^\infty : T_n \to T_\infty$. Of course it comes to the same

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thing to define T_∞ in one step, as the colimit of the diagram with all the T^n as vertices and all the $T^{i_1}\tau T^{i_2}\tau \ldots \tau T^{i_r} : T^n \to T^m$ as edges; but in some of the applications we shall want to separate the roles of the finite colimits and the filtered ones. Note that, as the colimit of cocontinuous functors, T_∞ is cocontinuous.

Fixing m in $\phi_{mn} : T_m T_n \to T_{m+n}$ and passing the colimit defines $\phi_{m\infty} : T_m T_\infty \to T_\infty$, while fixing n instead defines $\phi_{\infty n} : T_\infty T_n \to T_\infty$; a second passage to the colimit defines $\phi_{\infty\infty} : T_\infty T_\infty \to T_\infty$, which is easily seen to be independent of the order of the passages. Now (6) holds even if some of m, n, m', n' are ∞, provided we set $m + \infty = \infty + n = \infty + \infty = \infty$; whence (2) also holds if some of m, n, k are ∞. Moreover we have $\phi_{0\infty} = \phi_{\infty 0} = 1 : T_\infty \to T_\infty$. Thus $\left(T_n, T_n^m\right)$ and ϕ_{mn} are now extended to a monoidal functor from $\omega + 1 = \omega + \{\infty\}$ to End A. The monoid ∞ in $\omega + 1$ is sent by this to the monoid T_∞ in End A, with multiplication $\phi_{\infty\infty}$ and unit $T_0^\infty : 1 \to T_\infty$. Passage to the colimit in (4) gives

$$\phi_{1\infty}.\tau T_\infty = 1 , \tag{7}$$

exhibiting $\phi_{1\infty} : TT_\infty \to T_\infty$ as an *action* of T on T_∞.

In the light of Sections 14, 17, 22, and 23 of [1], the following result is a consequence of the more general Theorem 8 below.

THEOREM 5. *For a pointed cocontinuous* (T, τ) *, the free-algebra sequence at* $A \in A$ *is given by the* $T_n A$ *with the connecting maps* $T_n^m A$ *and the "approximate actions"* $\phi_{1n}A : TT_nA \to T_{n+1}A$ *. The free* T*-algebra on* A *is the* $T_\infty A$ *to which this converges, with the action* $\phi_{1\infty}$ *; and the unit of this adjunction is* $T_0^\infty A : A \to T_\infty A$ *. The (algebraically) free monad on* T *is given by* $(T_\infty, T_0^\infty, \phi_{\infty\infty})$ *, the unit of this adjunction being* $T_1^\infty : T \to T_\infty$ *. When* A *is monoidal biclosed and* (T, τ) *has the form* $(D \otimes -, \delta \otimes -)$ *, we have* $T_n = D_n \otimes -$ *and* $T_\infty = D_\infty \otimes -$ *, where* D_n *and* D_∞ *are constructed from* $D^{(n)} = D \otimes D \otimes \ldots \otimes D$ *just as* T_n *and* T_∞ *are from* T^n *; and* D_∞ *is the algebraically free monoid on* D *.* □

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4. The algebra-reflexion sequence in the cocontinuous pointed case

We now consider the reflexion into T-Alg of an arbitrary object $(A,\ a : TA \to B)$ of T/A. Again we give the sequence itself as well as the algebra it converges to: partly, once again, to separate the roles of finite and filtered colimits; and partly as a simple way of proving the result.

We set $X_0 = A$ and define X_{n+1} for $n \geq 0$ by the pushout

(8)
$$\begin{array}{ccc} T_nTA & \xrightarrow{\ T_n a\ } & T_nB \\ {\scriptstyle \phi_{n1}A}\downarrow & & \downarrow{\scriptstyle z_n} \\ T_{n+1}A & \xrightarrow[\ y_{n+1}\ ]{} & X_{n+1} \end{array} \ ;$$

noting that, since $\phi_{01} = 1$, we have

(9)
$$X_1 = B\ , \quad y_1 = a : TA \to B\ , \quad z_0 = 1 : B \to B\ .$$

We now set $x_0 : TX_0 \to X_1$ equal to $a : TA \to B$, and define $x_{n+1} : TX_{n+1} \to X_{n+2}$ for $n \geq 0$ by the diagram

(10)
$$\begin{array}{ccccc} TT_{n+1}A & \xrightarrow{\ Ty_{n+1}\ } & TX_{n+1} & \xleftarrow{\ Tz_n\ } & TT_nB \\ {\scriptstyle \phi_{1,n+1}A}\downarrow & & \downarrow{\scriptstyle x_{n+1}} & & \downarrow{\scriptstyle \phi_{1n}B} \\ T_{n+2}A & \xrightarrow[\ y_{n+2}\ ]{} & X_{n+2} & \xleftarrow[\ z_{n+1}\ ]{} & T_{n+1}B \end{array} \ .$$

This defines a unique x_{n+1} since T of (8) is again a pushout and since, by (2), (8), and naturality, we have

$$y_{n+2}.\phi_{1,n+1}A.T\phi_{n1}A = y_{n+2}.\phi_{n+1,1}A.\phi_{1n}TA$$
$$= z_{n+1}.T_{n+1}a.\phi_{1n}TA = z_{n+1}.\phi_{1n}B.TT_na\ .$$

We shall prove:

PROPOSITION 6. *X_n and x_n are the objects and maps so named in*

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Section 17.2 *of* [1].

First, a remark and a lemma. Similar reasoning to that which gave the existence of x_{n+1} in (10), except that this time we compare the pushout T_2 of (8) with the case $n+2$ of (8), gives a diagram like (10) of which we record only the right-hand square:

(11)
$$\begin{array}{ccc} T_2X_{n+1} & \xleftarrow{\;T_2z_n\;} & T_2T_nB \\ {\scriptstyle w_{n+1}}\downarrow & & \downarrow{\scriptstyle \phi_{2n}B} \\ X_{n+3} & \xleftarrow[\;z_{n+2}\;]{} & T_{n+2}B \end{array} \quad .$$

LEMMA 7. *The right-hand square of* (10) *is a pushout for* $n > 0$. *Not so for* $n = 0$ *: we have* $x_1 = z_1$ *, although* $Tz_0 = 1$ *and* $\phi_{10}B = 1$.

Proof. It suffices to prove that for $n > 0$ the outside of

$$\begin{array}{ccccc} TT_nTA & \xrightarrow{\;TT_na\;} & TT_nB & \xrightarrow{\;\phi_{1n}B\;} & T_{n+1}B \\ {\scriptstyle T\phi_{n1}A}\downarrow & & \downarrow{\scriptstyle Tz_n} & & \downarrow{\scriptstyle z_{n+1}} \\ TT_{n+1}A & \xrightarrow[\;Ty_{n+1}\;]{} & TX_{n+1} & \xrightarrow[\;x_{n+1}\;]{} & X_{n+2} \end{array}$$

is a pushout, since the left square is so by (8) and by the cocontinuity of T . But by naturality and (10), this is also the outside of

$$\begin{array}{ccccc} TT_nTA & \xrightarrow{\;\phi_{1n}TA\;} & T_{n+1}TA & \xrightarrow{\;T_{n+1}a\;} & T_{n+1}B \\ {\scriptstyle T\phi_{n1}A}\downarrow & & \downarrow{\scriptstyle \phi_{n+1,1}A} & & \downarrow{\scriptstyle z_{n+1}} \\ TT_{n+1}A & \xrightarrow[\;\phi_{1,n+1}A\;]{} & T_{n+2}A & \xrightarrow[\;y_{n+2}\;]{} & X_{n+2} \end{array} \quad ;$$

and here the right square is a pushout by (8), and the left square by Proposition 2 if $n > 0$. For $n = 0$, we have $z_0 = 1$ by (9) and

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$\phi_{10} = 1$ by Proposition 2, so that $x_1 = z_1$ by (10). □

Proof of Proposition 6. The values of X_0, X_1, and x_0 are those required by (17.3) of [1]. It remains to show that x_{n+1} is the coequalizer in

(12) $$TX_n \overset{\tau TX_n}{\underset{T\tau X_n}{\rightrightarrows}} T^2X_n \xrightarrow{Tx_n} TX_{n+1} \xrightarrow{x_{n+1}} X_{n+2} ,$$

as required by (17.4) of [1]. Since the coequalizer of τT and $T\tau$ is σ_2, which by (1) is ϕ_{11} because $\sigma_1 = 1$, we have equivalently to show that we have a pushout

(13) $$\begin{array}{ccc} T^2X_n & \xrightarrow{Tx_n} & TX_{n+1} \\ {\scriptstyle \phi_{11}X_n}\downarrow & & \downarrow{\scriptstyle x_{n+1}} \\ T_2X_n & \longrightarrow & X_{n+2} \end{array}$$

for $n \geq 0$. When $n = 0$ this reduces to the case $n = 1$ of the pushout (8), since $x_0 = a$ by definition and $x_1 = z_1$ by Lemma 7. To deal with the case $n > 0$ we replace n by $n + 1$ in (13), and compose with $T^2z_n : T^2T_nB \to T^2X_{n+1}$; it suffices to prove this composite a pushout, since T^2z_n is epimorphic - z_n is the pushout (8) of the epimorphism $\phi_{n1}A$, and T^2 preserves epimorphisms. Now the composite $Tx_{n+1}.T^2z_n$ is $Tz_{n+1}.T\phi_{1n}B$ by (10), while the composite $\phi_{11}X_{n+1}.T^2z_n$ is $T_2z_n.\phi_{11}T_nB$ by naturality; so that we must show the outside of

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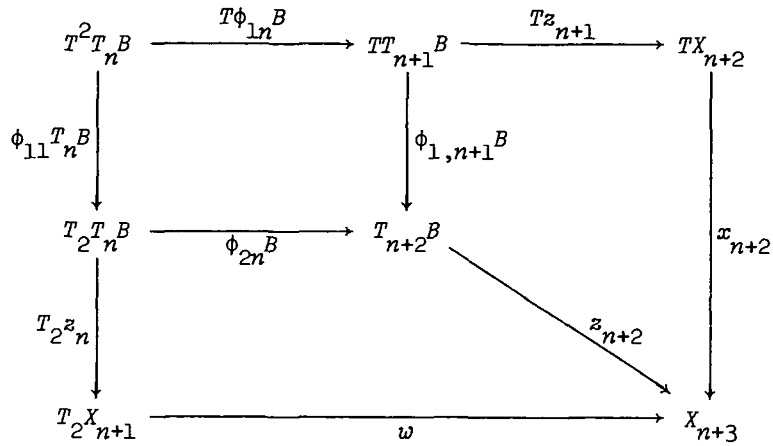
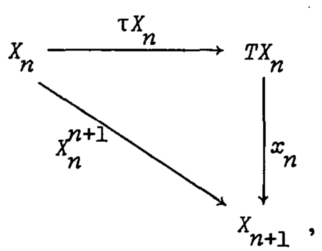
(14)



to be a pushout for some w . But here the top left region is a pushout by Proposition 2, and the right region is a pushout by Lemma 7; so that the two top regions express x_{n+2} as the pushout of $\phi_{11}T_nB$ by $Tz_{n+1}.T\phi_{1n}B$. Moreover the bottom region commutes by (11) if we take w_{n+1} for w , and T_2z_n is epimorphic. Hence the outside of (14) is indeed a pushout. □

Now, as in Section 17.2 of [1], we define the transition-map $X_n^{n+1} : X_n \to X_{n+1}$ by

(15)



making X into a functor $\omega \to A$ with transition-maps X_n^m . Composing (10) with $\tau T_{n+1}A$, using the naturality of τ , and using (4) and (15), we see that *y and z are natural transformations*. This being so, passing to the colimit in (8) gives a closed formula for the colimit X_∞ of $X : \omega \to A$, as the pushout

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(16)
$$\begin{array}{ccc} T_\infty TA & \xrightarrow{T_\infty a} & T_\infty B \\ {\scriptstyle \phi_{\infty 1}A}\big\downarrow & & \big\downarrow{\scriptstyle z_\infty} \\ T_\infty A & \xrightarrow[y_\infty]{} & X_\infty \end{array} .$$

Note that the naturality of z and of y gives, since $z_0 = 1$ and $y_1 = a$, that $X_1^\infty = z_\infty . T_0^\infty B$ and $X_1^\infty . a = y_\infty . T_1^\infty A$. Composing the first of these with $X_0^1 = x_0 . \tau X_0 = a . \tau A$ and the second with $\tau A = T_0^1 A$, we have

(17) $$X_0^\infty = z_\infty . T_0^\infty B . a . \tau A = y_\infty . T_0^\infty A \;,\quad X_1^\infty = z_\infty . T_0^\infty B \;.$$

That the x_n constitute a natural transformation x follows from the general theory in [1], or directly from (15) and (12). We can therefore define $x_\infty : TX_\infty \to X_\infty$ as the colimit of $x_n : TX_n \to X_{n+1}$; which is equally to define x_∞ by the case $n = \infty$ of (10), the right square of which is still a pushout by passage to the colimit in Lemma 7. Now passage to the colimit in (15) gives

(18) $$x_\infty . \tau X_\infty = 1 \;,$$

exhibiting X_∞ as a T-algebra with action x_∞. It follows from (18) that x_∞ is the coequalizer of $\tau X_\infty . x_\infty$ and 1 ; so that by (17.6) of [1] the $x_\omega : TX_\omega \to X_{\omega+1} = X_\omega$ where the sequence converges is indeed x_∞.

Thus:

THEOREM 8. *The algebra-reflexion sequence of* [1] *Section* 17 *at* $(A,\, a : TA \to B)$ *in* T/A *is given by the* X_n *above, with the connecting maps* X_n^m *and the "approximate actions"* $x_n : TX_n \to X_{n+1}$. *The* T*-algebra* $x_\infty : TX_\infty \to X_\infty$ *to which this converges is the reflexion of* $(A,\, a : TA \to B)$ *in* T-Alg, *the unit of this reflexion being* (X_0^∞, X_1^∞). □

We complete the proof of Theorem 5 by observing that the free-algebra sequence at $A \in \mathbf{A}$ is the algebra-reflexion sequence at $(A,\, 1 : TA \to TA)$, and that when $a = 1$ we have $y_{n+1} = 1$ and $z_n = \phi_{n1} A$ by (8), giving

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$x_n = \phi_{1n}A$ by (10) and the definition of x_0 .

5. The case of an unpointed cocontinuous endofunctor

Let H now be any cocontinuous endofunctor of A , and write T for the cocontinuous endofunctor $1 + H$, pointed by the coprojection $\tau : 1 \to 1 + H$. As we said, an H-algebra, given by an action $a : HA \to A$ subject to no axioms, is the same thing as a (T, τ)-algebra. Thus to give the algebra-reflexion sequence explicitly for the reflexion of T/A into H-Alg , we have only to translate the results above.

Since H is cocontinuous, $T^n = (1+H)^n$ is given by the binomial series $\sum \binom{n}{r} H^r$, and it is clear that $\sigma_n : T^n \to T_n$ just identifies the $\binom{n}{r}$ copies of H^r , so that we have

(19) $$T_n = \sum_{r=0}^{n} H^r .$$

Moreover the map $T_n^m : T_n \to T_m$ for $m \geq n$ is the evident coprojection, so that T_∞ is given by the case $n = \infty$ of (19). It is immediate that the maps $\phi_{mn} : T_m T_n \to T_{m+n}$, for finite or infinite m and n , are just those which map the summand $H^r H^s$ of $T_m T_n$ identically onto the summand H^{r+s} of T_{m+n} . Note that T_m and T_n now commute, and that $\phi_{mn} = \phi_{nm}$. As in Theorem 5, if A is monoidal biclosed and H is $E \otimes -$, we have $T_\infty = D_\infty \otimes -$, where now $D_\infty = \sum_{r=0}^{\infty} E^{(r)}$; which is the classical expression for the algebraically-free monoid D_∞ on the unpointed object E of a biclosed A .

Consider now the algebra-reflexion sequence (X_n, x_n) of Section 4 at a general object $a : A + HA \to B$ of T/A , where a has components $u : A \to B$ and $v : HA \to B$. When we simplify (8) by using (19), we easily see that $z_n : T_n B \to X_{n+1}$ is the universal map whose components

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$z_{nr} : H^r B \to X_{n+1}$ render commutative the diagrams

$$\begin{array}{ccc} H^{r+1}A & \xrightarrow{H^{r+1}u} & H^{r+1}B \\ {\scriptstyle H^r v}\downarrow & & \downarrow{\scriptstyle z_{n,r+1}} \\ H^r B & \xrightarrow[z_{nr}]{} & X_{n+1} \end{array}$$

for $0 \le r \le n-1$; which is to say that z_n is the coequalizer in

$$(20) \qquad T_{n-1}HA \underset{T_n^{n+1}HA}{\overset{\theta_{n-1}A}{\rightrightarrows}} \begin{matrix} T_nA \\ T_nHA \end{matrix} \underset{T_n v}{\overset{T_n u}{\rightrightarrows}} T_nB \xrightarrow[z_n]{} X_{n+1} ,$$

where $\theta_{n-1} : T_{n-1}H \to T_n$ is the obvious coprojection.

By (15), the first component of $x_n : X_n + HX_n \to X_{n+1}$ is $X_n^{n+1} : X_n \to X_{n+1}$; write $v_n : HX_n \to X_{n+1}$ for the second component, so that $v_0 = v : HA \to B$. The right square of (10), which since Tz_n is epimorphic is sufficient to fix the value of x_{n+1} , reduces on using $TT_nB = T_nB + HT_nB$ to two squares. One of these merely asserts the naturality of z ; the other fixes the value of v_{n+1} by the commutativity of

$$(21) \qquad \begin{array}{ccc} HT_nB & \xrightarrow{Hz_n} & HX_{n+1} \\ {\scriptstyle \theta_nB}\downarrow & & \downarrow{\scriptstyle v_{n+1}} \\ T_{n+1}B & \xrightarrow[z_{n+1}]{} & X_{n+2} \end{array}$$

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Passing to the colimit in (20) and in (21), we see that the reflexion into H-Alg of $(u, v) : A + HA \to B$ is X_∞ given as the coequalizer in

(22)
$$\begin{array}{ccccc} & & T_\infty A & & \\ & \overset{\theta_\infty A}{\nearrow} & & \overset{T_\infty u}{\searrow} & \\ T_\infty HA & & \xrightarrow[T_\infty v]{} & & T_\infty B \xrightarrow[z_\infty]{} X_\infty , \end{array}$$

with the action $v_\infty : HX_\infty \to X_\infty$ determined by

(23)
$$\begin{array}{ccc} HT_\infty B & \xrightarrow{Hz_\infty} & HX_\infty \\ {\scriptstyle \theta_\infty B}\downarrow & & \downarrow{\scriptstyle v_\infty} \\ T_\infty B & \xrightarrow[z_\infty]{} & X_\infty \end{array} .$$

The unit (X_0^∞, X_1^∞) of the reflexion is given by (17), which here becomes

(24)
$$X_0^\infty = z_\infty . T_0^\infty B . u \ , \quad X_1^\infty = z_\infty . T_0^\infty B \ .$$

6. The case of a cocontinuous monad

The category T-Alg of algebras for a monad $\mathsf{T} = (T, \tau, \mu)$ with unit τ and multiplication μ is a full subcategory of the category T-Alg of algebras for the pointed endofunctor (T, τ), and is hence like T-Alg fully embedded in T/A. Section 24 of [1] gives a new algebra-reflexion sequence $(X_n, x_n : TX_n \to X_{n+1})$ for each $(A, a : TA \to B)$ of T/A, which when it converges gives the reflexion of (A, a) into T-Alg (and no longer into T-Alg). This new sequence again starts with $X_0 = A$, $X_1 = B$, and $x_0 = a : TA \to B$; but now x_{n+1} is the coequalizer in

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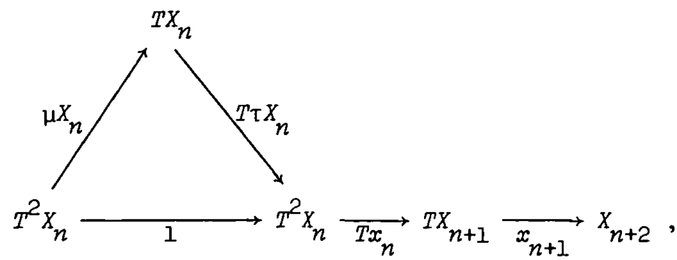
(25) 

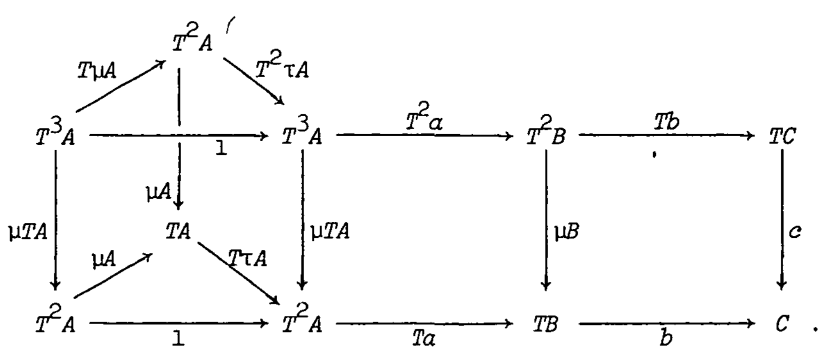
while the connecting-map $x_n^{n+1} : X_n \to X_{n+1}$ is still given by (15).

As remarked in Section 25.2 of [1], the free-algebra sequence, obtained by calculating the algebra-reflexion sequence at $(A, 1 : TA \to TA)$, is now trivial; $X_1 = TA$ is already the free T-algebra on A, and $x_1^2 = 1$. However the algebra-reflexion sequence at a general (A, a), which when it converges allows us to construct colimits in T-Alg and left adjoints to algebraic functors T-Alg → T′-Alg, does not usually converge at a finite index.

It is however otherwise in the special case of a cocontinuous T :

THEOREM 9. *The algebra-reflexion sequence for a cocontinuous monad* T *stops and gives the reflexion into* T-Alg *at the term* X_2 *, the map* x_2^3 *being the identity.*

Proof. We have $X_0 = A$, $X_1 = B$, $x_0 = a$. To avoid subscripts write C for X_2 and $b : TB \to C$ for x_1. Consider the diagram

(26) 

The bottom is the case $n = 0$ of (25), and the top is T of this; since T is cocontinuous, Tb is the coequalizer in the top as b is in the

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bottom. The vertical squares, except the last, commute by naturality and the associative law for μ . It follows that $b.\mu B$ coequalizes T^2a and $T^2a.T^2\tau A.T\mu a$, and hence factorizes through their coequalizer Tb as $c.Tb$ for some $c : TC \to C$.

We claim that c is the coequalizer x_2 of Tb and $Tb.T\tau B.\mu B$. For suppose that some $f : C \to D$ satisfies $f.Tb = f.Tb.T\tau B.\mu B$. Then $f.Tb$ factorizes through the coequalizer μB of 1 and $T\tau B.\mu B$ as $f.Tb = g.\mu B$ for some $g : TB \to D$. Now $f.Tb$ coequalizes T^2a and $T^2a.T^2\tau A.T\mu A$, since Tb already does so; hence $g.\mu B$ does so. By the commutativity in (26), it follows that g coequalizes $Ta.\mu TA$ and $Ta.T\tau A.\mu A.\mu TA$. Since μTA is a retraction, g already coequalizes Ta and $Ta.T\tau A.\mu A$, and hence factorizes through their coequalizer b as $g = hb$ for some $h : C \to D$. Thus $f.Tb = g.\mu B = hb.\mu B = hc.Tb$, giving $f = hc$ since Tb is epimorphic. Since c too is epimorphic because b and μB are, the factorization of f through c is unique, and c is indeed the coequalizer x_2 .

It remains to show that $X_2^3 = c.\tau C$ is 1 . But $c.\tau C.b = c.Tb.\tau TB$ by naturality, which by the commutativity of the right-most square in (26) is $b.\mu B.\tau TB$, which is b . Since b is epimorphic, we do have $c.\tau C = 1$. □

Reference

[1] G.M. Kelly, "A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on", *Bull. Austral. Math. Soc.* 22 (1980), 1-83.

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https://doi.org/10.1017/S0004972700005724 Published online by Cambridge University Press