LETTER TO THE EDITOR

Dear Editor,

Asymptotics of the Luria–Delbrück distribution via singularity analysis

Dedicated to Kerry Livgren

We consider the Luria–Delbrück distribution, as described in Kemp (1994) and previous writings cited therein. It is given by the probability generating function

$$G(z)=(1-z)^{m(1-z)/z}=\sum_{n\geq 0} p_n z^n,$$

where m is a parameter.

We show that with *singularity analysis of generating functions* (Flajolet and Odlyzko 1990), in principle, a full asymptotic expansion of p_n and $P_n := \Sigma_{k>n} p_k$ can be achieved. In this way we generalize results given in Pakes (1993), Kemp (1994), and Goldie (1995).

The basic idea of this powerful method is to realize that $G(z)$ has radius of convergence 1 and that 1 is the only singularity on the radius of convergence. Then there is a *local expansion* about $z=1$ (produced by Maple),

$$G(z)=1-m(1-z)\log\frac{1}{1-z}-m(1-z)^2\log\frac{1}{1-z}$$

(1)
$$+\frac{m^2}{2}(1-z)^2\log^2\frac{1}{1-z}+\mathcal{O}\left((1-z)^3\log^3\frac{1}{1-z}\right),$$

which can be translated *term-by-term* into an asymptotic expansion of the coefficients under some mild technical conditions which are satisfied in our instance; we refer again to Flajolet and Odlyzko (1990) for a full treatment.

Therefore we only have to know about the expansion of the standard functions appearing in (1).

We use the following notation: $[z^n]f(z)$ means the coefficient of z^n in (the Taylor expansion of) $f(z)$, the symbol $(n)_k$ means $n(n-1)\cdots(n-k+1)$ and $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+1/n$ means the nth harmonic number. Then we have for n large enough

$$[z^n](1-z)^i \log \frac{1}{1-z} = \frac{(-1)^i\, i!}{(n)_{i+1}},$$

$$[z^n]\log^2 \frac{1}{1-z} = \frac{2}{n} H_{n-1},$$

$$[z^n](1-z)^2 \log^2 \frac{1}{1-z} = \frac{2}{n} H_{n-1} - \frac{4}{n-1} H_{n-2} + \frac{2}{n-2} H_{n-3}.$$

Hence

$$p_n = \frac{m}{(n)_2} + \frac{2m^2}{(n)_3} + \frac{m^2}{2}\left(\frac{2}{n} H_{n-1} - \frac{4}{n-1} H_{n-2} + \frac{2}{n-2} H_{n-3}\right) + \mathcal{O}\left(\frac{\log^2 n}{n^4}\right).$$

Maple also translates this expansion into more traditional terms:

$$p_n = \frac{m}{n^2} + 2m^2 \frac{\log n}{n^3} + \frac{2m^2\gamma - 3m^2 - m}{n^3} + \mathcal{O}\left(\frac{\log^2 n}{n^4}\right).$$

To get the asymptotic expansions for the tails P_n, we just note that their generating function is

$$\frac{1}{1-z} - \frac{1}{1-z} G(z) = m \log \frac{1}{1-z} + m(1-z)\log \frac{1}{1-z}$$
$$- \frac{m^2}{2}(1-z)\log^2 \frac{1}{1-z} + \mathcal{O}\left((1-z)^2 \log^3 \frac{1}{1-z}\right),$$

from which we conclude

$$P_n = \frac{m}{n} - \frac{m}{(n)_2} - \frac{m^2}{2}\left(\frac{2}{n} H_{n-1} - \frac{2}{n-1} H_{n-2}\right) + \mathcal{O}\left(\frac{\log^2 n}{n^3}\right)$$
$$= \frac{m}{n} + m^2 \frac{\log n}{n^2} + \frac{m^2\gamma - m^2 - m}{n^2} + \mathcal{O}\left(\frac{\log^2 n}{n^3}\right).$$

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Yours sincerely,

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