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ISOMETRIES BETWEEN UNIT SPHERES OF THE ℓ^∞-SUM OF STRICTLY CONVEX NORMED SPACES

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Abstract

We prove that any surjective isometry between unit spheres of the ℓ^∞-sum of strictly convex normed spaces can be extended to a linear isometry on the whole space, and we solve the isometric extension problem affirmatively in this case.

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1. Introduction and preliminaries

In 1987, Tingley [8] proposed the following problem.

Problem 1.1. Let E and F be real normed spaces. Suppose that V is a surjective isometry between the unit spheres $S(E)$ and $S(F)$. Is V necessarily the restriction of a linear isometry on the whole space?

We only consider this problem in real normed spaces, since it is clearly negative in the complex case. For example, consider $V : \mathbb{C} \to \mathbb{C}$ which maps x to $\bar{x}$.

The isometric extension problem arises from the famous Mazur–Ulam theorem. This problem is interesting and easy to understand. Moreover, it is very important. If this problem has a positive answer, then the local geometric property of a mapping on the unit sphere will determine the property of the mapping on the whole space. However, it is very difficult to solve. In [8], Tingley only proved that any isometry V between the unit spheres of real finite-dimensional normed spaces $X_{(n)}$ and $Y_{(m)}$ necessarily maps the antipodal points to antipodal points, that is, $V(-x) = -V(x)$ for any $x \in S(X_{(n)})$.

In the past decade, the isometric extension problem has been considered in various classical Banach spaces and many good results have been obtained, through studying the concrete form of the norm (see [2]). The isometric extension problem has been

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solved affirmatively if X is a classical normed space and Y is a general normed space. However, there has been little progress between two general normed spaces, even in the two-dimensional case. Recently, this problem was solved affirmatively between finite-dimensional polyhedral Banach spaces (see [6]). There is still much to be done regarding this challenging problem.

In particular, this problem was solved affirmatively between the unit spheres of the ℓ^1-sum of strictly convex normed spaces (see [9]), through the representation of the surjective isometry. The isometries on the unit spheres of the space $\ell^\infty(\Gamma)$ (see [3–5, 7]) were also considered. In this paper, we consider the isometric extension problem between the ℓ^∞-sum of a family of strictly convex normed spaces, and solve the isometric extension problem affirmatively in this case. Furthermore, we can obtain the main results of [4] as a corollary.

Before we start, we introduce some notation. Let Γ be an index set with at least two elements. Let $\{E_\gamma : \gamma \in \Gamma\}$ be a collection of strictly convex normed spaces, and let the linear space $\oplus E_\gamma$ be the direct sum of these normed spaces (see [1]). Suppose that

$$\bigoplus_{\ell^\infty} E_\gamma := \{u = (u_\gamma)_{\gamma\in\Gamma} \in \oplus E_\gamma : \|u\| = \sup\{\|u_\gamma\| : \gamma \in \Gamma\} < \infty\}$$

is a subspace of $\oplus E_\gamma$ with the ℓ^∞-norm. For any $\gamma_0 \in \Gamma$ and $x \in E_{\gamma_0}$, we denote $\hat{x} \in \oplus_{\ell^\infty} E_\gamma$ by

$$\hat{x}_\gamma = \begin{cases} x & \text{if } \gamma = \gamma_0, \\ 0 & \text{if } \gamma \neq \gamma_0. \end{cases}$$

Note that $S(E) := \{u \in E : \|u\| = 1\}$ and $B_1(E) := \{u \in E : \|u\| \le 1\}$. For any $u \in S(\oplus_{\ell^\infty} E_\gamma)$ and $0 < \varepsilon < 1$, we write

$$\operatorname{supp} u = \{\gamma \in \Gamma : u_\gamma \neq 0\},$$

and denote $u^\varepsilon \in S(\oplus_{\ell^\infty} E_\gamma)$ by

$$u^\varepsilon_\gamma = \begin{cases} u_\gamma & \text{if } \|u_\gamma\| > 1 - \varepsilon, \\ 0 & \text{if } \|u_\gamma\| \le 1 - \varepsilon. \end{cases}$$

2. Isometries between $S(\oplus_{\ell^\infty} E_\gamma)$ and $S(\oplus_{\ell^\infty} F_\delta)$

Let Δ be an index set with at least two elements. In this section, we similarly write $\oplus_{\ell^\infty} F_\delta$ for the ℓ^∞-sum of a collection of strictly convex normed spaces $\{F_\delta : \delta \in \Delta\}$. We begin by considering the isometry between $S(\oplus_{\ell^\infty} E_\gamma)$ and $S(\oplus_{\ell^\infty} F_\delta)$. In the following result, we prove that any surjective isometry between them necessarily maps antipodal points to antipodal points.

PROPOSITION 2.1. *Let $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ be a surjective isometry. Then $V(-u) = -V(u)$ for any $u \in S(\oplus_{\ell^\infty} E_\gamma)$.*

Proof. We first prove that $V(-\hat{x}) = -V(\hat{x})$ for any $\gamma_0 \in \Gamma$ and $x \in S(E_{\gamma_0})$. Note that V is surjective. There exists $u \in S(\oplus_{\ell^\infty} E_\gamma)$ such that $V(u) = -V(\hat{x})$. Then

$$\|u - \hat{x}\| = \|V(u) - V(\hat{x})\| = \| -2V(\hat{x})\| = 2,$$

and thus $\|u_{\gamma_0} - x\| = 2$. Note that

$$2 = \|u_{\gamma_0} - x\| \leq \|u_{\gamma_0}\| + \|x\| \leq 2,$$

and E_1 is strictly convex. We get that $u_{\gamma_0} = -x$. For any $y \in S(E_{\gamma_1})$ with $\gamma_1 \neq \gamma_0$, there exists $v \in S(\oplus_{\ell^\infty} E_\gamma)$ such that $V(v) = -V(\hat{y})$. Note that

$$\begin{aligned}\|v_{\gamma_0} - u_{\gamma_0}\| \leq \|v - u\| &= \| -V(\hat{y}) - (-V(\hat{x}))\| \\ &= \|V(\hat{y}) - V(\hat{x})\| = \|\hat{x} - \hat{y}\| = 1.\end{aligned}$$

We have

$$\|v_{\gamma_0} + u_{\gamma_0}\| \geq \|2 \cdot u_{\gamma_0}\| - \|v_{\gamma_0} - u_{\gamma_0}\| \geq 1, \tag{2.1}$$

and thus

$$\begin{aligned}\|u - \hat{y}\| &= \|V(\hat{y}) + V(\hat{x})\| = \|\hat{x} - v\| \\ &\geq \|x - v_{\gamma_0}\| = \|u_{\gamma_0} + v_{\gamma_0}\| \geq 1.\end{aligned}$$

If $\|u - \hat{y}\| = 1$, we get that $\|u_{\gamma_0} + v_{\gamma_0}\| = 1$, and thus $\|u_{\gamma_0} - v_{\gamma_0}\| = 1$ by (2.1). Note that E_{γ_0} is strictly convex and

$$\|v_{\gamma_0} + u_{\gamma_0}\| + \|v_{\gamma_0} - u_{\gamma_0}\| = 2\|u_{\gamma_0}\|.$$

Then $v_{\gamma_0} = 0$ or $u_{\gamma_0} = 0$, which is a contradiction. Therefore, $\|u - \hat{y}\| > 1$, and thus $\|u_{\gamma_1} - y\| > 1$. Note that $y \in S(E_{\gamma_1})$ is arbitrary. We get that $u_{\gamma_1} = 0$ and thus $u = -\hat{x}$.

Now we prove that $V(-u) = -V(u)$ for any $u \in S(\oplus_{\ell^\infty} E_\gamma)$. Note that V is surjective. There exists $v \in S(\oplus_{\ell^\infty} E_\gamma)$ such that $V(v) = -V(u)$. For any $\gamma_0 \in \Gamma$ with $u_{\gamma_0} \neq 0$,

$$1 + \|u_{\gamma_0}\| = \left\|\frac{\widehat{u_{\gamma_0}}}{\|u_{\gamma_0}\|} + u\right\| = \left\|V\left(\frac{\widehat{u_{\gamma_0}}}{\|u_{\gamma_0}\|}\right) - (-V(u))\right\| = \left\|\frac{\widehat{u_{\gamma_0}}}{\|u_{\gamma_0}\|} - v\right\|,$$

and thus

$$1 + \|u_{\gamma_0}\| = \left\|\frac{u_{\gamma_0}}{\|u_{\gamma_0}\|} - v_{\gamma_0}\right\| \leq 1 + \|v_{\gamma_0}\|. \tag{2.2}$$

Therefore, $v_{\gamma_0} \neq 0$ and thus

$$\begin{aligned}1 + \|v_{\gamma_0}\| &= \left\|\frac{\widehat{v_{\gamma_0}}}{\|v_{\gamma_0}\|} + v\right\| = \left\|V\left(\frac{\widehat{v_{\gamma_0}}}{\|v_{\gamma_0}\|}\right) - (-V(v))\right\| \\ &= \left\|\frac{\widehat{v_{\gamma_0}}}{\|v_{\gamma_0}\|} - u\right\| = \left\|\frac{v_{\gamma_0}}{\|v_{\gamma_0}\|} - u_{\gamma_0}\right\| \leq 1 + \|u_{\gamma_0}\|.\end{aligned}$$

Then $\|v_{\gamma_0}\| = \|u_{\gamma_0}\|$. Note that E_{γ_0} is strictly convex and see (2.2). We get $u_{\gamma_0} = -v_{\gamma_0}$. For any $\gamma_0 \in \Gamma$ with $v_{\gamma_0} \neq 0$, we can get a similar result by the methods above, and complete the proof. □

Proposition 2.2. *Let* $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ *be a surjective isometry. For any* $u \in S(\oplus_{\ell^\infty} E_\gamma)$*,* supp u *is a single point if and only if* supp $V(u)$ *is a single point.*

Proof. Assume that there exist $\gamma_0 \in \Gamma$ and $x \in S(E_{\gamma_0})$ such that $u = \hat{x}$. Let $h = V(u)$. We first prove that $h^\varepsilon = h$ for any $0 < \varepsilon < 1$. In other words, $\|h_\delta\| = 0$ or $\|h_\delta\| = 1$ for any $\delta \in \Delta$. Otherwise, there exists $\varepsilon_0 > 0$ such that $h^{\varepsilon_0} \neq h$. Note that V is surjective. There exist $v, w \in S(\oplus_{\ell^\infty} E_\gamma)$ such that

$$V(v) = h^{\varepsilon_0}, \qquad V(w) = \frac{h - h^{\varepsilon_0}}{\|h - h^{\varepsilon_0}\|}.$$

Note that E_{γ_0} is strictly convex and

$$\|v_{\gamma_0} + x\| = \|v + \hat{x}\| = \|h + h^{\varepsilon_0}\| = 2,$$

by Proposition 2.1.

We have

$$2 = \|v_{\gamma_0} + x\| \leq \|v_{\gamma_0}\| + \|x\| \leq 2,$$

and thus $v_{\gamma_0} = x$. It is now the case that

$$\|w_{\gamma_0} + x\| = \|w + \hat{x}\| = 1 + \|h - h^{\varepsilon_0}\|$$

and

$$\|w_{\gamma_0} + v_{\gamma_0}\| \leq \|w + v\| = \left\| h^{\varepsilon_0} + \frac{h - h^{\varepsilon_0}}{\|h - h^{\varepsilon_0}\|} \right\| = 1,$$

by Proposition 2.1. This is a contradiction.

We now prove that supp h is a single point. Suppose that there exist $\delta_1, \delta_2 \in \Delta$ such that $\|h_{\delta_1}\| = \|h_{\delta_2}\| = 1$. Note that V is surjective. There exist $v, w \in S(\oplus_{\ell^\infty} E_\gamma)$ such that

$$V(v) = \widehat{h_{\delta_1}}, \qquad V(w) = \widehat{h_{\delta_2}}.$$

Note that E_{γ_0} is strictly convex and

$$\|v_{\gamma_0} + x\| = \|v + \hat{x}\| = \|\widehat{h_{\delta_1}} + h\| = 2,$$

by Proposition 2.1. We get that

$$2 = \|v_{\gamma_0} + x\| \leq \|v_{\gamma_0}\| + \|x\| \leq 2,$$

and thus $v_{\gamma_0} = x$. Similarly, we have $w_{\gamma_0} = x$. By Proposition 2.1,

$$\|v_{\gamma_0} + w_{\gamma_0}\| \leq \|v + w\| = \|\widehat{h_{\delta_1}} + \widehat{h_{\delta_2}}\| = 1,$$

which is a contradiction. This completes the proof. □

Proposition 2.3. *Let* $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ *be a surjective isometry. For any* $\gamma \in \Gamma$*, there exists* $\delta \in \Delta$ *such that* $V(S(E_\gamma)) = S(F_\delta)$*.*

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PROOF. Note that V is surjective and recall Proposition 2.2. We only need to prove that $\operatorname{supp} V(\hat{x}) = \operatorname{supp} V(\hat{x'})$, for any $\gamma_0 \in \Gamma$ and $x, x' \in S(E_{\gamma_0})$. Otherwise, there exist $y \in S(F_\delta)$ and $y' \in S(F_{\delta'})$ with $\delta \neq \delta'$ such that

$$V(\hat{x}) = \hat{y}, \quad V(\hat{x'}) = \hat{y'}.$$

Then

$$\|x - x'\| = \|\hat{x} - \hat{x'}\| = \|\hat{y} - \hat{y'}\| = 1$$

and

$$\|x + x'\| = \|\hat{x} + \hat{x'}\| = \|\hat{y} + \hat{y'}\| = 1,$$

by Proposition 2.1. Note that E_{γ_0} is strictly convex and

$$\|x + x'\| + \|x - x'\| = 2 = \|(x + x') + (x - x')\|.$$

Then $x = 0$ or $x' = 0$, which is a contradiction. This completes the proof. □

REMARK 2.4. Let $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ be a surjective isometry. By Proposition 2.3, there exists a bijection $\sigma : \Gamma \to \Delta$ such that $V(S(E_\gamma)) = S(F_{\sigma(\gamma)})$. We define $V_\gamma : S(E_\gamma) \to S(F_{\sigma(\gamma)})$ by

$$V_\gamma(x) = V(\hat{x}), \ \forall x \in S(E_\gamma).$$

Then $V_\gamma(S(E_\gamma)) = S(F_{\sigma(\gamma)})$.

PROPOSITION 2.5. *Let $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ be a surjective isometry, and let $\sigma : \Gamma \to \Delta$ be the bijection in Remark 2.4. Suppose that $u \in S(\oplus_{\ell^\infty} E_\gamma)$ and $V(u) = v$. Then*

$$v_{\sigma(\gamma)} = \begin{cases} \|u_\gamma\| V\Big(\widehat{\dfrac{u_\gamma}{\|u_\gamma\|}}\Big) & \text{if } u_\gamma \neq 0, \\ 0 & \text{if } u_\gamma = 0. \end{cases}$$

PROOF. We first prove that $\|u_\gamma\| \leq \|v_{\sigma(\gamma)}\|$ for any $\gamma \in \Gamma$. If $\|u_\gamma\| > 0$, there exists $y \in S(F_{\sigma(\gamma)})$ such that $V(\widehat{u_\gamma/\|u_\gamma\|}) = \hat{y}$ by Proposition 2.3 and Remark 2.4. Then

$$\begin{aligned} 1 + \|u_\gamma\| &= \Big\| u + \widehat{\frac{u_\gamma}{\|u_\gamma\|}} \Big\| = \|v + \hat{y}\| \\ &= \|v_{\sigma(\gamma)} + y\| \leq 1 + \|v_{\sigma(\gamma)}\|, \end{aligned}$$

and thus $\|u_\gamma\| \leq \|v_{\sigma(\gamma)}\|$.

If $\|v_{\sigma(\gamma)}\| = 0$, we have $\|u_\gamma\| = 0$. If $\|v_{\sigma(\gamma)}\| \neq 0$, there exists $x \in S(E_\gamma)$ such that $V(\hat{x}) = \widehat{v_{\sigma(\gamma)}/\|v_{\sigma(\gamma)}\|}$. Therefore,

$$1 + \|v_{\sigma(\gamma)}\| = \Big\| \frac{\widehat{v_{\sigma(\gamma)}}}{\|v_{\sigma(\gamma)}\|} + v \Big\| = \|\hat{x} + u\|,$$

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by Proposition 2.1 and thus

$$1 + \|v_{\sigma(\gamma)}\| = \|x + u_\gamma\| \le 1 + \|u_\gamma\|.$$

Then $\|v_{\sigma(\gamma)}\| = \|u_\gamma\|$ and thus

$$1 + \|u_\gamma\| = \|x + u_\gamma\|.$$

Note that E_γ is strictly convex. We get that $u_\gamma = \|u_\gamma\|x$ and thus

$$\widehat{v_{\sigma(\gamma)}} = \|v_{\sigma(\gamma)}\|V(\hat{x}) = \|u_\gamma\|V\Big(\widehat{\frac{u_\gamma}{\|u_\gamma\|}}\Big),$$

which completes the proof. □

Theorem 2.6. *Let $V : S(\oplus_{\ell^\infty} E_\gamma) \to S(\oplus_{\ell^\infty} F_\delta)$ be a surjective isometry. Then V can be extended to a linear isometry on the whole space.*

Proof. For any $\gamma \in \Gamma$, $V_\gamma : S(E_\gamma) \to S(F_{\sigma(\gamma)})$ defined in Remark 2.4 can be seen as a surjective isometry between $S(E_\gamma)$ and $S(F_{\sigma(\gamma)})$. We then define $\tilde{V}_\gamma : E_\gamma \to F_{\sigma(\gamma)}$ by

$$\tilde{V}_\gamma(x) = \begin{cases} 0 & \text{if } x = 0; \\ \|x\|V_\gamma\Big(\dfrac{x}{\|x\|}\Big) & \text{if } x \neq 0. \end{cases}$$

It is clear that $\tilde{V}_\gamma$ is surjective. Now we prove that $\tilde{V}_\gamma$ is an isometry. Note that

$$\tilde{V}_\gamma(\lambda x) = \lambda \tilde{V}_\gamma(x), \quad \forall x \in E_\gamma, \lambda \in \mathbb{R},$$

by Proposition 2.1. We only need to prove that

$$\|\tilde{V}_\gamma(x) - \tilde{V}_\gamma(x')\| = \|x - x'\|$$

for any $x, x' \in B_1(E_\gamma)$. In fact, for any $\gamma_1 \neq \gamma$ and $y \in S(E_{\gamma_1})$, let $u = \hat{x} + \hat{y}$ and $v = \hat{x}' + \hat{y}$. It is clear that $u, v \in S(\oplus_{\ell^\infty} E_\gamma)$. Then

$$\begin{aligned} \|\tilde{V}_\gamma(x) - \tilde{V}_\gamma(x')\| &= \|V(u) - V(v)\| \\ &= \|u - v\| = \|x - x'\| \end{aligned}$$

for any $x, x' \in B_1(E_\gamma)$, by Proposition 2.5. By the Mazur–Ulam theorem, $\tilde{V}_\gamma$ is a linear map.

We now define $\tilde{V} : \oplus_{\ell^\infty} E_\gamma \to \oplus_{\ell^\infty} E_\delta$ by

$$\tilde{V}(u)_\delta = \widetilde{V_{\sigma^{-1}(\delta)}}(u_{\sigma^{-1}(\delta)})$$

for any $\delta \in \Delta$ and $u \in \oplus_{\ell^\infty} E_\gamma$. By Proposition 2.5, $\tilde{V}|_{S(\oplus_{\ell^\infty} E_\gamma)} = V$ and $\tilde{V}$ is linear. This completes the proof. □

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