BULL. AUSTRAL. MATH. SOC. 05C20, 05C35
VOL. 19 (1978), 205-216.

Path decompositions of digraphs

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Let $G = (X, U)$ be a digraph of order n. $P(G)$ denotes the minimal cardinal of a path-partition of the arcs of G.

Oystein Ore, *Theory of graphs* (Amer. Math. Soc., Providence, Rhode Island, 1962) has proved that $P(G) \geq \sum_{x \in X_G^+} \left(d_G^+(x) - d_G^-(x)\right)$, where $X_G^+ = \left\{x \in X \mid d_G^+(x) > d_G^-(x)\right\}$. We say that G satisfies Q if the preceeding inequality is an equality.

We give some properties of the digraphs satisfying Q, and in particular the case where G is strongly connected. Then we prove that $P(G) \leq \left[n^2/4\right] - 2$, and that this result is the best possible. Next we exhibit the existence of digraphs with circuits such that $P(G) = \left[n^2/4\right]$.

Finally we prove that if G is a strongly connected digraph of order n which satisfies Q, then there exists a strongly connected digraph H of order $n + 1$ having G as a sub-digraph and satisfying Q, too.

1. Introduction

1.1. The notations are those of Berge [5].

A digraph $G = (X, U)$ is a non-empty finite set X (the vertices), together with a finite family U of ordered pairs of vertices (the arcs). A simple digraph is a digraph without parallel arcs and loops.

In this paper we only consider simple digraphs. The digraph obtained

Received 8 August 1978. Communicated by Michel Chein.

https://doi.org/10.1017/S0004972700008698 Published online by Cambridge University Press

from G by deleting a vertex $x \in X$ and its adjacent arcs will be denoted by $G - x$.

We denote by $(x_1, x_2, \ldots, x_k)$ (respectively $(x_1, x_2, \ldots, x_k, x_1)$) the elementary path (respectively the circuit) containing the k distinct vertices $x_1, \ldots, x_k$. Let R be a family of elementary paths of G . If each arc of G lies on exactly one element of R then R is a path-partition of G . We denote by $P(G)$ the minimal cardinality of a path-partition of a digraph G .

From now on we denote

$$\Gamma_G^+(x) = \{y \in X \mid (x, y) \in U\} , \quad \left(d_G^+(x) = |\Gamma_G^+(x)|\right) ,$$

$$\Gamma_G^-(x) = \{y \in X \mid (y, x) \in U\} , \quad \left(d_G^-(x) = |\Gamma_G^-(x)|\right) ,$$

$$X_G^+ = \{x \in X \mid d_G^+(x) > d_G^-(x)\} ,$$

$$X_G^0 = \{x \in X \mid d_G^+(x) = d_G^-(x)\} ,$$

$$X_G^- = X - \left(X_G^+ \cup X_G^0\right) .$$

From Ore [9], we have

$$P(G) \geq \sum_{x \in X_G^+} \left(d_G^+(x) - d_G^-(x)\right) .$$

Alspach and Pullman [4], have conjectured that for any simple digraph G or order n , $P(G) \leq [n^2/4]$.

O'Brien [8] proved this conjecture. For a further detailed study of the index $P(G)$, we refer also to Chaty, Chein ([6], [7]).

DEFINITION 1.2. Let $G = (X, U)$ be a digraph of order n ; if $P(G) = \sum_{x \in X_G^+} \left(d_G^+(x) - d_G^-(x)\right)$ we say G has the property Q . In the following, we denote by $e(G)$ the sum $e(G) = \sum_{x \in X_G^+} \left(d_G^+(x) - d_G^-(x)\right)$.

2. Results

LEMMA 2.1. *Let* $G = (X, U)$ *be a digraph of order* n *and* $v \in X_G^+$. *If* G *satisfies the following conditions,*

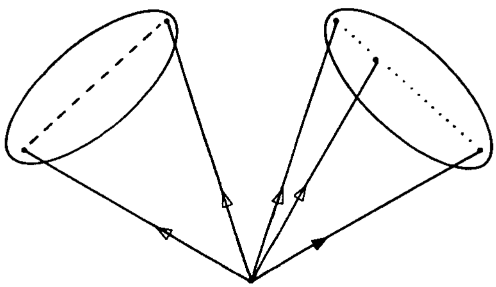
(i) $d_G^-(v) = 0$,

(ii) $P(G-v) = e(G-v)$ *(that is* $G - v$ *has the property* Q *)*,

then G *has the property* Q .

Proof. $X_{G-v}^+ = (X_G^+-\{v\}) \cup \left[X_G^0 \cap \Gamma_G^+(v)\right]$, and $P(G-v) = e(G-v)$. If $x \in X_{G-v}^+$, then $d_{G-v}^+(x) = d_G^+(x)$ and $d_{G-v}^-(x) = d_G^-(x) - 1$. Moreover, for $x \in X_{G-v}^+ - (X_{G-v}^+ \cap \Gamma_G^+(v))$, we have

$$d_{G-v}^+(x) = d_G^+(x) \quad \text{and} \quad d_{G-v}^-(x) = d_G^-(x) .$$



X_{G-v}^+ v $(X-\{v\}) - X_{G-v}^+$

But in $G - v$, through each vertex $x \in X_{G-v}^+ \cap \Gamma_G^+(v)$ there pass $d_{G-v}(x) - d_{G-v}^-(x) = (d_G^+(x)-d_G^-(x)) + 1$ elementary paths of origin x , which belong to a path-partition R of the arcs of the digraph, the cardinal of R being $P(G-v)$. Among those paths of origin x , consider the path $\lambda = (x, \ldots)$. Since $(v, x) \in U$, the path λ allows the construction in G of the path $\mu = (v, x, \ldots)$ of origin v . Thus the number of paths of origin x in G becomes $(d_G^+(x)-d_G^-(x))$. Moreover, for each $x \in \Gamma_G^+(v) - (X_{G-v}^+ \cap \Gamma_G^+(v))$, we construct the path (v, x) of origin v in G . Let R' be the set of elementary paths obtained from R by cancelling those paths λ which have been used to define the path μ of origin v in G . Let T be the following set of elementary paths:

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$$T = R' \cup \{\mu = (v, x, \ldots) \mid x \in X^+_{G-v} \cap \Gamma^+_G(v)\} \\ \cup \{(v, x) \mid x \in \Gamma^+_G(v) - (X^+_{G-v} \cap \Gamma^+_G(v))\} .$$

It is obvious that the set T partitions the arcs of G, and we have

$$|T| \le e(G) \le P(G) .$$

Therefore $P(G) = e(G)$.

From the preceding lemma, we deduce the following theorem.

THEOREM 2.2 (Ore [9]). *Let $G = (X, U)$ be a digraph without circuit; then*

$$P(G) = e(G) .$$

COROLLARY 1 (Alspach and Pullman [4]). *If TT_n is the transitive tournament of order n we have*

$$P(TT_n) = [n^2/4] .$$

REMARKS. (1) If we replace the condition *(i)* of the lemma by the condition *(i')*, $d^+_G(v) = 0$, we get a similar result. Moreover, the preceding lemma allows us to construct from a digraph of order $(n-1)$ satisfying Q, another digraph of order n still satisfying Q.

(2) By that lemma, we can define an algorithm which allows the construction of a path-partition of a digraph without circuit.

The following lemma is due to Alspach, Mason, Pullman [3].

LEMMA 2.3. *Let $G = (X, U)$ be a digraph of order n satisfying Q and (x, y) an arc of G such that $x \in X - X^+_G$ and $y \in X^+_G \cup X^0_G$. If H is the digraph obtained from G by reversing the arc (x, y), then H satisfies Q and $P(H) = P(G) + 2 = e(H)$.*

THEOREM 2.4. *Consider a strongly connected digraph $G = (X, U)$ satisfying $P(G) = e(G)$. Then we have*

$$P(G) \le [n^2/4] - 2 .$$

Proof. Suppose that $P(G) \ge [n^2/4] - 1$. Since G is strongly connected, there exist $x \in X - X^+_G$ and $y \in X^+_G$ such that $(x, y) \in U$.

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Denote by G_1 the digraph obtained from G by reversing the arc (x, y). By Lemma 2.3, G_1 satisfies Q and we have

$$P(G_1) = P(G) + 2 \geq [n^2/4] + 1 ,$$

which is a contradiction to the fact that $P(G_1) \leq [n^2/4]$. Thus we necessarily have $P(G) \leq [n^2/4] - 2$.

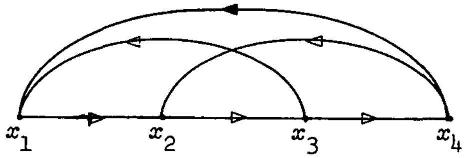
We show that the result of Theorem 2.4 is the best possible.

REMARKS. (1) Let $T_n = (X, U)$ be a tournament of order n. It is easy to verify that $P(T_n) \geq [(n+1)/2]$; therefore

$$[(n+1)/2] \leq P(T_n) \leq [n^2/4] .$$

(2) Let $A_n = (X, U)$ be the strongly connected c-minimal tournament[1] of order n (that is $A_n = (X, U)$ admits exactly $((n-1)(n-2))/2$ elementary circuits). Let us study some particular cases.

Case $n = 4$.



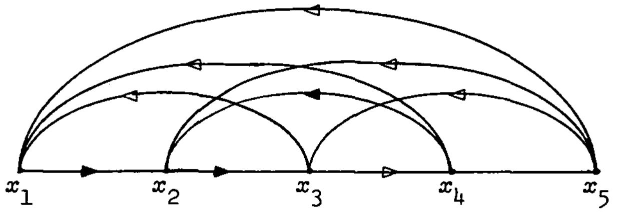
Let x_1, x_2, x_3, x_4 be the canonical indexation of the vertices of A_4. We have

$$P(A_4) \geq e(A_4) = 2 .$$

But the set $\{(x_4, x_2, x_3, x_1), (x_3, x_4, x_1, x_2)\}$ of elementary paths forms a partition of the arcs of A_4. Therefore $P(A_4) = [n^2/4] - 2 = 2$.

[1] A complete study of strongly connected c-minimal tournaments will be found in Abdul-Kader [1].

Case $n = 5$.



Let x_1, x_2, x_3, x_4, x_5 be the canonical indexation of the vertices of A_5 . We have

$$P(A_5) \geq e(A_5) = 4 .$$

But the set

$$\{(x_5, x_2, x_3, x_1), (x_5, x_3, x_4, x_2), (x_4, x_1, x_2), (x_4, x_5, x_1)\}$$

of elementary paths forms a partition of the arcs of A_5 . Therefore

$$P(A_5) = [n^2/4] - 2 = 4 = e(A_5) .$$

THEOREM 2.5. *Let $A_n = (X, U)$ be the strongly connected c-minimal tournament of order $n \geq 5$; then*

$$P(A_n) = [n^2/4] - 2 = e(A_n) .$$

Proof. We prove the theorem by induction on n . The theorem is already true for $n = 4, 5$. Suppose it is true for A_{n-1} ; we prove it for A_n $(n \geq 6)$.

Let $x_1, x_2, \ldots, x_n$ be the canonical indexation of the vertices of A_n .

(1) Let G be the digraph obtained from A_{n-1} by adding the vertex y and the arcs (y, x_i) for all $i = 1, \ldots, n-2$. By the induction hypothesis A_{n-1} satisfies Q , which implies that the digraph G satisfies Q (see Lemma 2.1).

(2) Let G_1 be the digraph obtained from G by adding the arc

(x_{n-1}, y) . But $d^-_{G_1}(x_{n-1}) = 1$ and $n > 5$; hence there exists at least one path λ of origin y which does not end up at the point x_{n-1} . From this path λ we can construct a path of origin x_{n-1} in G_1 ; therefore $P(G_1) = e(G_1)$ (G_1 isomorphic to A_n).

In A_n we have (see Abdul-Kader [1]),

$$X^+_{A_n} = \{x_i \mid i = [n/2]+1, \ldots, n\} ,$$

$$d^+_{A_n}(x_i) - d^-_{A_n}(x_i) = 2(i-1) - (n-1) \text{ for all } i = [n/2]+1, \ldots, n-1 ,$$

$$d^+_{A_n}(x_n) - d^-_{A_n}(x_n) = n - 3 .$$

Therefore

$$P(A_n) = \sum_{i=[n/2]+1}^{n-1} 2(i-1) - (n-1) + (n-3)$$

$$= e(A_n) = [n^2/4] - 2 .$$

The following corollary proves the existence, by exhibiting them, of digraphs G with circuits satisfying $P(G) = [n^2/4]$.

We denote by TT_n the transitive tournament of order n .

COROLLARY 1. *There exist tournaments* T_n *which are not isomorphic to* TT_n *and such that*

$$P(T_n) = [n^2/4] .$$

Proof. Let $A_n = (X, U)$ be the tournament strongly connected and c-minimal of order n . Let $x_1, x_2, \ldots, x_n$ be the canonical indexation of the vertices of A_n .

First Case: $n = 2k$.

In A_{2k} we have: if $d^+_{A_{2k}}(x_k) = k - 1$ and $d^-_{A_{2k}}(x_k) > k - 1$, then $d^+_{A_{2k}}(x_k) - d^-_{A_{2k}}(x_k) < 0$; moreover $d^+_{A_{2k}}(x_{k+1}) - d^-_{A_{2k}}(x_{k+1}) > 0$. We denote by T_n the tournament of order n obtained from A_n by reversing the arc (x_k, x_{k+1}) .

By Lemma 2.3, the tournament T_n satisfies Q and we have

$$P(T_n) = P(A_n) + 2 = [n^2/4] = e(T_n) .$$

Second Case: $n = 2k + 1$.

We have

$$d^+_{A_n}(x_k) - d^-_{A_n}(x_k) < 0 ,$$

$$d^+_{A_n}(x_{k+1}) - d^-_{A_n}(x_{k+1}) = 0 ,$$

and

$$d^+_{A_n}(x_{k+2}) - d^-_{A_n}(x_{k+2}) > 0 .$$

Let T_n be the tournament of order n obtained from A_n by reversing the arc (x_k, x_{k+1}) ; by Lemma 2.3, the tournament satisfies Q and $P(T_n) = P(A_n) + 2 - [n^2/4] = e(T_n)$. Similarly the tournament of order n obtained from A_n by reversing the arc (x_{k+1}, x_{k+2}) satisfies Q and $P(T_n) = P(A_n) + 2 = [n^2/4] = e(T_n)$. This proves our result.

By Abdul-Kader [2], if T_n is a tournament having a unique hamiltonian circuit, we have

(1) $P(T_n) \le [n^2/4] - 2$;

(2) this result is the best possible, that is, there exist tournaments having a unique hamiltonian circuit, which are not isomorphic to A_n , and which satisfy the equation

$$P(T_n) = [n^2/4] - 2 ;$$

(3) T_n does not satisfy the property Q , in general.

THEOREM 2.6. *Let* $G = (X, U)$ *be a strongly connected digraph of*

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order n satisfying Q ; then there exists a strongly connected digraph H of order $n+1$, satisfying Q and having G as a sub-digraph.

Proof. We have $X_G^+ \neq \emptyset$.

First Case: $|X_G^-| \geq |X_G^+|$.

Let B_1, B_2 be a partition of X_G^- such that $|B_1| = |X_G^+|$. Let $x_0 \notin X$ and H be the digraph generated by $X \cup \{x_0\}$ such that

(1) $G \subset H$,

(2) for all $x \in X_G^+ \cup B_1$, we consider the arcs (x, x_0) if $x \in X_G^+$ and (x_0, x) if not, as being arcs of H . We have then $X_H^+ = X_G^+$. Moreover, $d_H^+(x) = d_G^+(x) + 1$ and $d_H^-(x) = d_G^-(x)$ for all $x \in X_H^+$; then

$$P(H) \geq e(H) = e(G) + |X_G^+| .$$

If R is a path-partition of G such that $|R| = P(G)$, then the set $R_1 = R \cup \{(x_i, x_0, b_i) \mid x_i \in X_G^+, b_i \in B_1\}$ is a path-partition of the arcs of H and $|R_1| = P(G) + |X_H^+|$. One verifies easily that the digraph H is strongly connected; therefore

$$P(H) = e(H) .$$

Second Case: $|X_G^-| < |X_G^+|$.

Let C_1, C_2 be a partition of X_G^+ such that $|C_1| = |X_G^-|$, and let H denote the digraph generated by $X \cup \{x_0\}$ and satisfying

(1) $G \subset H$,

(2) for all $x \in C_1 \cup X_G^-$, we consider the arcs (x, x_0) if $x \in C_1$ and (x_0, x) if $x \in X_G^-$, as being arcs of H .

We have the relations

$$d_H^+(x) = d_G^+(x) + 1 , \quad d_H^-(x) = d_G^-(x) \quad \text{for all } x \in C_1 ,$$

$$d_H^+(x) = d_G^+(x) \quad , \quad d_H^-(x) = d_G^-(x) \quad \text{for all } x \in C_2 .$$

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Then

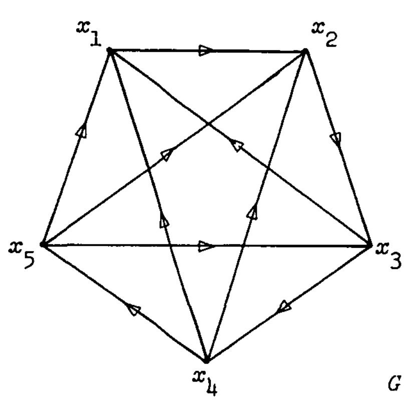
$$P(H) \geq e(H) = e(G) + |C_1| .$$

Moreover the set $R_1 = . R \quad \{(x_i, x_0, y_i) \mid x_i \in C_1, y_i \in X_G^-\}$ partitions the arcs of H and $|R_1| = |R| + |C_1| = P(G) + |C_1|$. Therefore $P(H) = e(H)$. As before, one easily verifies that H is strongly connected.

REMARK. This last theorem constitutes a procedure of extension permitting the construction, from a class of strongly connected digraphs satisfying Q , another class of strongly connected digraphs satisfying Q too.

EXAMPLES. Consider a strongly connected digraph G of order n satisfying $P(G) = e(G)$. We study two cases.

(1) $n = 5$.



By Theorem 2.6, we have

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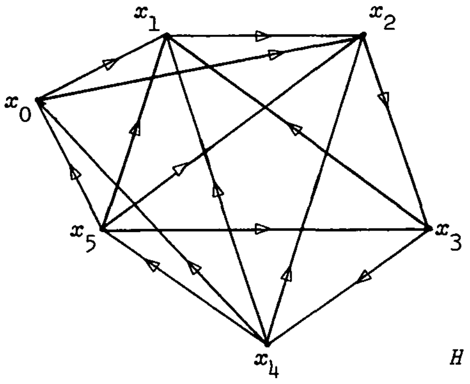

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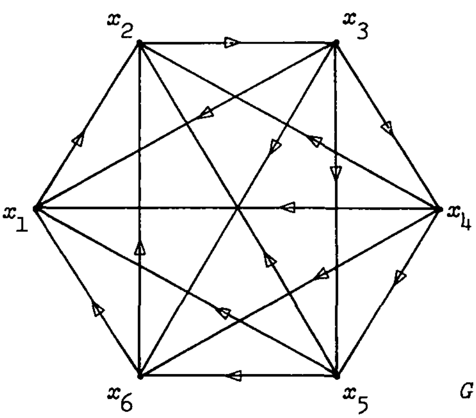
(2) $n = 6$.



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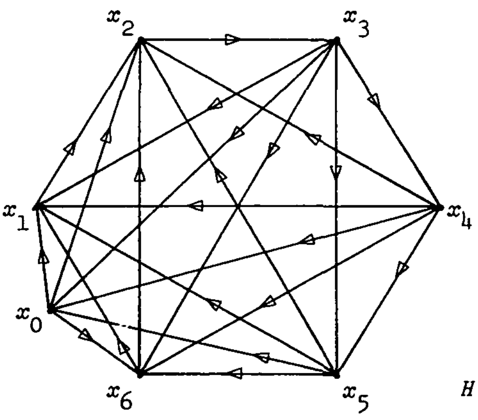
By Theorem 2.6 we have



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