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ESTIMATES FOR CONVEX INTEGRAL MEANS OF HARMONIC FUNCTIONS

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Abstract We prove that if f is an integrable function on the unit sphere S in $\mathbb{R}^n$, g is its symmetric decreasing rearrangement and u, v are the harmonic extensions of f, g in the unit ball $\mathbb{B}$, then v has larger convex integral means over each sphere rS, $0 < r < 1$, than u has. We also prove that if u is harmonic in $\mathbb{B}$ with $|u| < 1$ and $u(0) = 0$, then the convex integral mean of u on each sphere rS is dominated by that of U, which is the harmonic function with boundary values 1 on the right hemisphere and -1 on the left one.

Keywords: harmonic function; integral means; symmetric decreasing rearrangement; harmonic measure; polarization; Schwarz lemma

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1. Introduction

Suppose that f is a real-valued integrable function on the unit circle. Let g be the symmetric decreasing rearrangement of f. Denote by u and v the harmonic extensions in the unit disc of f and g; that is, u and v are the Poisson integrals of f and g, respectively. Baernstein [**4**] proved that, for $0 < r < 1$ and $1 \leqslant p < \infty$,

$$\int_0^{2\pi} |u(re^{i\theta})|^p \, d\theta \leqslant \int_0^{2\pi} |v(re^{i\theta})|^p \, d\theta. \tag{1.1}$$

Moreover, Essén and Shea [**11**] showed that the equality holds in (1.1) for some $0 < r < 1$ and some $p > 1$ if and only if there exists a θ_o such that $f(e^{i\theta}) = g(e^{i(\theta+\theta_o)})$ for almost every real θ.

Much earlier, Gabriel [**12**] (see also [**1**]) had proved, with the additional assumption $f \geqslant 0$, that, for every increasing, convex function $\Phi\colon [0,\infty) \to [0,\infty)$ and every $r \in (0,1)$,

$$\int_0^{2\pi} \Phi(u(re^{i\theta})) \, d\theta \leqslant \int_0^{2\pi} \Phi(v(re^{i\theta})) \, d\theta. \tag{1.2}$$

We shall prove an extension of these results in higher dimensions. First we need to introduce some notation. We shall denote by S the unit sphere, and by $\mathbb{B}$ the unit ball in

$\mathbb{R}^n$, $n \geqslant 2$. Let σ be the surface area measure on S. A point $x \in \mathbb{R}^n$ will also be denoted by $(x_1, x_2, \dots, x_n)$. We set $e_1 = (1, 0, \dots, 0)$. For a point $x \in \mathbb{R}^n$, let θ be defined by $\cos\theta = x_1$; that is, $\theta \in [0, \pi]$ is the angle formed at the origin 0 by the x_1-axis and the ray $0x$. Thus, $r = |x|$ and θ are the first two spherical coordinates of x; we shall not use the remaining $n-2$ spherical coordinates. A real function Φ is *affine* if it has the form $\Phi(t) = at + b$ with real constants a, b.

The cap-symmetric decreasing rearrangement g of $f \in L^1(S)$ is a rearrangement of f such that $g(x)$ depends only on θ and it is a decreasing function of θ. The precise definition will be given in § 2.

Theorem 1.1. *Let $f \in L^1(S)$ and let g be the cap-symmetric decreasing rearrangement of f. Let u and v be the harmonic extensions of f and g in $\mathbb{B}$, respectively. Then, for every $0 < r < 1$ and every convex function $\Phi\colon \mathbb{R} \to \mathbb{R}$,*

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(v(r\zeta))\sigma(\mathrm{d}\zeta). \tag{1.3}$$

If Φ is affine, then the equality holds in (1.3) for all $0 < r < 1$. If there is no interval on which Φ is affine, then the equality holds in (1.3) for some $0 < r < 1$ if and only if $f = g \circ T$ σ-almost everywhere (σ-a.e.) on S for some orthogonal transformation T.

For $n = 2$ and $f \geqslant 0$, Theorem 1.1 implies Gabriel's result (1.2), and for $n = 2$ and $\Phi(t) = |t|^p$, $p \geqslant 1$, it gives Baernstein's inequality (1.1). The proof of Theorem 1.1, given in § 3, uses a discretization argument, the approach to symmetrization via polarization and properties of harmonic measure.

Our second theorem is motivated by the harmonic Schwarz lemma. Schwarz himself had proved that if u is a harmonic function in the unit disc with $|u| < 1$ and $u(0) = 0$, then, for all z in the unit disc,

$$|u(z)| \leqslant \frac{4}{\pi} \tan^{-1} |z| \tag{1.4}$$

(see [**16**, pp. 189–199, 361–362]). Various extensions of this inequality have been discovered; see [**2**, pp. 123–128], [**10**, p. 77], [**9**], [**6**] and [**15**]. A higher-dimensional extension of (1.4) has been proved in [**2**, **6**, **9**]. If u is harmonic in $\mathbb{B}$ with $|u| < 1$ and $u(0) = 0$, then for $x \in \mathbb{B}$,

$$|u(x)| \leqslant U(|x|e_1), \tag{1.5}$$

where U is the harmonic extension of the function $F\colon S \to \mathbb{R}$ with

$$F(\zeta) = \begin{cases} 1, & \zeta_1 \geqslant 0, \\ -1, & \zeta_1 < 0. \end{cases}$$

Moreover, the equality holds in (1.5) for some non-zero $x \in \mathbb{B}$ if and only if $u = U \circ T$ for some orthogonal transformation T.

The function U is extremal for another problem (see [**2**, **6**, **9**]): if u is harmonic in $\mathbb{B}$ and $|u| < 1$, then

$$|\nabla u(0)| \leqslant |\nabla U(0)|, \tag{1.6}$$

with equality if and only if $u = U \circ T$. We shall now see that U is also extremal for a problem involving integral means.

Theorem 1.2. *Let u be harmonic in $\mathbb{B}$ and suppose that $|u| < 1$ and $u(0) = 0$. Then, for every $0 < r < 1$ and every convex function $\Phi\colon (-1, 1) \to \mathbb{R}$,*

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(U(r\zeta))\sigma(\mathrm{d}\zeta). \tag{1.7}$$

If Φ is affine, then (1.7) holds with equality for every $r \in (0, 1)$. If there is no interval on which Φ is affine, then the equality holds for some $r \in (0, 1)$ if and only if $u = U \circ T$ for some orthogonal transformation T.

Note that, for $n = 2$, U is the real part of the holomorphic function

$$H(z) = \frac{4}{\pi} \tan^{-1} z$$

that maps the unit disc conformally onto the strip $\{z\colon -1 < \operatorname{Re} z < 1\}$. In this case, (1.7) follows from the theory of subordination; see, for example, [**14**, Theorem 2.23]. We shall see in § 5 that (1.7) implies both (1.5) and (1.6).

The proof of Theorem 1.2 in § 4 uses Baernstein's star function method [**3**, **5**, **13**] and Theorem 1.1.

2. Preparation for the proofs

2.1. A convexity lemma

We shall use the following elementary lemma (cf. [**17**]).

Lemma 2.1. *Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ be such that*

$$a_2 + b_2 = a_1 + b_1 \quad \text{and} \quad \max\{a_2, b_2\} < b_1.$$

Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a convex function. Then

$$\Phi(a_2) + \Phi(b_2) \leqslant \Phi(a_1) + \Phi(b_1). \tag{2.1}$$

The equality holds in (2.1) if and only if Φ is affine on $[a_1, b_1]$.

2.2. Harmonic measure

Let E be a Borel set on S. The *harmonic measure* of E is the Poisson integral of the function χ_E. The harmonic measure of E at the point $x \in \mathbb{B}$ will be denoted by $\omega(x, E)$.

2.3. Polarization

We define the *polarization* with respect to the $(n-1)$-dimensional plane $\Pi = \{x\colon x_1 = 0\}$. For $A \subset S$, we denote by $\hat{A}$ the reflection of A in Π, i.e.

$$\hat{A} = \{(x_1, x_2, \ldots, x_n)\colon (-x_1, x_2, \ldots, x_n) \in A\}.$$

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We shall also use the notation $\hat{x} = (-x_1, x_2, \ldots, x_n)$, $A_+ = \{x \in A\colon x_1 > 0\}$, $A_- = \{x \in A\colon x_1 < 0\}$.

Let $E \subset S$. We divide E into three disjoint sets: the symmetric part $E_{\mathrm{sym}} = E \cap \hat{E}$, the right non-symmetric part $E_{\mathrm{r}} = E_+ \setminus E_{\mathrm{sym}}$ and the left non-symmetric part $E_{\mathrm{l}} = E_- \setminus E_{\mathrm{sym}}$. The polarization of E with respect to Π is the set

$$E^{\Pi} = E_{\mathrm{sym}} \cup E_{\mathrm{r}} \cup \hat{E}_{\mathrm{l}}.$$

It follows from symmetry and the maximum principle that, for every Borel set $E \subset S$,

$$\omega(x, E) + \omega(\hat{x}, E) = \omega(x, E^{\Pi}) + \omega(\hat{x}, E^{\Pi}), \quad x \in \mathbb{B}, \tag{2.2}$$

and

$$\max\{\omega(x, E), \omega(\hat{x}, E), \omega(\hat{x}, E^{\Pi})\} \leqslant \omega(x, E^{\Pi}), \quad x \in \mathbb{B}_+. \tag{2.3}$$

The polarization of a function $f\colon S \to \mathbb{R}$ with respect to Π is the function $f^{\Pi}\colon S \to \mathbb{R}$ given by

$$f^{\Pi}(x) = \begin{cases} \max\{f(x), f(\hat{x})\}, & x \in S_+ \cup \Pi, \\ \min\{f(x), f(\hat{x})\}, & x \in S_-. \end{cases}$$

Note that $(\chi_E)^{\Pi} = \chi_{(E^{\Pi})}$.

In a similar way, we define the polarization E^H of $E \subset S$ and the polarization f^H of a function f with respect to any oriented plane H passing through the origin.

2.4. Cap-symmetric decreasing rearrangement

We give here the definition of the cap-symmetric decreasing rearrangement g of a function $f \in L^1(S)$. The function $g\colon S \to \mathbb{R}$ depends only on θ, is decreasing as θ increases from 0 to π and has the same distribution function as f: for all $t \in \mathbb{R}$,

$$\sigma(\{x \in S\colon g(x) > t\}) = \sigma(\{x \in S\colon f(x) > t\}).$$

These conditions determine g uniquely, except for sets of σ-measure zero. The function g may be expressed by the formula

$$g(\theta) = \inf\{t\colon \sigma(\{x\colon f(x) > t\}) \leqslant \sigma(C(\theta))\}, \quad \theta \in [0, \pi].$$

Here and below, $C(\theta_o)$ is the spherical cap on S centred at e_1 given by $C(\theta_o) = \{x \in S\colon 0 \leqslant \theta < \theta_o\}$.

3. Proof of Theorem 1.1

Let $f \in L^1(S)$ and let g be the cap-symmetric decreasing rearrangement of f. Let u and v be the harmonic extensions in $\mathbb{B}$ of f and g, respectively. Suppose first that f is a simple function taking a finite number of values

$$a_1 > a_2 > \cdots > a_k.$$

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Then f has the representation

$$f = \sum_{j=1}^{k} a_j \chi_{A_j} \quad \text{with } A_j = \{x \in S\colon f(x) = a_j\}.$$

We modify the above representation of f as follows. Set

$$\begin{aligned} E_1 &= A_1, \\ E_2 &= A_1 \cup A_2, \\ &\vdots \\ E_{k-1} &= A_1 \cup \cdots \cup A_{k-1}, \\ E_k &= A_1 \cup A_2 \cup \cdots \cup A_k = S \end{aligned}$$

and

$$c_1 = a_1 - a_2, \quad c_2 = a_2 - a_3, \quad \ldots, \quad c_{k-1} = a_{k-1} - a_k, \quad c_k = a_k.$$

Then

$$f = \sum_{j=1}^{k} c_j \chi_{E_j}.$$

Moreover, for the polarization f^{Π} of f with respect to the hyperplane Π, we have

$$f^{\Pi} = \sum_{j=1}^{k} c_j \chi_{E_j^{\Pi}}.$$

Let h be the harmonic extension of f^{Π} in $\mathbb{B}$. Then

$$u(x) = \sum_{j=1}^{k} c_j \omega(x, E_j) \quad \text{and} \quad h(x) = \sum_{j=1}^{k} c_j \omega(x, E_j^{\Pi}).$$

It follows from (2.2) and (2.3) that

$$u(x) + u(\hat{x}) = h(x) + h(\hat{x}), \quad x \in \mathbb{B}, \tag{3.1}$$

and

$$\max\{u(x), u(\hat{x}), h(\hat{x})\} \leqslant h(x), \quad x \in \mathbb{B}_+. \tag{3.2}$$

By a standard approximation argument, (3.1) and (3.2) continue to hold for general $f \in L^1(S)$.

Lemma 2.1 and (3.1) imply that, for every convex Φ,

$$\Phi(u(x)) + \Phi(u(\hat{x})) \leqslant \Phi(h(x)) + \Phi(h(\hat{x})), \quad x \in \mathbb{B}, \tag{3.3}$$

which yields

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(h(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{3.4}$$

By another approximation argument (involving the approximation of symmetrization by a sequence of polarizations with respect to suitable hyperplanes passing through the origin [**7**]), (3.4) implies

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(v(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{3.5}$$

Thus, (1.3) is proved.

(We note here that [**7**] treats the case of Steiner symmetrization, not the cap symmetrization that we use here. However, the arguments of [**7**, § 6] can be slightly modified to prove the corresponding results for cap symmetrization.)

We prove now the equality statement of Theorem 1.1. If Φ is affine, then the equality holds in (3.3) for every $x \in \mathbb{B}$ and therefore the equality holds in (3.4) for every $r \in (0,1)$. Suppose there is no interval on which Φ is affine.

Claim 3.1. *The equality holds in (3.3) for some $x_o \in \mathbb{B} \setminus \Pi$ if and only if either $f(\zeta) = f^{\Pi}(\zeta)$ for σ-almost every $\zeta \in S$ or $f(\zeta) = f^{\Pi}(\hat{\zeta})$ for σ-almost every $\zeta \in S$.*

Proof of Claim 3.1. If $f(\zeta) = f^{\Pi}(\zeta)$ for σ-almost every $\zeta \in S$ (or if $f(\zeta) = f^{\Pi}(\hat{\zeta})$ for σ-almost every $\zeta \in S$), then $u(x) = h(x)$ in $\mathbb{B}$ (or $u(x) = h(\hat{x})$ in $\mathbb{B}$) and (3.3) holds with equality for every $x \in \mathbb{B}$. Suppose conversely that we have the equality in (3.3) for some $x_o \in \mathbb{B} \setminus \Pi$. By (3.1), (3.2) and Lemma 2.1, either

$$u(x_o) = h(x_o) \quad \text{and} \quad u(\hat{x}_o) = h(\hat{x}_o)$$

or

$$u(x_o) = h(\hat{x}_o) \quad \text{and} \quad u(\hat{x}_o) = h(x_o).$$

Suppose that $u(x_o) = h(x_o)$ and $u(\hat{x}_o) = h(\hat{x}_o)$ and $x_o \in \mathbb{B}_+$. By (3.2), $u \leqslant h$ in $\mathbb{B}_+$. By the maximum principle, $u = h$ in $\mathbb{B}_+$ and, by the identity principle for harmonic functions, $u = h$ in $\mathbb{B}$. In all other cases, we conclude similarly that either $u(x) = h(x)$ for every $x \in \mathbb{B}$ or $u(x) = h(\hat{x})$ for every $x \in \mathbb{B}$. By Fatou's theorem for the Poisson integrals (see, for example, [**2**, p. 135]), the limit of $u(x)$ as x tends non-tangentially to $\zeta \in S$ is $f(\zeta)$ for σ-almost every $\zeta \in S$, and similarly for h. Hence, either $f(\zeta) = f^{\Pi}(\zeta)$ for σ-almost every $\zeta \in S$ or $f(\zeta) = f^{\Pi}(\hat{\zeta})$ for σ-almost every $\zeta \in S$. So Claim 3.1 is proved. □

Suppose now that the equality holds in (1.3) for some $r \in (0,1)$. Suppose also that $f \neq g \circ T$ on a set of positive σ-measure for any orthogonal transformation. By a standard result in the theory of symmetrization (see [**7**, Lemma 6.3] for the corresponding result for Steiner symmetrization), there exists an $(n-1)$-dimensional plane H passing through the origin such that $f(\zeta) \neq f^H(\zeta)$ and $f(\zeta) \neq f^H(\hat{\zeta})$ for all $\zeta \in E \subset S$ with $\sigma(E) > 0$. Let h denote the harmonic extension of f^H in $\mathbb{B}$ and let the hat denote reflection in H. By Claim 3.1, for every $x \in \mathbb{B} \setminus H$,

$$\Phi(u(x)) + \Phi(u(\hat{x})) < \Phi(h(x)) + \Phi(h(\hat{x})). \tag{3.6}$$

Therefore,

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) < \int_S \Phi(h(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{3.7}$$

But the cap-symmetric decreasing rearrangement of f^H is again g. By (1.3),

$$\int_S \Phi(h(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(v(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{3.8}$$

By (3.7) and (3.8),

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) < \int_S \Phi(h(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1, \tag{3.9}$$

which is a contradiction.

4. Proof of Theorem 1.2

Let $\Phi\colon (-1,1) \to \mathbb{R}$ be a convex function. Since u is bounded, there exists a function $f \in L^\infty(S)$ such that u is the Poisson integral of f (see, for example, [**2**, Chapter 6]). Let g be the cap-symmetric decreasing rearrangement of f, and let v be the Poisson integral of g. By Theorem 1.1,

$$\int_S \Phi(u(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(v(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{4.1}$$

Consider the star functions v^* and U^* of v and U, respectively. These functions are defined on the upper half D of the unit disc by the formulae

$$v^*(r\mathrm{e}^{\mathrm{i}\theta}) = \sup_E \int_E v(r\zeta)\sigma(\mathrm{d}\zeta)$$

and

$$U^*(r\mathrm{e}^{\mathrm{i}\theta}) = \sup_E \int_E U(r\zeta)\sigma(\mathrm{d}\zeta),$$

where the supremum is taken over all sets $E \subset S$ with $\sigma(E) = \sigma(C(\theta))$. The function v is symmetric decreasing on each of the spheres rS, $0 < r < 1$; this fact can be proved easily by an approximation argument and by inequality (2.3). Therefore (see [**5**, p. 246]),

$$v^*(r\mathrm{e}^{\mathrm{i}\theta}) = \int_{C(\theta)} v(r\zeta)\sigma(\mathrm{d}\zeta), \quad 0 < r < 1,\ 0 < \theta < \pi.$$

Similarly,

$$U^*(r\mathrm{e}^{\mathrm{i}\theta}) = \int_{C(\theta)} U(r\zeta)\sigma(\mathrm{d}\zeta), \ \ 0 < r < 1,\ 0 < \theta < \pi.$$

Since v and U are harmonic functions in $\mathbb{B}$ and depend only on the spherical coordinates r, θ (and not on the rest of the spherical coordinates), a calculation (see [**5**, p. 247]) shows that both v^* and U^* are L-harmonic functions. This means that $Lv^* = 0 = LU^*$, where L is the elliptic partial differential operator (written in polar coordinates)

$$L = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{\sin^{n-2}\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^{n-2}\theta}\frac{\partial}{\partial\theta}\right).$$

We shall prove that, for all $z \in D$,

$$v^*(z) \leqslant U^*(z). \tag{4.2}$$

We extend v^* and U^* to the closure $\bar{D}$ of D by taking limits as z tends to boundary points of D. It is easy to see that the extended functions are both continuous on $\bar{D}$. By the fundamental result of the star function theory (see [**3**, Theorem A], [**13**, Theorem 1], [**5**, Theorem 5]), to prove (4.2) it suffices to show that (4.2) holds for every $z \in \partial D$. We distinguish four cases for the location of $z \in \partial D$.

Case 1. z lies on the interval $(0, 1)$.
Then $v^*(z) = U^*(z) = 0$.

Case 2. z lies on the interval $(-1, 0)$.
Then, by the mean-value theorem for harmonic functions,

$$v^*(z) = \int_{C(\pi)} v(r\zeta)\sigma(\mathrm{d}\zeta) = \int_S v(r\zeta)\sigma(\mathrm{d}\zeta) = v(0) = 0,$$

and similarly $U^*(z) = 0$.

Case 3. $z = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in [0, \frac{1}{2}\pi]$.
Then

$$v^*(z) = \int_{C(\theta)} g(\zeta)\sigma(\mathrm{d}\zeta) \leqslant \int_{C(\theta)} 1\sigma(\mathrm{d}\zeta) = \int_{C(\theta)} F(\zeta)\sigma(\mathrm{d}\zeta) = U^*(z).$$

Case 4. $z = \mathrm{e}^{\mathrm{i}\theta}$ for some $\theta \in (\frac{1}{2}\pi, \pi]$.
Then

$$\begin{aligned} v^*(z) &= \int_{C(\theta)} g(\zeta)\sigma(\mathrm{d}\zeta) = -\int_{S\setminus C(\theta)} g(\zeta)\sigma(\mathrm{d}\zeta) \\ &\leqslant -\int_{S\setminus C(\theta)} (-1)\sigma(\mathrm{d}\zeta) = -\int_{S\setminus C(\theta)} F(\zeta)\sigma(\mathrm{d}\zeta) \\ &= \int_{C(\theta)} F(\zeta)\sigma(\mathrm{d}\zeta) = U^*(z). \end{aligned}$$

Therefore, (4.2) is proved. The inequality (4.2) implies that

$$\int_S \Phi(v(r\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(U(r\zeta))\sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{4.3}$$

This follows from a modification of [**3**, Proposition 3]. This proposition, in fact, has the additional assumption that Φ is increasing; however, this assumption is not necessary when the two functions ($v(r\cdot)$ and $U(r\cdot)$ in our case) have the same integral, which is the case for the above application of the proposition.

Now (1.7) follows from (4.1) and (4.3).

We proceed to prove the statement of the equality. If Φ is affine, then it follows from the mean-value property that (1.7) holds with equality for every $r \in (0,1)$. Suppose from now on that there is no interval on which Φ is affine. If $u = U \circ T$ for some orthogonal transformation, then clearly (1.7) holds with equality for every $r \in (0,1)$. Assume, conversely, that (1.7) holds with equality for some $r_o \in (0,1)$, i.e.

$$\int_S \Phi(u(r_o\zeta))\sigma(\mathrm{d}\zeta) = \int_S \Phi(U(r_o\zeta))\sigma(\mathrm{d}\zeta). \tag{4.4}$$

Seeking a contradiction, suppose that $g \neq F$ on a set of positive σ-measure. Then there exists a small $\delta > 0$ such that

$$\operatorname{ess\,sup}\{g(x)\colon \tfrac{1}{2}\pi - \delta < \theta < \tfrac{1}{2}\pi\} =: \alpha < 1$$

and

$$\operatorname{ess\,inf}\{g(x)\colon \tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi + \delta\} =: \beta > -1.$$

Recall that $\theta \in [0,\pi]$ is the first angular spherical coordinate of x. We set $\eta = \max\{\alpha, -\beta\}$ and note that $\eta \in [0,1)$. Consider the function $q\colon S \to \mathbb{R}$, which depends only on θ and is given by

$$q(x) = \begin{cases} 1-\eta, & \tfrac{1}{2}\pi - \delta < \theta < \tfrac{1}{2}\pi, \\ \eta - 1, & \tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi + \delta, \\ 0, & \text{elsewhere}. \end{cases}$$

Let $\mathcal{P}[\cdot]$ denote the Poisson integral. By symmetry,

$$\mathcal{P}[q](x) = 0, \quad x \in \mathbb{B} \cap \mathit{\Pi}. \tag{4.5}$$

So the maximum principle gives

$$\mathcal{P}[q](x) = -\mathcal{P}[q](\hat{x}) > 0, \quad x \in \mathbb{B}_+. \tag{4.6}$$

The function $\mathcal{P}[g+q]$ satisfies the assumptions of Theorem 1.2, namely $|\mathcal{P}[g+q]| < 1$ and $\mathcal{P}[g+q](0) = 0$. Hence,

$$\int_S \Phi(\mathcal{P}[g+q](r_o\zeta))\sigma(\mathrm{d}\zeta) \leqslant \int_S \Phi(U(r_o\zeta))\sigma(\mathrm{d}\zeta). \tag{4.7}$$

On the other hand, by (4.6), for $x \in \mathbb{B}$,

$$\begin{aligned} \mathcal{P}[g+q](x) + \mathcal{P}[g+q](\hat{x}) &= \mathcal{P}[g](x) + \mathcal{P}[q](x) + \mathcal{P}[g](\hat{x}) + \mathcal{P}[q](\hat{x}) \\ &= \mathcal{P}[g](x) + \mathcal{P}[g](\hat{x}) \\ &= v(x) + v(\hat{x}), \end{aligned} \tag{4.8}$$

and, for $x \in \mathbb{B}_+$,

$$v(x) = \mathcal{P}[g](x) < \mathcal{P}[g](x) + \mathcal{P}[q](x) = \mathcal{P}[g+q](x). \tag{4.9}$$

Hence, Lemma 2.1 implies that, for $x \in \mathbb{B}_+$,

$$\Phi(v(x)) + \Phi(v(\hat{x})) < \Phi(\mathcal{P}[g+q](x)) + \Phi(\mathcal{P}[g+q](\hat{x})), \tag{4.10}$$

and therefore

$$\int_S \Phi(v(r_o\zeta))\sigma(\mathrm{d}\zeta) < \int_S \Phi(\mathcal{P}[g+q](r_o\zeta))\sigma(\mathrm{d}\zeta). \tag{4.11}$$

By (4.1), (4.7) and (4.11),

$$\int_S \Phi(u(r_o\zeta))\sigma(\mathrm{d}\zeta) < \int_S \Phi(U(r_o\zeta))\sigma(\mathrm{d}\zeta), \tag{4.12}$$

which contradicts (4.4). We conclude that $g = F$ σ-a.e. on S, and therefore $v = U$ in $\mathbb{B}$. By the equality statement of Theorem 1.1, there exists an orthogonal transformation T such that $f = g \circ T = F \circ T$ σ-a.e. on S. It follows that $u = U \circ T$ in $\mathbb{B}$.

5. Concluding remarks

Remark 5.1. Theorem 1.2 implies both inequalities (1.5) and (1.6). Indeed, by applying (1.7) with $\Phi(t) = |t|^p$, $1 \leqslant p < \infty$, we obtain

$$\int_S |u(r\zeta)|^p \sigma(\mathrm{d}\zeta) \leqslant \int_S |U(r\zeta)|^p \sigma(\mathrm{d}\zeta), \quad 0 < r < 1. \tag{5.1}$$

Letting $p \to \infty$, we get

$$\max_{\zeta \in S} |u(r\zeta)| \leqslant \max_{\zeta \in S} |U(r\zeta)| = U(re_1), \quad 0 < r < 1,$$

which is equivalent to (1.5). To prove (1.6), we use (5.1) for $p = 2$. By expanding u and U in a series of homogeneous polynomials (see [**2**, pp. 24, 140]), we see that (5.1) for $p = 2$ implies

$$c|\nabla u(0)|^2 r^2 + c_2 r^4 + c_3 r^6 + \cdots \leqslant c|\nabla U(0)|^2 r^2 + C_2 r^4 + C_3 r^6 + \cdots,$$

where $c > 0$ and c_j, C_j, $j = 2, 3, \ldots$, are real constants. By dividing by r^2 and letting $r \to 0$, we obtain (1.6).

Remark 5.2. We can replace the assumption $u(0) = 0$ in Theorem 1.2 by the weaker assumption $u(0) = c$ for a fixed $c \in (-1, 1)$. Then the extremal function is the harmonic extension U_c of the function $F_c = \chi_{C(\theta_c)} - \chi_{S \setminus C(\theta_c)}$, where $\theta_c \in (0, \pi)$ is chosen so that $U_c(0) = c$. The proof of this version of Theorem 1.2 is similar but more technical. Analogous modifications of the inequalities (1.5) and (1.6) have been proved in [**9**] and [**6**].

Remark 5.3. By a basic symmetrization inequality for harmonic measure [**3**, **17**], if $x \in \mathbb{B}$ and E is a Borel subset of S, then

$$\omega(-|x|e_1, E^\sharp) \leqslant \omega(x, E) \leqslant \omega(|x|e_1, E^\sharp), \tag{5.2}$$

where $E^\sharp$ is the spherical cap centred at e_1 with $\sigma(E^\sharp) = \sigma(E)$. It follows from (5.2) and a discretization argument (as in the proof of Theorem 1.1) that if $f \in L^1(S)$ and g is its cap-symmetric decreasing rearrangement, then, for $x, y \in r\mathbb{B}$,

$$|\mathcal{P}[f](x) - \mathcal{P}[f](y)| \leqslant \mathcal{P}[g](re_1) - \mathcal{P}[g](-re_1) \leqslant \mathcal{P}[F](re_1) - \mathcal{P}[F](-re_1). \tag{5.3}$$

An immediate consequence of this inequality is the following diameter version of the harmonic Schwarz lemma; see [**8**] and references therein for the corresponding result for holomorphic functions.

Proposition. *If u is a harmonic function in $\mathbb{B}$ and $\operatorname{diam}(u(\mathbb{B})) = 2$, then*

$$\operatorname{diam}(u(r\mathbb{B})) \leqslant \operatorname{diam}(U(r\mathbb{B})), \quad 0 < r < 1. \tag{5.4}$$

The equality holds in (5.4) for some $r \in (0, 1)$ if and only if $u = \pm U \circ T + c$ for some orthogonal transformation T and some real constant c.

This easily extends to complex-valued harmonic functions. We do not give a detailed account of this result because it can easily be obtained from the methods in [**2**, **6**, **9**], which use only the properties of the Poisson integral.

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