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ADDITIVE SELECTIONS AND THE STABILITY OF THE CAUCHY FUNCTIONAL EQUATION

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Abstract

The main result of this paper offers a necessary and sufficient condition for the existence of an additive selection of a weakly compact convex set-valued map defined on an amenable semigroup. As an application, we obtain characterisations of the solutions of several functional inequalities, including that of quasi-additive functions.

1. Introduction

The Cauchy functional equation has numerous applications such as to information theory and information measures, the problem of aggregated allocations, geometric objects, Hamel bases, harmonic analysis and stochastic processes. For the last-mentioned see, for example, [17, 18].

The stability properties of the Cauchy functional equation have attracted the attention of many mathematicians. A cornerstone result is the so-called Hyers-Ulam stability theorem obtained by Hyers [10]. An account of the progress and developments in this field can be found in a recent survey papers by Forti [4], Ger [7], Hyers and Rassias [12] and Hyers, Isac and Rassias [11].

The motivation for this paper has its origins in three sources. In 1988, Tabor [20, 21] introduced the notion of quasi-additive functions and showed, that from the point of view of regularity, they behave very similarly to additive functions. In 1992, Ger [6]—using invariant mean techniques, or selection theorems—obtained stability

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results which yielded a deeper explanation of the regularity-irregularity properties of quasi-additive functions. Recently, Páles [16] has obtained a characterisation of real-valued quasi-additive functions by showing that they can be factorised as the composition of an odd strictly monotonic globally Lipschitz function and an additive function. The aim of this paper is to combine the methods of Ger and Páles to obtain analogous statements in the vector-valued setting.

In the first part, we show that left (right) invariant means (defined on the space of real-valued bounded functions over a semigroup) can always be extended to Hausdorff locally convex space-valued functions whose range has weakly compact convex closure. Thus, we generalise the results of Székelyhidi [19] and Gajda [5] who obtained analogous statements for the semi-reflexive locally convex space-valued setting. As an application, we derive a necessary and sufficient condition for the existence of an additive selection of a weakly compact convex set-valued map. (In the real-valued case, such results have recently been obtained by Páles [16].) The proof of this selection theorem is an adaptation of the methods of Ger [6]. This result has direct consequences in the stability theory of the Cauchy functional equation.

In the second part of this paper, we consider functional inequalities that are related to the quasi-additivity property of vector-valued functions. Our main results show that such functions can always be written as the compositions of an additive function and of a globally Lipschitz function. As an application, we also offer a complete characterisation of the quasi-additivity property. The results so derived generalise those of Páles [16].

2. Means for locally convex space-valued functions

Let S be a nonempty set and denote by $\mathscr{B}(S, X)$ the space of bounded X-valued functions defined on S, where X is an arbitrary topological vector space over the field of real numbers.

DEFINITION 1. A mapping $M : \mathscr{B}(S, \mathbb{R}) \to \mathbb{R}$ is called a *mean on* $\mathscr{B}(S, \mathbb{R})$ if it satisfies the following properties:

(M1) M is *linear*.

(M2) M has the *mean value property*, that is, for all $f \in \mathscr{B}(S, \mathbb{R})$,

$$\inf_{s\in S} f(s) \le M(f) \le \sup_{s\in S} f(s).$$

We note that, (M2) can be replaced by the following two conditions:

(M3) M is *nonnegative*, that is, $M(f) \ge 0$ if $f \ge 0$.

(M4) M is *normalised*, that is, $M(1) = 1$.

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If $f = (f_1, \dots, f_n) : S \to \mathbb{R}^n$ is a bounded vector-valued function then a real-valued mean can naturally be extended to such functions by

$$M(f) := (M(f_1), \dots, M(f_n)).$$

It is obvious that $M : \mathscr{B}(S, \mathbb{R}^n) \to \mathbb{R}^n$ is still linear and the mean value property (M2) easily yields that

(M5) for all $f \in \mathscr{B}(S, \mathbb{R}^n)$, $M(f) \in \overline{\text{co}} f(S)$.

This latter property will be called the mean value property in the vector-valued setting. That is, we have the following definition.

DEFINITION 2. Let X be a topological vector space. An X-valued map M defined on a subspace of $\mathscr{B}(S, X)$ is called a mean if it is linear and satisfies (M5) for all functions f from the domain of M.

Our aim now is to extend a given real-valued mean M to general vector space-valued functions. Such an extension was constructed by Székelyhidi [19] and by Gajda [5] when the target space is a semi-reflexive locally convex space. The idea in [19] and [5] is to consider the mapping $x^* \mapsto M(x^* \circ f)$, $x^* \in X^*$, and observe that it is a continuous linear functional on X^*; therefore, by semi-reflexivity, there exists an element $m \in X$ such that $x^*(m) = M(x^* \circ f)$, $x^* \in X^*$. This element m is then called the mean of f and is denoted by $M(f)$.

Another approach is due to Badora [1] who extends M to normed spaces with the Hahn-Banach extension property. Ger [6] considered boundedly complete Banach lattices with a strong unit element and showed that a mean M on $\mathscr{B}(S, \mathbb{R})$ admits a continuous linear extension on the space of bounded lattice-valued functions. In these cases however the extension may not have the mean value property and satisfies only weaker inclusions than (M5).

In what follows, we generalise the approaches of Székelyhidi and Gajda. Assume that X is a Hausdorff locally convex space. For a bounded function $f : S \to X$, the mean $M(f)$ of f is defined by the following formula:

$$x^*(M(f)) = M(x^* \circ f) \quad (x^* \in X^*). \tag{2.1}$$

The uniqueness of $M(f)$ follows from the fact that X^* separates the points of X. The existence of $M(f)$ requires conditions on the range of the function f (see Theorem 2).

THEOREM 1. *Assume that X is a Hausdorff locally convex space. Let M be a mean on $\mathscr{B}(S, \mathbb{R})$. Denote by $\mathscr{B}_M(S, X)$ the set of all bounded functions $f : S \to X$ such that $M(f)$ exists. Then $\mathscr{B}_M(S, X)$ is a linear subspace of $\mathscr{B}(S, X)$ and $M : \mathscr{B}_M(S, X) \to X$ is a linear map satisfying*

$$M(f) \in \overline{\text{co}} f(S) \quad (f \in \mathscr{B}_M(S, X)). \tag{2.2}$$

PROOF. Let $f, g \in \mathscr{B}(S, X)$ such that $M(f)$ and $M(g)$ exist. Then, for all $x^* \in X^*$,

$$x^*(M(f) + M(g)) = x^*(M(f)) + x^*(M(g)) = M(x^* \circ f) + M(x^* \circ g)$$
$$= M(x^* \circ f + x^* \circ g) = M(x^* \circ (f + g)).$$

Hence by uniqueness, $M(f+g)$ exists and is equal to $M(f)+M(g)$. Thus $\mathscr{B}_M(S, X)$ is closed under addition and M is additive on this group. A statement concerning homogeneity can be proved analogously.

Contrary to (2.2), assume that $M(f)$ does not belong to $\overline{\text{co}} f(S)$. Then the strict version of the Hahn-Banach separation theorem yields the existence of a linear functional $x^* \in X^*$ such that $\sup_{x \in \overline{\text{co}} f(S)} x^*(x) < x^*(M(f))$. Therefore there exists a positive ε satisfying

$$x^*(x) + \varepsilon \leq x^*(M(f)) \quad (x \in \overline{\text{co}} f(S)).$$

Hence

$$x^*(f(s)) + \varepsilon \leq x^*(M(f)) \quad (s \in S),$$

or, in other words, $x^* \circ f + \varepsilon \leq x^*(M(f))$. Applying M to the functions on both sides of this inequality, we obtain

$$M(x^* \circ f) + \varepsilon \leq x^*(M(f)) = M(x^* \circ f),$$

which is an obvious contradiction.

The next result offers a sufficient condition for the existence of the vector-valued mean.

THEOREM 2. *Let M be a mean on $\mathscr{B}(S, \mathbb{R})$ and let $f : S \to X$ be a function such that $\overline{\text{co}} f(S)$ is weakly compact in X. Then there exists a unique element $M(f) \in \overline{\text{co}} f(S)$ such that* (2.1) *holds.*

PROOF. We note first, that for convex sets in locally convex spaces, the notions of strong and weak closedness coincide, hence $\overline{\text{co}} f(S)$ is always weakly closed.

Let $\Phi = \{x_1^*, \dots, x_n^*\}$ be a finite subset of X^* and let M_Φ be the set of vectors $m \in \overline{\text{co}} f(S)$ such that $x_i^*(m) = M(x_i^* \circ f)$, $i \in \{1, \dots, n\}$. Clearly, M_Φ is a weakly closed subset of $\overline{\text{co}} f(S)$, and hence it is weakly compact. If all the sets M_Φ ($\Phi \subset X^*$, finite) are nonempty, then this family of sets has the finite intersection property. Hence the intersection of all such sets is also nonempty and its (unique) element clearly satisfies (2.1). Thus it suffices to show that M_Φ is nonempty for all finite subsets $\Phi \subset X^*$.

Let $\Phi = \{x_1^*, \dots, x_n^*\} \subset X^*$ and construct a linear map $L : X \to \mathbb{R}^n$ by

$$L(x) = (x_1^*(x), \dots, x_n^*(x)) \quad (x \in X).$$

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We are going to show that the vector $(p_1, \dots, p_n) = (M(x_1^* \circ f), \dots, M(x_n^* \circ f))$ belongs to the set $K = L(\overline{\text{co}} f(S))$. This will immediately yield that M_Φ is nonvoid.

On the contrary, assume that $(p_1, \dots, p_n)$ does not belong to K. Since the set $\overline{\text{co}} f(S)$ is weakly compact and convex, K yields a compact convex subset of $\mathbb{R}^n$. Thus, by the Hahn-Banach separation theorem, there exists a vector $c = (c_1, \dots, c_n)$ which, considered as a linear functional on $\mathbb{R}^n$, strictly separates $(p_1, \dots, p_n)$ from K, that is, there exists a positive ε such that

$$\sum_{i=1}^n c_i x_i^*(x) + \varepsilon \le \sum_{i=1}^n c_i p_i$$

for all $x \in \overline{\text{co}} f(S)$. Hence, for $s \in S$,

$$\sum_{i=1}^n c_i x_i^* \circ f(s) + \varepsilon \le \sum_{i=1}^n c_i p_i.$$

Applying M to both sides of this inequality, and using the properties (M1), (M3) and (M4), we get

$$\sum_{i=1}^n c_i p_i + \varepsilon = \sum_{i=1}^n c_i M(x_i^* \circ f) + \varepsilon \le \sum_{i=1}^n c_i p_i,$$

which is a contradiction. Thus we have proved that $(p_1, \dots, p_n) \in L(\overline{\text{co}} f(S))$.

Therefore there exists a point $m \in \overline{\text{co}} f(S)$ such that $M(x_i^* \circ f) = p_i = x_i^*(m)$, $i \in \{1, \dots, n\}$. Hence $m \in M_\Phi$, proving that M_Φ is nonempty.

In what follows, the class of X-valued functions on S such that the closed convex hull of the range is weakly compact will play a significant role. We shall denote this class of functions by $\mathscr{C}_{\mathscr{W}}(S, X)$. It is not difficult to see that this class of functions forms a vector space, moreover it is a subspace of all bounded X-valued functions on S. In many cases (for example, when X with the weak topology is a quasi-complete locally convex space), the weak relative compactness of $f(S)$ yields the weak relative compactness of $\text{co} f(S)$ (see Holmes [9, Theorem 11B, p. 61]).

If X is a semi-reflexive locally convex space, then bounded sets are always weakly relatively compact (see Yosida [22, Chapter V, Theorem 3.1, p. 140]). Hence if $f(S)$ is bounded then $\overline{\text{co}} f(S)$ is weakly compact. Therefore in this case $\mathscr{C}_{\mathscr{W}}(S, X)$ is identical to the space of bounded functions, that is, to $\mathscr{B}(S, X)$.

COROLLARY 1. *Let X be a semi-reflexive locally convex space, M be a mean on $\mathscr{B}(S, \mathbb{R})$ and let $f : S \to X$ be a bounded function. Then there exists a unique element $M(f) \in \overline{\text{co}} f(S)$ such that* (2.1) *holds.*

(See also Székelyhidi [19] and Gajda [5].)

Now we turn our attention to the case where S admits a semigroup structure, that is, we assume that $(S, +)$ is a (not necessarily commutative) semigroup. The elements of S induce the notion of *left* and *right translations* for functions $f : S \to X$ in the following way. If $t \in S$, then denote

$$ {}_tf(s) := f(t+s), \quad f_t(s) := f(s+t) \quad (s \in S). $$

The functions ${}_tf$ and f_t so defined are called *left* and *right translates of* f.

DEFINITION 3. A semigroup S is called *left* (respectively *right*) *amenable* if there exists a mean M on $\mathscr{B}(S, \mathbb{R})$ which is *left* (respectively *right*) *invariant with respect to the left* (respectively *right*) *translations*, that is, if it satisfies $M({}_tf) = M(f)$ (respectively $M(f_t) = M(f)$) for all $f \in \mathscr{B}(S, \mathbb{R})$ and $t \in S$.

If both left and right invariant means exist, then S is called *amenable*.

It is well-known that any commutative semigroup is amenable (see for example Hewitt-Ross [8, Chapter 4, Theorem 17.5] and Day [3]).

Exactly as Theorem 1 was proved, one can obtain the following result.

THEOREM 3. *Let $(S, +)$ be a semigroup and assume that M is a left (respectively right) invariant mean on $\mathscr{B}(S, \mathbb{R})$. In addition, let X be a Hausdorff locally convex space. Then $\mathscr{B}_M(S, X)$ is closed under left (respectively right) translations and, for all $f \in \mathscr{B}_M(S, X)$ and $t \in S$, $M({}_tf) = M(f)$ (respectively $M(f_t) = M(f)$).*

In other words, if M is a left (right) invariant mean on $\mathscr{B}(S, \mathbb{R})$, then its extension is also a left (right) invariant mean on $\mathscr{B}_M(S, X)$.

The main result of this section is contained in the following theorem.

THEOREM 4. *Let $(S, +)$ be a left (respectively right) amenable semigroup and let X be a Hausdorff locally convex linear space. Then the space $\mathscr{C}_{\mathscr{W}}(S, X)$ of all X-valued functions whose range has a weakly compact closed convex hull admits a left (respectively right) invariant mean.*

PROOF. Let S be left amenable and M be a left invariant mean on $B(S, \mathbb{R})$. Then, by Theorem 2, it can be extended to be a mean on $\mathscr{C}_{\mathscr{W}}(S, X)$. Due to Theorem 3, this extension will also be left invariant.

The above theorem is a direct generalisation of the following result of Székelyhidi [19] and Gajda [5].

COROLLARY 2. *Let $(S, +)$ be a left (respectively right) amenable semigroup and let X be a semi-reflexive locally convex linear space. Then the space $\mathscr{B}(S, X)$ of all bounded X-valued functions admits a left (respectively right) invariant mean.*

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3. Additive selections and the stability of the Cauchy functional equation

The main result of this section offers necessary and sufficient conditions for the existence of additive selections from a set-valued mapping with nonempty weakly compact convex values. The proof uses the vector-valued invariant mean from the previous section and the ideas of the proof of Theorem 2 in Ger [6]. The result obtained answers the problem of Páles [16] affirmatively.

THEOREM 5. *Let $(S, +)$ be a left amenable semigroup and let X be a Hausdorff locally convex linear space. Let $F : S \to 2^X$ be a set-valued map such that, for all $s \in S$, $F(s)$ is nonempty, convex and weakly compact. Then F admits an additive selection $A : S \to X$ if and only if there exists a function $f : S \to X$ such that*

$$f(s+t) - f(t) \in F(s) \quad (s, t \in S). \tag{3.1}$$

PROOF. If A is an additive selection of F, then the function $f = A$ clearly satisfies (3.1).

Assume that (3.1) is valid with a certain function $f : S \to X$. Due to Theorem 4, we have the existence of a left invariant mean M on $\mathscr{C}_{\mathscr{W}}(S, \mathbb{R})$. For a fixed element $s \in S$ define $g^s(t) = f(s+t) - f(t)$, $t \in S$. Then, by (3.1), the range of g^s is contained in $F(s)$, which is convex and weakly compact. Hence g^s belongs to $\mathscr{C}_{\mathscr{W}}(S, X)$ and we may apply M to the function g^s. Define the mapping $A : S \to X$ by

$$A(s) = M(g^s) \quad (s \in S).$$

Due to (3.1) and the properties of F,

$$A(s) \in \overline{\mathrm{co}}\, g^s(S) \subset \overline{\mathrm{co}}\, F(s) = F(s) \quad (s \in S).$$

That is, A is a selection of F. To obtain the additivity of A, let $s_1, s_2 \in S$ be arbitrary. Then, for $t \in S$,

$$\begin{aligned} g^{s_1+s_2}(t) &= f(s_1 + s_2 + t) - f(t) \\ &= (f(s_1 + s_2 + t) - f(s_2 + t)) + (f(s_2 + t) - f(t)) \\ &= {}_{s_2}g^{s_1}(t) + g^{s_2}(t). \end{aligned}$$

Hence using the left invariance of M, we get

$$\begin{aligned} A(s_1 + s_2) &= M(g^{s_1+s_2}) = M({}_{s_2}g^{s_1} + g^{s_2}) \\ &= M({}_{s_2}g^{s_1}) + M(g^{s_2}) = M(g^{s_1}) + M(g^{s_2}) = A(s_1) + A(s_2). \end{aligned}$$

The proof is completed.

REMARK 1. In the case where the image space is $\mathbb{R}$ and F is a compact interval-valued set-valued map from a commutative semigroup S, then Theorem 5 reduces to a recent result of Páles [16] which was proved by a completely different technique based on the sandwich theorems obtained by Nikodem, Páles and Wąsowicz [15].

COROLLARY 3. *Let* $(S, +)$ *be a left amenable semigroup and let* X *be a reflexive Banach space. In addition, let* $\rho : S \to [0, \infty[$ *and* $g : S \to X$ *be arbitrary functions. Then there exists an additive function* $A : S \to X$ *such that*

$$\|A(s) - g(s)\| \leq \rho(s) \quad (s \in S) \tag{3.2}$$

if and only if there exists a function $f : S \to X$ *such that*

$$\|f(s+t) - f(t) - g(s)\| \leq \rho(s) \quad (s, t \in S). \tag{3.3}$$

PROOF. Define a set-valued map $F : S \to 2^X$ by

$$F(s) = B(g(s), \rho(s)) = \{x \in X : \|x - g(s)\| \leq \rho(s)\} \quad (s \in S).$$

Then, due to the reflexivity of X, F has weakly compact nonempty convex values. Observe that (3.2) means that A is a selection of F, and (3.3) is equivalent to (3.1). Thus the statement immediately follows from Theorem 5.

The next result can be interpreted as a stability result for the Cauchy functional equation. It is due to Ger [6, Theorem 2].

COROLLARY 4. *Let* $(S, +)$ *be a left amenable semigroup,* X *be a reflexive Banach space and let* $\rho : S \to [0, \infty)$ *be an arbitrary function. Assume that a function* $f : S \to X$ *satisfies*

$$\|f(s+t) - f(t) - f(s)\| \leq \rho(s) \quad (s, t \in S). \tag{3.4}$$

Then there exists an additive function $A : S \to X$ *such that*

$$\|A(s) - f(s)\| \leq \rho(s) \quad (s \in S). \tag{3.5}$$

PROOF. Taking $g = f$, we can see that condition (3.3) of the previous corollary is fulfilled. Hence an additive function A such that (3.2) holds exists.

REMARK 2. In fact, the result of Ger [6, Theorem 2] contains analogous statements also for the case when X has the Hahn-Banach extension property, or when X is a boundedly complete Banach lattice with a strong unit element. Starting from Corollary 4 above, other types of stability results can also be obtained, for example when the right-hand side of (3.4) is replaced by expressions of the form $\rho(s) + \rho(t) - \rho(s+t)$ and $\rho(s+t)$, respectively. The details and the corresponding statements can be found in [6, Theorems 3 and 4].

4. Quasi-additive functions

We recall the notion of quasi-additive functions introduced by Tabor [20, 21]. These functions turned out to behave very similarly to additive functions (see also Baran [2]). A better understanding of the either very regular or very irregular behaviour was obtained by Ger [6]. Recently, Páles [16] proved for real-valued quasi-additive functions that they can be factorised as the composition of a very regular function and an additive function. The main goal of this section is to obtain analogous results for the vector-valued case as well.

DEFINITION 4. Let $(S, +)$ be a semigroup and X be a linear normed space. A function $f : S \to X$ is said to be *quasi-additive* if, for some $\varepsilon \in [0, 1)$,

$$\|f(s+t) - f(s) - f(t)\| \leq \varepsilon \min(\|f(s+t)\|, \|f(s) + f(t)\|) \quad (s, t \in S).$$

If ε is given, then the functions satisfying this inequality will be called ε-quasiadditive.

Clearly, the above inequality can be split into the following two inequalities:

$$\|f(s+t) - f(s) - f(t)\| \leq \varepsilon \|f(s+t)\| \quad (s, t \in S), \tag{4.1}$$

and

$$\|f(s+t) - f(s) - f(t)\| \leq \varepsilon \|f(s) + f(t)\| \quad (s, t \in S). \tag{4.2}$$

In what follows, we first consider the functional inequality (4.3) below that is strictly related to (4.1). We shall obtain a complete characterisation for the solution of this inequality.

THEOREM 6. *Let $(G, +)$ be an amenable group, X be a reflexive Banach space and let $f, g : G \to X$ be arbitrary functions. If, for some $\varepsilon \in [0, 1)$, f and g satisfy the functional inequality*

$$\|f(s+t) - f(t) - g(s)\| \leq \varepsilon \|g(s)\| \quad (s, t \in G), \tag{4.3}$$

then there exist an additive function $A : G \to X$ and a function $\varphi : A(G) \to X$ such that

$$f(s) = \varphi(A(s)) \quad (s \in G), \tag{4.4}$$

$$\|A(s) - g(s)\| \leq \gamma \|g(s)\| \quad (s \in G) \tag{4.5}$$

and

$$\|\varphi(u) - \varphi(v) - (u - v)\| \leq \delta \|u - v\| \quad (u, v \in A(G)) \tag{4.6}$$

hold with $\gamma = \varepsilon$ and $\delta = 2\varepsilon/(1 - \varepsilon)$.

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Conversely, if $A : G \to X$ is an additive function, $\varphi : A(G) \to X$ satisfies (4.6) *with $\delta \geq 0$, g satisfies* (4.5) *with $\gamma \geq 0$ and f is given by* (4.4)*, then the functions f and g satisfy* (4.3) *with $\varepsilon = \gamma\delta + \gamma + \delta$.*

PROOF. Assume that f and g satisfy (4.3). Then, by Corollary 3, we have the existence of an additive function $A : G \to X$ such that (4.5) holds with $\gamma = \varepsilon$. Using the inequality $\big|\|u\| - \|v\|\big| \leq \|u - v\|$, it follows from (4.3) and (4.5) that

$$(1-\varepsilon)\|g(s)\| \leq \|A(s)\| \quad (s \in G) \tag{4.7}$$

and, respectively,

$$\|f(s+t) - f(t)\| \leq (1+\varepsilon)\|g(s)\| \quad (s, t \in G). \tag{4.8}$$

From (4.7) and (4.8), we obtain

$$\|f(s+t) - f(t)\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|A(s)\| \quad (s, t \in G).$$

Since G is a group, after some obvious substitutions, this inequality implies

$$\|f(s) - f(t)\| \leq \frac{1+\varepsilon}{1-\varepsilon}\|A(s) - A(t)\| \quad (s, t \in G).$$

Hence if $A(s) = A(t)$ for some $s, t \in G$, then $f(s) = f(t)$. Therefore the equation

$$\varphi(A(s)) = f(s) \quad (s \in S)$$

correctly defines a function $\varphi : A(G) \to X$. Thus we have (4.4). To see that (4.6) is also valid, we need the following estimate which follows from (4.3) and (4.5):

$$\begin{aligned}\|f(s+t) - f(t) - A(s)\| &\leq \|f(s+t) - f(t) - g(s)\| + \|g(s) - A(s)\| \\ &\leq \varepsilon\|g(s)\| + \varepsilon\|g(s)\| \leq \frac{2\varepsilon}{1-\varepsilon}\|A(s)\|.\end{aligned}$$

Making obvious substitutions again and using (4.4), we get

$$\|\varphi(A(s)) - \varphi(A(t)) - (A(s) - A(t))\| \leq \frac{2\varepsilon}{1-\varepsilon}\|A(s) - A(t)\| \quad (s, t \in G).$$

Thus, we have proved (4.6).

To show the reversed statement, assume that (4.4), (4.5) and (4.6) are valid with $\gamma, \delta \geq 0$. Then, using (4.4) and substituting $u = A(s+t)$, $v = A(t)$ into (4.6), we obtain $\|f(s+t) - f(t) - A(s)\| \leq \delta\|A(s)\|$, $s, t \in G$. On the other hand, (4.5) yields

$$\|A(s)\| \leq (1+\gamma)\|g(s)\| \quad (s \in G).$$

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Applying the above inequalities, we get

$$\begin{aligned}\|f(s+t)-f(t)-g(s)\| &\le \|f(s+t)-f(t)-A(s)\| + \|A(s)-g(s)\| \\ &\le \delta\|A(s)\| + \gamma\|g(s)\| \le (\gamma\delta+\gamma+\delta)\|g(s)\|.\end{aligned}$$

Thus (4.3) is valid with $\varepsilon = \gamma\delta + \gamma + \delta$.

REMARK 3. Equation (4.4) states that if f is a solution of (4.3), then it is the composition of an additive function and a function φ which is, by (4.6), globally Lipschitz (with Lipschitz constant $L = 1 + \delta = (1+\varepsilon)/(1-\varepsilon)$).

Inequality (4.5) yields, for g, the following more explicit condition (provided that $\gamma < 1$):

$$\|A(s) - g(s)\| \le \frac{\gamma}{1-\gamma}\|A(s)\| \quad (s \in G),$$

that is, if g is a solution of (4.3), then it must be a selection of the set-valued function $F(s) = B(A(s), (\gamma/(1-\gamma))\|A(s)\|)$. Conversely, the inequality

$$\|A(s) - g(s)\| \le \frac{\gamma}{1+\gamma}\|A(s)\| \quad (s \in G),$$

implies (4.5). In the case where X is a Hilbert space (and $\gamma < 1$) (4.5) can equivalently be written as

$$\left\| g(s) - \frac{A(s)}{1-\gamma^2} \right\| \le \frac{\gamma}{1-\gamma^2}\|A(s)\| \quad (s \in G).$$

The inequality (4.6) states that the map $\varphi - I$, where I stands for the identity mapping, is Lipschitzian. Therefore, if $\varphi : X \to X$ is Fréchet differentiable everywhere, then (4.6) holds for all $u, v \in X$ if and only if $\|\varphi'(u) - I\| \le \delta$, $u \in X$, that is, the above inequality is sufficient for (4.6) to hold.

If G is the additive group of a real linear space, then the closure of $A(G)$ is a closed linear space of X. The function φ, being globally Lipschitz, admits a unique continuous extension to this closed subspace of X. Clearly this extension also satisfies (4.6).

Now we specialise Theorem 6 to the case when $f = g$.

COROLLARY 5. *Let $(G, +)$ be an amenable group, X be a reflexive Banach space and let $f : G \to X$ be an arbitrary function. If, for some $\varepsilon \in [0, 1)$, the function f satisfies the inequality*

$$\|f(s+t) - f(t) - f(s)\| \le \varepsilon\|f(s)\| \quad (s, t \in G), \tag{4.9}$$

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then there exist an additive function $A : G \to X$ and a function $\varphi : A(G) \to X$ with $\varphi(0) = 0$ such that $f = \varphi \circ A$ and (4.6) *holds with $\delta = 2\varepsilon/(1-\varepsilon)$.*

Conversely, if $A : G \to X$ is an additive function, $\varphi : A(G) \to X$ satisfies (4.6) *with $\delta \in [0, 1)$, $\varphi(0) = 0$, then $f = \varphi \circ A$ satisfies* (4.9) *with $\varepsilon = 2\delta/(1-\delta)$.*

PROOF. Assume that f is a solution of (4.9). Then f and $g = f$ satisfy (4.3). Due to Theorem 6, we have an additive function A and a function φ such that $f = \varphi \circ A$, and (4.5), (4.6) are valid with $\gamma = \varepsilon$ and $\delta = 2\varepsilon/(1-\varepsilon)$. Substituting $s = 0$ into (4.5), we get that $g(s) = f(s) = 0$. Hence $\varphi(0) = 0$.

Conversely, assume that $\varphi(0) = 0$, $f = \varphi \circ A$ and (4.6) is satisfied. Then, putting $v = 0$ and $u = A(s)$ into (4.6), we get $\|f(s) - A(s)\| \le \delta\|A(s)\|$, $s \in G$. It follows from this inequality that $(1-\delta)\|A(s)\| \le \|f(s)\|$, hence

$$\|f(s) - A(s)\| \le \frac{\delta}{1-\delta}\|f(s)\| \quad (s \in G).$$

Thus we can see that (4.5) is satisfied with $g = f$ and $\gamma = \delta/(1-\delta)$. Applying the reversed statement of Theorem 6, we get (4.9) with $\varepsilon = \gamma\delta + \gamma + \delta = 2\delta/(1-\delta)$. Thus the proof is complete.

The above corollary immediately yields the following result concerning the functional inequality (4.1).

THEOREM 7. *Let $(G, +)$ be an amenable group, X be a reflexive Banach space and let $f : G \to X$ satisfy, for some $\varepsilon \in [0, 1)$, the functional inequality* (4.1). *Then there exist an additive function $A : G \to X$ and an odd function $\varphi : A(G) \to X$ such that $f = \varphi \circ A$ and* (4.6) *holds with $\delta = 2\varepsilon/(1-\varepsilon)$.*

Conversely, if $A : G \to X$ is an additive function, $\varphi : A(G) \to X$ is an odd function satisfying (4.6) *with $\delta \in [0, 1)$, and f is of the form $\varphi \circ A$, then f satisfies* (4.1) *with $\varepsilon = 2\delta/(1-\delta)$.*

PROOF. Assume that f satisfies (4.1). Letting $s = t = 0$, we get that $f(0) = 0$. Substituting $t = -s$, we get that f is an odd function. Now put $s = x + y$, $t = -y$ into (4.1). Then, using the oddness of f, we obtain that f satisfies (4.9). Thus the existence of φ, A such that $f = \varphi \circ A$ and (4.6) is valid is a consequence of the previous corollary. The oddness of φ follows from that of the function f.

To prove the reversed statement, we apply the previous corollary again. Clearly we can obtain that f satisfies (4.9). Putting $s = x + y$, $t = -y$ into (4.9) and using the oddness of f, we derive (4.1) easily.

In our next result, we consider the functional inequality (4.2). Using Corollary 5, we obtain a characterisation for the solutions of (4.2).

THEOREM 8. *Let $(G, +)$ be an amenable group, X be a reflexive Banach space and let $f : G \to X$ satisfy, for some $\varepsilon \in [0, 1/2)$, the functional inequality* (4.2), *then there exist an additive function $A : G \to X$ and an odd function $\varphi : A(G) \to X$ such that $f = \varphi \circ A$ and* (4.6) *holds with $\delta = 2\varepsilon/(1 - 2\varepsilon)$.*

Conversely, if $A : G \to X$ is an additive function, $\varphi : A(G) \to X$ is an odd function satisfying (4.6) *with $\delta \in [0, 1)$, and f is of the form $\varphi \circ A$, then f satisfies* (4.2) *with $\varepsilon = 2\delta/(1 - \delta)$.*

PROOF. Assume that f satisfies (4.2). Then we get that

$$(1 - \varepsilon)\|f(s) + f(t)\| \leq \|f(s + t)\| \quad (s, t \in G).$$

Hence it follows from (4.2) that f fulfills the following type of inequality (4.1):

$$\|f(s + t) - f(s) - f(t)\| \leq \tilde{\varepsilon}\|f(s + t)\| \quad (s, t \in G),$$

where $\tilde{\varepsilon} = \varepsilon/(1 - \varepsilon)$. Since $\varepsilon \in [0, 1/2)$, we have that $\tilde{\varepsilon} \in [0, 1)$. Therefore Corollary 5 yields the existence of an additive function A and an odd function φ such that f has the decomposition $f = \varphi \circ A$ and (4.6) holds with

$$\delta = \frac{2\tilde{\varepsilon}}{1 - \tilde{\varepsilon}} = \frac{2\varepsilon}{1 - 2\varepsilon}.$$

In the proof of the sufficiency, we shall give a direct computation without the use of Corollary 5. Assume that A is an additive and φ is an odd function satisfying (4.6). Let f be of the form $\varphi \circ A$. Then, substituting $u = A(s)$, $v = A(-t)$ into (4.6) and using the oddness of φ, we obtain

$$\|f(s) + f(t) - A(s + t)\| \leq \delta\|A(s + t)\| \quad (s, t \in G). \tag{4.10}$$

The following estimate follows directly from this inequality:

$$(1 - \delta)\|A(s + t)\| \leq \|f(s) + f(t)\|.$$

Setting $s + t$ and 0 in place of s and t, respectively, in (4.10), we also have

$$\|f(s + t) - A(s + t)\| \leq \delta\|A(s + t)\| \quad (s, t \in G).$$

Hence we can obtain (4.2) in the following way:

$$\begin{aligned}\|f(s + t) - f(s) - f(t)\| &\leq \|f(s + t) - A(s + t)\| + \|A(s + t) - f(s) - f(t)\| \\ &\leq 2\delta\|A(s + t)\| \leq \frac{2\delta}{1 - \delta}\|f(s) + f(t)\|.\end{aligned}$$

Thus the proof has been completed.

REMARK 4. We note that $\varepsilon < 1/2$ is not only a technical condition in the above result. If $\varepsilon \geq 1/2$, then the function $f = c$ (= constant) satisfies (4.2), but f is not odd (except in the trivial case $c = 0$). On the other hand, it is not clear if $\delta = 2\varepsilon/(1-2\varepsilon)$ is the best constant in (4.6).

In our final result, we offer a characterisation of quasi-additivity.

COROLLARY 6. *Let* $(G, +)$ *be an amenable group,* X *be a reflexive Banach space and let* $f : G \to X$ *be an* ε*-quasi-additive function. Then there exist an additive function* $A : G \to X$ *and an odd function* $\varphi : A(G) \to X$ *such that* $f = \varphi \circ A$ *and* (4.6) *holds with* $\delta = 2\varepsilon/(1-\varepsilon)$.

Conversely, if $A : G \to X$ *is an additive function,* $\varphi : A(G) \to X$ *is an odd function satisfying* (4.6) *with* $\delta \in [0, 1)$, *and* f *is of the form* $\varphi \circ A$, *then* f *is* ε*-quasi-additive with* $\varepsilon = 2\delta/(1-\delta)$.

PROOF. If f is ε-quasiadditive, then f satisfies (4.1). Hence by Theorem 7, we have an additive function A and an odd function φ such that $f = \varphi \circ A$, and (4.6) is valid.

Conversely, if f is of the form $\varphi \circ A$, and (4.6) holds, then Theorems 7 and 8 yield that (4.1) and (4.2) are satisfied with $\varepsilon = 2\delta/(1-\delta)$. Hence f is ε-quasiadditive.

REMARK 5. Due to the decomposition $f = \varphi \circ A$, the regularity and irregularity properties of quasi-additive functions, very similar to those observed and proved by Tabor [20], [21] and Baran [2] for additive functions, can easily be obtained.

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