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RESEARCH ARTICLE

Global F-regularity for weak del Pezzo surfaces

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Abstract
Let k be an algebraically closed field of characteristic $p > 0$. Let X be a normal projective surface over k with canonical singularities whose anticanonical divisor is nef and big. We prove that X is globally F-regular except for the following cases: (1) $K_X^2 = 4$ and $p = 2$, (2) $K_X^2 = 3$ and $p \in \{2, 3\}$, (3) $K_X^2 = 2$ and $p \in \{2, 3\}$, (4) $K_X^2 = 1$ and $p \in \{2, 3, 5\}$. For each degree K_X^2, the assumption of p is optimal.

Contents

1. Introduction

We work over an algebraically closed field of characteristic $p > 0$. Fano varieties play a significant role in the classification of algebraic varieties. In positive characteristic, properties defined by the Frobenius morphism such as (global) F-splitting or global F-regularity are useful. Therefore, it is natural to ask when Fano varieties are F-split or globally F-regular. For smooth del Pezzo surfaces, Hara [Har98a] proved the following result:

Theorem 1.1 [Har98a, Example 5.5]. *Let X be a smooth del Pezzo surface over an algebraically closed field of characteristic $p > 0$. Then X is globally F-regular except for the following cases:*

(1) $K_X^2 = 3$ *and* $p = 2$.
(2) $K_X^2 = 2$ *and* $p \in \{2, 3\}$.
(3) $K_X^2 = 1$ *and* $p \in \{2, 3, 5\}$.

Our aim is to generalize Hara's result to the case when $-K_X$ is nef and big, or equivalently, X is canonical. Here, we say that a variety is *canonical* if it has only canonical singularities. In fact, the following theorem holds.

Theorem A. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a canonical projective surface over k whose anticanonical divisor is nef and big. Then X is globally F-regular except for the following cases:*

(1) $K_X^2 = 4$ *and* $p = 2$.
(2) $K_X^2 = 3$ *and* $p \in \{2, 3\}$.
(3) $K_X^2 = 2$ *and* $p \in \{2, 3\}$.
(4) $K_X^2 = 1$ *and* $p \in \{2, 3, 5\}$.

Theorem A has an important role for investigating of global F-regularity of smooth Fano threefolds and smooth del Pezzo varieties (see [KT24b, KT24c] for details).

Remark 1.2. For each degree K_X^2, the assumption on p is optimal. In fact, for each case listed above, there exists a canonical del Pezzo surface that is not strongly F-regular (see [KN23, Table 1] and [KT24a, Table 1]).

The assumption on p is still optimal even if we assume that X is smooth since taking the minimal resolution does not change the degree and globally F-regularity (Corollary 2.5). Similarly, even if we replace 'nef and big' by 'ample', the conclusion of Theorem A is the same. This is because X is globally F-regular if and only if its anticanonical model is globally F-regular (Proposition 2.4). In particular, Theorem A does not imply Theorem 1.1.

Remark 1.3. For each prime number p, there exists a Kawamata log terminal (klt) del Pezzo surface X (i.e., a normal projective surface such that $(X, 0)$ is klt and $-K_X$ is ample) that is not F-split [CTW18, Theorem 1.1].

We now focus on the proof of Theorem A. We first investigate when X as in Theorem A is F-split. The proof is divided into two cases: the case where $K_X^2 \geq 5$ and the case where $K_X^2 \leq 4$.

First, we consider the case where $K_X^2 \geq 5$. We may assume that X is obtained by taking a blowup $f: X \to \mathbb{P}^2$ along some points. Recall that if there exists an effective divisor $\Delta_{\mathbb{P}^2}$ on $\mathbb{P}^2$ such that the divisor Δ on X defined by $K_X + \Delta = f^*(K_{\mathbb{P}^2} + \Delta_{\mathbb{P}^2})$ is effective, then the following holds (Proposition 2.4):

$$(\mathbb{P}^2, \Delta_{\mathbb{P}^2}) \text{ is } F\text{-split} \Leftrightarrow (X, \Delta) \text{ is } F\text{-split} \Rightarrow X \text{ is } F\text{-split}.$$

Such a divisor $\Delta_{\mathbb{P}^2}$ can be found by utilizing an inversion of adjunction for F-splitting (Proposition 2.6). However, since we can only assume that the blowup points are in *almost general position*, the situation is more complicated than the smooth del Pezzo cases, which are obtained by blowing up points in *general position*. For more details, see Proposition 3.8.

Next, we consider the case where $K_X^2 \leq 4$. In the case of smooth del Pezzo surfaces, Hara [Har98a] investigated F-splitting using the following two steps:

(i) The reduction of F-splitting to the vanishing $H^1(X, \Omega_X^1(pK_X)) = 0$ via the Cartier operator.
(ii) Embedding X as a hypersurface or a complete intersection of a weighted projective space and proving the vanishing $H^1(X, \Omega_X^1(pK_X)) = 0$.

For a smooth weak del Pezzo surface X, we can also reduce the F-splitting of X to the vanishing $H^1(X, \Omega_X^1(pK_X)) = 0$. However, Step (ii), proving $H^1(X, \Omega_X^1(pK_X)) = 0$, is not easy because X is not embedded in a weighted projective space via $|-mK_X|$ for $m \in \mathbb{Z}_{>0}$. Therefore, by replacing X with its anticanonical model, we embed X into a weighted projective space via $|-mK_X|$. However, in this case, Step (i), the reduction of the F-splitting of X to the vanishing $H^1(X, \Omega_X^1(pK_X)) = 0$, is not

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straightforward since Cartier operator is defined on smooth schemes. To address this issue, we use the reflexive Cartier operator introduced in [Kaw22b]. Indeed, we utilize the fact that the reflexive Cartier operator behaves well on F-pure klt surfaces (Lemma 3.1).

Combining the above results, we obtain the following theorem.

Theorem B. *Let k be an algebraically closed field of characteristic $p > 0$. Let X be a canonical F-pure projective surface over k whose anticanonical divisor is ample. Then X is F-split except for the following cases:*

(1) $K_X^2 = 3$ *and* $p = 2$.
(2) $K_X^2 = 2$ *and* $p \in \{2, 3\}$.
(3) $K_X^2 = 1$ *and* $p \in \{2, 3, 5\}$.

Remark 1.4.

(1) When $K_X^2 = 4$ and $p = 2$, there exists a a canonical del Pezzo surface with D_5^0-singularity, which is not F-pure [KN23, Proposition 3.16].
(2) When $K_X^2 = 3$ and $p = 3$, there exists a canonical del Pezzo surface with E_6^0-singularity, which is not F-pure [KN23, Proposition 3.22].

Finally, we now overview how to deduce Theorem A from Theorem B. Let X be a canonical weak del Pezzo surface. Replacing X with its anticanonical model, we can assume that $-K_X$ is ample. For each degree K_X^2, we will find an optimal bound on p that ensures all the singularities of X are strongly F-regular (see Lemma 3.9). We then conclude by the following well-known fact that asserts the equivalence between F-splitting and global F-regularity for strongly F-regular $\mathbb{Q}$-Gorenstein Fano varieties:

Theorem 1.5 (cf. [KT23, Proof of Theorem 6.2]). *Let X be a normal projective variety such that $-K_X$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Suppose that X is strongly F-regular. If X is F-split, then X is globally F-regular.*

2. Preliminaries

2.1. Notation and terminology

In this subsection, we summarize notation and basic definitions used in this article.

(1) Throughout the paper, p denotes a prime number and we work over an algebraically closed field k of characteristic $p > 0$. We set $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. We denote by $F\colon X \to X$ the absolute Frobenius morphism on an $\mathbb{F}_p$-scheme X.
(2) We say that X is a *variety* (over k) if X is an integral scheme that is separated and of finite type over k. We say that X is a *curve* (resp. *surface*) if X is a variety of dimension one (resp. two).
(3) For a variety X, we define the *function field* $K(X)$ of X as the stalk $\mathcal{O}_{X,\xi}$ at the generic point ξ of X.
(4) We say that a $\mathbb{Q}$-divisor D on a normal variety X is *simple normal crossing* if for every point $x \in \operatorname{Supp} D$, the local ring $\mathcal{O}_{X,x}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d$ of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,x}$ and $1 \leq r \leq d$ such that $\operatorname{Supp}(D|_{\operatorname{Spec} \mathcal{O}_{X,x}}) = \operatorname{Spec}(\mathcal{O}_{X,x}/(x_1 \cdots x_r))$.
(5) Given a variety X, a projective birational morphism $\pi\colon Y \to X$ is called a *log resolution of* X if Y is a smooth variety and $\operatorname{Exc}(f)$ is a simple normal crossing divisor.
(6) Given a variety X and a closed subscheme Z, we denote by $\operatorname{Bl}_Z X$ the blowup of X along Z.
(7) Given a normal variety X and a $\mathbb{Z}$-divisor D on X, we define a reflexive sheaf $\Omega_X^{[i]}(D)$ by $j_*(\Omega_U^i \otimes \mathcal{O}_U(D))$, where $j\colon U \hookrightarrow X$ is the open immersion from the smooth locus U of X.

2.1.1. Singularities in minimal model program

For the definitions of singularities in minimal model program (e.g., canonical and klt), we refer to [KM98, Section 2.3]. Take a normal surface X. Let $f\colon Y \to X$ be the minimal resolution. We only need the following characterizations in this paper.

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(1) X is canonical if and only if K_X is $\mathbb{Q}$-Cartier and $K_Y \sim_{\mathbb{Q}} f^*K_X$.
(2) X is klt if and only if K_X is $\mathbb{Q}$-Cartier, f is a log resolution of X and all the coefficients of the $\mathbb{Q}$-divisor Γ defined by $K_Y + \Gamma \sim_{\mathbb{Q}} f^*K_X$ are < 1.

By definition, we have

$$\text{canonical} \Rightarrow \text{klt}.$$

Moreover, the following implications hold for the surface case:

- canonical $\Rightarrow$ Gorenstein.
- klt $\Rightarrow$ $\mathbb{Q}$-factorial.

2.1.2. Weak del Pezzo surfaces

Given a normal projective Gorenstein surface X, we say that X is *del Pezzo* (resp. *weak del Pezzo*) if $-K_X$ is ample (resp. nef and big). In what follows, we summarize some properties on (weak) del Pezzo surfaces for later usage.

Let Z be a canonical weak del Pezzo surface. The *anticanonical model* Y of Z is defined as the Stein factorisation of the morphism $\varphi_{|-mK_Z|} \colon Z \to \mathbb{P}^{h^0(Z,-mK_Z)-1}$ induced by the complete linear system $|-mK_Z|$, where m is a positive integer such that $|-mK_X|$ is base point free (whose existence is guaranteed by [Tan15, Theorem 0.4]). Then Y is canonical, because have $K_Z \sim h^*K_Y$ for the induced morphism $h \colon Z \to Y$. Moreover, h is obtained by contracting all the (-2)-curves on Y. In particular, the minimal resolution X of Z coincides with the minimal resolution of Y:

$$f \colon X \xrightarrow{g} Z \xrightarrow{h} Y.$$

Moreover, X is a smooth weak del Pezzo surface.

There is a natural one-to-one correspondence between

- smooth weak del Pezzo surfaces and
- canonical del Pezzo surfaces.

Indeed, if Y is a canonical del Pezzo surface, then its minimal resolution X is a smooth weak del Pezzo surface. Conversely, given a smooth weak del Pezzo surface X, its anticanonical model Y is a canonical del Pezzo surface.

2.2. F-splitting and global F–regularity

In this subsection, we gather basic facts on F-splitting and global F-regularity.

Definition 2.1. Let X be a normal variety, and let Δ be an effective $\mathbb{Q}$-divisor on X.

(1) We say that (X, Δ) is *F-split* if

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lfloor (p^e-1)\Delta \rfloor)$$

splits as an $\mathcal{O}_X$-module homomorphism for every $e \in \mathbb{Z}_{>0}$.
(2) We say that (X, Δ) is *sharply F-split* if

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil)$$

splits as an $\mathcal{O}_X$-module homomorphism for some $e \in \mathbb{Z}_{>0}$.

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(3) We say that (X, Δ) is *globally F-regular* if, given an effective $\mathbb{Z}$-divisor E, there exists $e \in \mathbb{Z}_{>0}$ such that

$$\mathcal{O}_X \xrightarrow{F^e} F^e_*\mathcal{O}_X \hookrightarrow F^e_*\mathcal{O}_X(\lceil (p^e-1)\Delta \rceil + E)$$

splits as an $\mathcal{O}_X$-module homomorphism.

We say that X is *F-split* (resp. *globally F-regular*) if so is $(X, 0)$.

Remark 2.2. We have the following implications:

$$\text{globally } F\text{-regular} \Longrightarrow \text{sharply } F\text{-split} \Longrightarrow F\text{-split}$$

where the former implication is easy and the latter one holds by the same argument as in [Sch08, Proposition 3.3]. Moreover, if the condition (★) holds, then (X, Δ) is sharply F-split if and only if (X, Δ) is F-split.

(★) $(p^e - 1)\Delta$ is a $\mathbb{Z}$-divisor for some $e \in \mathbb{Z}_{>0}$.

In particular, X is F-split if and only if $F\colon \mathcal{O}_X \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism. In this paper, we only treat the case when (★) holds, and hence being F-split is equivalent to being sharply F-split. For more foundational properties, we refer to [SS10].

We shall also use the local versions of F-splitting and global F-regularity.

Definition 2.3. Given a normal variety X, we say that X is *F-pure* (resp. *strongly F-regular*) if there exists an open cover $X = \bigcup_{i \in I} X_i$ such that X_i is F-split (resp. globally F-regular) for every $i \in I$.

In what follows, we summarize some F-splitting criteria, which are well known to experts.

Proposition 2.4. *Let $f : X \to Y$ be a birational morphism of normal projective varieties. Take an effective $\mathbb{Q}$-divisor Δ_Y on Y such that $(p^e - 1)(K_Y + \Delta_Y)$ is Cartier for some $e \in \mathbb{Z}_{>0}$. Assume that the $\mathbb{Q}$-divisor Δ defined by $K_X + \Delta = f^*(K_Y + \Delta_Y)$ is effective. Then (X, Δ) is F-split (resp. globally F-regular) if and only if (Y, Δ_Y) is F-split (resp. globally F-regular).*

Proof. If (X, Δ) is F-split, then so is (Y, Δ_Y), which can be checked by taking the push-forward. As for the opposite implication, the same argument as in the first paragraph of the proof of [HX15, Proposition 2.11] works. □

Corollary 2.5. *Let Y be a canonical projective surface, and let $f : X \to Y$ be its minimal resolution. Then X is F-split (resp. globally F-regular) if and only if Y is F-split (resp. globally F-regular).*

Proof. The assertion immediately follows from Proposition 2.4 by using $K_X = f^*K_Y$. □

Proposition 2.6. *Let X be a normal projective Gorenstein variety. Take a normal prime Cartier divisor S and an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor B on X such that $S \not\subset \operatorname{Supp} B$. Assume that*

(1) *$(S, B|_S)$ is F-split, and*
(2) *there is a positive integer $e \in \mathbb{Z}_{>0}$ such that $(p^e - 1)(K_X + S + B)$ is Cartier and*

$$H^1(X, \mathcal{O}_X(-S - (p^e - 1)(K_X + S + B))) = 0.$$

Then $(X, S + B)$ is F-split.

Proof. The same argument as in [CTW17, Lemma 2.7] works. □

Example 2.7. We now summarize some easy cases for later usage, although all of them are well known to experts,

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(1) If $P, Q \in \mathbb{P}^1$ are distinct points, then $(\mathbb{P}^1, P + Q)$ is F-split (Proposition 2.6).
(2) Let L, L', L'' be three lines on $\mathbb{P}^2$ such that $L + L' + L''$ is simple normal crossing. Then $(\mathbb{P}^2, L + L' + L'')$ is F-split by (1) and Proposition 2.6.
(3) For each $i \in \{1, 2\}$, let F_i and F_i' be distinct fibers of the i-th projection $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then $(\mathbb{P}^1 \times \mathbb{P}^1, F_1 + F_1' + F_2 + F_2')$ is F-split by (1) and Proposition 2.6.

2.3. *Reflexive Cartier operator*

Throughout this subsection, we use the following convention unless stated otherwise.

Convention 2.8. *Let X be a normal variety and D a $\mathbb{Z}$-divisor on X. Let U be the smooth locus of X and $j : U \hookrightarrow X$ the inclusion. By abuse of notation, $D|_U$ is denoted by D.*

The Frobenius pushforward of the de Rham complex

$$F_*\Omega_U^\bullet : F_*\mathcal{O}_U \xrightarrow{F_*d} F_*\Omega_U^1 \xrightarrow{F_*d} \cdots$$

is a complex of $\mathcal{O}_U$-modules. Tensoring with $\mathcal{O}_U(D)$, we obtain a complex

$$F_*\Omega_U^\bullet : F_*\mathcal{O}_U(pD) \xrightarrow{F_*d\otimes\mathcal{O}_U(D)} F_*\Omega_U^1(pD) \xrightarrow{F_*d\otimes\mathcal{O}_U(D)} \cdots .$$

We define coherent $\mathcal{O}_U$-modules $B_U^i(pD)$ and $Z_U^i(pD)$ by

$$B_U^i(pD) \coloneqq \mathrm{Im}(F_*d \otimes \mathcal{O}_U(D) : F_*\Omega_U^{i-1}(pD) \to F_*\Omega_U^i(pD)),$$
$$Z_U^i(pD) \coloneqq \mathrm{Ker}(F_*d \otimes \mathcal{O}_U(D) : F_*\Omega_U^i(pD) \to F_*\Omega_U^{i+1}(pD)),$$

for all $i \geq 0$. Then $B_U^i(pD)$ and $Z_U^i(pD)$ are locally free [Kaw22b, Lemma 3.2]. When $D = 0$, we simply denote $B_U^i(pD)$ and $Z_U^i(pD)$ by B_U^i and Z_U^i, respectively. Then $B_U^i(pD) = B_U^i \otimes \mathcal{O}_U(D)$ and $Z_U^i(pD) = Z_U^i \otimes \mathcal{O}_U(D)$ holds [Kaw22b, Remark 3.3]. In particular, we note that $B_U^i(pD)$ and $Z_U^i(pD)$ do not mean $B_U^i \otimes \mathcal{O}_U(pD)$ and $Z_U^i \otimes \mathcal{O}_U(pD)$.

By [Kaw22b, Lemma 3.2], there exists an exact sequence

$$0 \to B_U^i(pD) \to Z_U^i(pD) \xrightarrow{C_U^i(D)} \Omega_U^i(D) \to 0, \tag{2.8.1}$$

and the map $C_U^i(D)$ coincides with $C_U^i \otimes \mathcal{O}_U(D)$, where C_U^i is the usual Cartier operator.

Definition 2.9. We define reflexive $\mathcal{O}_X$-modules $B_X^{[i]}(pD)$ and $Z_X^{[i]}(pD)$ by

$$B_X^{[i]}(pD) \coloneqq j_*B_U^i(pD) \text{ and}$$
$$Z_X^{[i]}(pD) \coloneqq j_*Z_U^i(pD)$$

for all $i \geq 0$. The *i-th reflexive Cartier operator*

$$C_X^{[i]}(D) : Z_X^{[i]}(pD) \to \Omega_X^{[i]}(D)$$

associated to D is defined as $j_*C_U^i(D)$ for all $i \geq 0$.

Lemma 2.10. *There exist the following exact sequences:*

$$0 \to Z_X^{[i]}(pD) \to F_*\Omega_X^{[i]}(pD) \xrightarrow{d'} B_X^{[i+1]}(pD), \tag{2.10.1}$$

$$0 \to B_X^{[i]}(pD) \to Z_X^{[i]}(pD) \xrightarrow{C_X^{[i]}(D)} \Omega_X^{[i]}(D), \tag{2.10.2}$$

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for all $i \geq 0$. Moreover, $d'|_U\colon F_*\Omega_X^{[i]}(pD)|_U \to B_X^{[i+1]}(pD)|_U$ and $C_X^{[i]}(D)|_U\colon Z_X^{[i]}(pD)|_U \to \Omega_X^{[i]}(D)|_U$ are surjective, and the homomorphism $C_X^{[i]}(D)|_U$ coincides with $C_U^i \otimes \mathcal{O}_U(D)$.

Proof. Taking $B = 0$ in [Kaw22b, Lemma 3.5], we obtain the assertion. □

Remark 2.11. Taking $i = 0$ in equation (2.10.1), we obtain an exact sequence

$$0 \to \mathcal{O}_X(D) \to F_*\mathcal{O}_X(pD) \to B_X^{[1]}(pD),$$

and the first map is induced by the Frobenius homomorphism. In particular,

$$B_X^{[1]}(pD) = j_*\mathrm{Coker}(F\colon \mathcal{O}_U(D) \to F_*\mathcal{O}_U(pD))$$

holds.

3. Proofs of main theorems

3.1. Criterion of the F-splitting of klt surfaces

In this subsection, we provide a criterion for the F-splitting of klt surfaces (Proposition 3.2).

Lemma 3.1. *Let X be an F-pure klt surface and D a $\mathbb{Z}$-divisor. Then the sequence*

$$0 \to B_X^{[1]}(pD) \to Z_X^{[1]}(pD) \xrightarrow{C_X^{[1]}(D)} \Omega_X^{[1]}(D) \to 0 \tag{3.1.1}$$

is exact.

Proof. It is enough to show that $C_X^{[1]}(D)$ is surjective, as the other parts has been settled in Lemma 2.10. Since the assertion is local on X, we may assume that X is affine and has a unique singular point Q. If $p \neq 5$ or the singularity Q is not rational double point (RDP) of type E_8^1, then X is F-liftable by [KT24a, Theorem A]. Then the surjectivity of $C^{[1]}(D)$ follows from [Kaw22b, Lemma 3.8].

Suppose that $p = 5$ and the singularity Q is of type E_8^1. Then we may assume that $D = 0$ by [Lip69, Section 24] (see also [LMM21, Table 2]). Then the desired surjectivity follows from [Kaw22b, Proposition 4.4] and [KT24a, Theorem B]. □

Proposition 3.2. *Let X be an F-pure klt projective surface. Suppose that the following conditions hold:*

(1) $H^0(X, \Omega_X^{[1]}(K_X)) = 0$.
(2) $H^1(X, \Omega_X^{[1]}(pK_X)) = 0$.
(3) $H^0(X, \mathcal{O}_X((p+1)K_X)) = 0$.

Then X is F-split.

Proof. Recall that X is F-split if and only if the evaluation map

$$\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \xrightarrow{F^*} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)(\cong H^0(X, \mathcal{O}_X))$$

is surjective. Then, as in [BK05, 1.3.9 Remarks (ii)], the surjectivity is equivalent to the injectivity of the map

$$F\colon H^2(X, \mathcal{O}_X(K_X)) \to H^2(X, \mathcal{O}_X(pK_X)) \tag{3.2.1}$$

induced by Frobenius by Serre duality. Let U be the smooth locus of X. Since U is F-pure, the exact sequence

$$0 \to \mathcal{O}_U \to F_*\mathcal{O}_U \to B_U^1 \to 0$$

splits locally by definition. Tensoring with $\mathcal{O}_U(K_X)$, we obtain a locally split exact sequence

$$0 \to \mathcal{O}_U(K_U) \to F_*\mathcal{O}_U(pK_U) \to B_U^{[1]}(pK_U) \to 0.$$

Since the above exact sequence splits locally, taking pushforward preserves exactness on the right. Thus, taking the pushforward by the inclusion $U \hookrightarrow X$, we obtain the following locally split exact sequence

$$0 \to \mathcal{O}_X(K_X) \to F_*\mathcal{O}_X(pK_X) \to B_X^{[1]}(pK_X) \to 0.$$

Therefore, for the injectivity of equation (3.2.1), it suffices to show that $H^1(X, B_X^{[1]}(pK_X)) = 0$. By Lemma 3.1 and the condition (1), it is enough to prove that $H^1(X, Z_X^{[1]}(pK_X)) = 0$. By equation (2.10.1), we have an exact sequence

$$0 \to Z_X^{[1]}(pK_X) \to F_*\Omega_X^{[1]}(pK_X) \to B_X^{[2]}(pK_X).$$

Let $\mathcal{B} := \mathrm{Im}(F_*\Omega_X^{[1]}(pK_X) \to B_X^{[2]}(pK_X))$. Then we obtain an exact sequence

$$H^0(X, \mathcal{B}) \to H^1(X, Z_X^{[1]}(pK_X)) \to H^1(X, \Omega_X^{[1]}(pK_X)) \overset{(2)}{=} 0.$$

Since we have

$$\mathcal{B} \subset B_X^{[2]}(pK_X) \subset F_*\Omega_X^{[2]}(pK_X) = H^0(X, \mathcal{O}_X((p+1)K_X)) \overset{(3)}{=} 0,$$

we conclude that $H^1(X, Z_X^{[1]}(pK_X)) = 0$. □

The condition (3) of Proposition 3.2 is satisfied if $-K_X$ is big. In what follows, we see when the condition (1) of Proposition 3.2 is satisfied.

Definition 3.3 (Log liftability). Let X be a normal projective surface. We say that X is *log liftable* if there exists a log resolution $f: Y \to X$ of X such that $(Y, \mathrm{Exc}(f))$ lifts to the ring $W(k)$ of Witt vectors. For the definition of liftability of a log smooth pair, we refer to [Kaw22a, Definition 2.6].

Lemma 3.4. *Let X be a normal projective F-pure surface such that $-K_X$ is a nef and big $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor. Then X is log liftable if and only if $H^0(X, \Omega_X^{[1]}(K_X)) = 0$.*

Proof. Since $H^2(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X(K_X)) = 0$, the 'if' direction is [Kaw22c, Theorem 2.8]. We prove the 'only if' direction. Let $f: Y \to X$ be a log resolution such that $(Y, E := \mathrm{Ex}(f))$ lifts to $W(k)$. Since $f_*(\Omega_Y^1(\log E) \otimes \mathcal{O}_Y(f^*K_X)) = \Omega_X^{[1]}(K_X)$ by [KT24a, Theorem B], we have $H^0(X, \Omega_X^{[1]}(K_X)) = H^0(Y, \Omega_Y^1(\log E) \otimes \mathcal{O}_Y(f^*K_X))$. Then the vanishing follows from [Kaw22a, Theorem 2.11]. □

3.2. Global F-splitting: Proof of Theorem B

In the following proposition, we investigate F-splitting of F-pure canonical del Pezzo surfaces. For the proof, we confirm when the condition (2) of Proposition 3.2 is satisfied.

Proposition 3.5. *Let X be an F-pure canonical del Pezzo surface. Suppose that one of the following holds.*

(1) $K_X^2 = 1$ *and* $p > 5$.
(2) $K_X^2 = 2$ *and* $p > 3$.
(3) $K_X^2 = 3$ *and* $p > 2$.
(4) $K_X^2 = 4$.

Then X is F-split.

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Proof. In each case, X is log liftable by [KN22, Theorem 1.7 (1)], and thus the condition (1) of Proposition 3.2 is satisfied. Thus, it suffices to confirm the condition (2) of Proposition 3.2, that is, $H^1(X, \Omega_X^{[1]}(pK_X)) = 0$. By Serre duality of Cohen–Macaulay sheaves [KM98, Theorem 5.71], we have $H^1(X, \Omega_X^{[1]}(-pK_X)) \cong H^1(X, \Omega_X^{[1]}(pK_X))$. Since X has only hypersurface singularities, Ω_X^1 is torsion-free by [Lip65, Section 8 (1)], and the natural map $\Omega_X^1 \to \Omega_X^{[1]}$ is injective. Since $\mathcal{O}_X(-pK_X)$ is Cartier, we have an exact sequence

$$0 \to \Omega_X^1 \otimes \mathcal{O}_X(-pK_X) \to \Omega_X^{[1]}(-pK_X) \to \mathcal{C} \to 0$$

for some coherent sheaf $\mathcal{C}$ satisfying $\dim \operatorname{Supp}(\mathcal{C}) = 0$. Since $H^1(X, \mathcal{C}) = 0$, it suffices to show that

$$H^1(X, \Omega_X^1 \otimes \mathcal{O}_X(-pK_X)) = 0.$$

In what follows, we divide the proof into the cases according to (1)–(4) in the proposition.

The case (1): In this case, X is a hypersurface of $P := \mathbb{P}(1,1,2,3)$ of degree 6 [BT22, Theorem 2.15]. By [Mor75, Theorem 1.7], the non-Gorenstein locus of P is $\{[0:0:0:1], [0:0:1:0]\}$, and this locus coincides with the singular locus of P (Remark 3.6). Thus, X is contained in the smooth locus of P since it is Gorenstein. We define invertible sheaves $\mathcal{O}_X(n)$ by $\mathcal{O}_P(n)|_X$ for all $n \in \mathbb{Z}$.

By adjunction, we have $\omega_X = \mathcal{O}_X(-1)$, and thus we aim to show that

$$H^1(X, \Omega_X^1 \otimes \mathcal{O}_X(p)) = 0.$$

By the conormal exact sequence, we have an exact sequence

$$\mathcal{O}_X(-X+p) = \mathcal{O}_X(p-6) \to \Omega_P^1|_X \otimes \mathcal{O}_X(p) \to \Omega_X^1 \otimes \mathcal{O}_X(p) \to 0.$$

Since $\mathcal{O}_X(p-6)$ is torsion-free and the first map is injective outside the singular points of X, we obtain an exact sequence

$$0 \to \mathcal{O}_X(p-6) \to (\Omega_P^1 \otimes \mathcal{O}_P(p))|_X \to \Omega_X^1 \otimes \mathcal{O}_X(p) \to 0.$$

Since $p \geq 7$, we have $H^2(X, \mathcal{O}_X(p-6)) = 0$ by Serre duality, and hence it suffices to show that $H^1(X, (\Omega_P^1 \otimes \mathcal{O}_P(p))|_X) = 0$. We have an exact sequence

$$\Omega_P^{[1]}(p-6) = \Omega_P^{[1]}(p) \otimes \mathcal{O}_P(-6) \to \Omega_P^{[1]}(p) \to \Omega_P^{[1]}(p)|_X \to 0.$$

Here, we obtain the first equality as follows:

$$\Omega_P^{[1]}(p) \otimes \mathcal{O}_P(-6) = (\Omega_P^{[1]} \otimes \mathcal{O}_P(p))^{**} \otimes \mathcal{O}_P(-6) = (\Omega_P^{[1]} \otimes \mathcal{O}_P(p-6))^{**} = \Omega_P^{[1]}(p-6)$$

since $\mathcal{O}_P(-6)$ is Cartier. In particular, the first term of the above exact sequence is torsion-free, and thus the first map is injective since it is injective outside the singular points of P.

Moreover, since X is contained in the smooth locus of P, it follows that $\Omega_P^{[1]}(p)|_X = (\Omega_P^1 \otimes \mathcal{O}_P(p))|_X$. Thus, we obtain an exact sequence

$$0 \to \Omega_P^{[1]}(p-6) \to \Omega_P^{[1]}(p) \to (\Omega_P^1 \otimes \mathcal{O}_P(p))|_X \to 0.$$

By Bott vanishing on P [Fuj07, Corollary 1.3], we have

$$H^1(P, \Omega_P^{[1]}(p)) = H^2(P, \Omega_P^{[1]}(p-6)) = 0$$

since $p \geq 7$. Therefore, we obtain $H^1(X, (\Omega_P^1 \otimes \mathcal{O}_P(p))|_X) = 0$.

The case (2): In this case, X is a hypersurface of $P := \mathbb{P}(1,1,1,2)$ of degree 4 [BT22, Theorem 2.15]. By [Mor75, Theorem 1.7], the non-Gorenstein locus of P is $\{[0:0:0:1]\}$, and this locus coincides with the singular locus of P (Remark 3.6). Thus, X is contained in the smooth locus of P since it is Gorenstein.

By adjunction, we have $\omega_X = \mathcal{O}_X(-1)$, and thus we aim to show that

$$H^1(X, \Omega^1_X \otimes \mathcal{O}_X(p)) = 0.$$

As in the case (1), by the conormal exact sequence and the torsion-freeness of $\mathcal{O}_X(p-4)$, we have an exact sequence

$$0 \to \mathcal{O}_X(p-4) \to \Omega^1_P|_X \otimes \mathcal{O}_X(p) \to \Omega^1_X \otimes \mathcal{O}_X(p) \to 0.$$

Since $p \geq 5$, we have $H^2(X, \mathcal{O}_X(p-4)) = 0$, and it suffices to show that

$$H^1(X, \Omega^1_P|_X \otimes \mathcal{O}_X(p)) = 0.$$

As in the case (1), we have an exact sequence

$$0 \to \Omega^{[1]}_P(p-4) \to \Omega^{[1]}_P(p) \to \Omega^1_P|_X \otimes \mathcal{O}_X(p) \to 0.$$

By Bott vanishing [Fuj07, Corollary 1.3], we have

$$H^1(P, \Omega^{[1]}_P(p)) = H^2(P, \Omega^{[1]}_P(p-4)) = 0$$

since $p \geq 5$. Therefore, we obtain $H^1(X, \Omega^1_P(p)|_X) = 0$.

The case (3): In this case, X is a hypersurface of $P := \mathbb{P}^3$ of degree 3 [BT22, Theorem 2.15]. By adjunction, we have $\omega_X = \mathcal{O}_X(-1)$, and thus we aim to show that $H^1(X, \Omega^1_X(p)) = 0$. By the conormal exact sequence and the torsion-freeness of $\mathcal{O}_X(p-3)$, we have an exact sequence

$$0 \to \mathcal{O}_X(p-3) \to \Omega^1_P(p)|_X \to \Omega^1_X(p) \to 0.$$

Since $p \geq 3$, we have $H^2(X, \mathcal{O}_X(p-3)) = 0$, and it suffices to show that $H^1(X, \Omega^1_P(p)|_X) = 0$. We have an exact sequence

$$0 \to \Omega^1_P(p-3) \to \Omega^1_P(p) \to \Omega^1_P(p)|_X \to 0.$$

By Bott vanishing [Fuj07, Corollary 1.3], we have $H^1(P, \Omega^1_P(p)) = 0$. By [Tot24, Proposition 1.3], we also have $H^2(P, \Omega^1_P(p-3)) = 0$ since $p \geq 3$. Therefore, we obtain $H^1(X, \Omega^1_P(p)|_X) = 0$.

The case (4): In this case, X is a complete intersection of two quadric hypersurfaces Q and Q' of $P := \mathbb{P}^4$ [BT22, Theorem 2.15]. By adjunction, we have $\omega_X = \mathcal{O}_X(-1)$, and thus we aim to show that $H^1(X, \Omega^1_X(p)) = 0$. By the conormal exact sequence and the torsion-freeness of $\mathcal{O}_X(p-2)$, we have an exact sequence

$$0 \to \mathcal{O}_X(p-2) \to (\Omega^1_Q \otimes \mathcal{O}_Q(p))|_X \to \Omega^1_X \otimes \mathcal{O}_X(p) \to 0.$$

Since $p \geq 2$, we have $H^2(X, \mathcal{O}_X(p-2)) = 0$, and hence it suffices to show that $H^1(X, (\Omega^1_Q \otimes \mathcal{O}_Q(p))|_X) = 0$.

We define invertible sheaves $\mathcal{O}_Q(n)$ as $\mathcal{O}_P(n) \otimes \mathcal{O}_Q$ for all $n \in \mathbb{Z}$. We have an exact sequence

$$\Omega^1_Q \otimes \mathcal{O}_Q(p-2) \to \Omega^1_Q \otimes \mathcal{O}_Q(p) \to (\Omega^1_Q \otimes \mathcal{O}_Q(p))|_X \to 0.$$

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Since X is regular in codimension one, Ω^1_Q is torsion-free by [Lip65, Section 8 (1)]. Thus, we have an exact sequence

$$0 \to \Omega^1_Q \otimes \mathcal{O}_Q(p-2) \to \Omega^1_Q \otimes \mathcal{O}_Q(p) \to (\Omega^1_Q \otimes \mathcal{O}_Q(p))|_X \to 0.$$

Therefore, it suffices to show that

$$H^1(Q, \Omega^1_Q \otimes \mathcal{O}_Q(p)) = 0 \text{ and } H^2(Q, \Omega^1_Q \otimes \mathcal{O}_Q(p-2)) = 0,$$

and in particular, the following claim finishes the proof of the case (4):

Claim. We have

(i) $H^1(Q, \Omega^1_Q \otimes \mathcal{O}_Q(n)) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$ and
(ii) $H^2(Q, \Omega^1_Q \otimes \mathcal{O}_Q(n)) = 0$ for every $n \in \mathbb{Z}$.

We have

(a) $H^i(P, \Omega^1_P(n)) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$ and $i \in \{1, 2, 3\}$, and
(b) $H^j(P, \Omega^1_P(n)) = 0$ for every $n \in \mathbb{Z}$ and $j \in \{2, 3\}$.

Indeed, (a) follows from Bott vanishing and Serre duality. Then (a), together with [Tot24, Proposition 1.3], implies (b). By the following exact sequence

$$0 \to \Omega^1_P(n-2) \to \Omega^1_P(n) \to \Omega^1_P(n)|_Q \to 0,$$

we get

(i)' $H^1(Q, \Omega^1_P(n)|_Q) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$, and
(ii)' $H^2(Q, \Omega^1_P(n)|_Q) = 0$ for every $n \in \mathbb{Z}$.

By the conormal exact sequence and the torsion-freeness of $\mathcal{O}_P(n-2)$, we have an exact sequence

$$0 \to \mathcal{O}_P(n-2) \to \Omega^1_P(n)|_Q \to \Omega^1_Q \otimes \mathcal{O}_Q(n) \to 0$$

for every $n \in \mathbb{Z}$. Since $H^2(P, \mathcal{O}_P(n)) = H^3(P, \mathcal{O}_P(n)) = 0$ for every $n \in \mathbb{Z}$, we have the claim. □

Remark 3.6. Take positive integers q_1, q_2, q_3 such that $\gcd(q_1, q_2, q_3) = 1$. Set $P := \mathbb{P}(1, q_1, q_2, q_3)$. Then it is well known (cf. [Ful93, Section 2.2]) that P coincides with the projective $\mathbb{Q}$-factorial toric threefold associated to the fan in $\mathbb{R}^3$ that is generated by four rays $\mathbb{R}u, \mathbb{R}e_1, \mathbb{R}e_2, \mathbb{R}e_3$, where e_1, e_2, e_3 is the standard basis of $\mathbb{Z}^3$ and

$$u := -(q_1e_1 + q_2e_2 + q_3e_3).$$

In the above proof, we have used the results (1) and (2).

(1) $\mathbb{P}(1, 1, 2, 3)$ has exactly two singular points, which corresponds to the cones $\mathbb{R}u + \mathbb{R}e_1 + \mathbb{R}e_2$ and $\mathbb{R}u + \mathbb{R}e_1 + \mathbb{R}e_3$ [CLS11, Theorem 1.3.12].
(2) $\mathbb{P}(1, 1, 1, 2)$ has a unique singular point, which corresponds to the cone $\mathbb{R}u + \mathbb{R}e_1 + \mathbb{R}e_2$ [CLS11, Theorem 1.3.12].

From now on, we focus on the case where $K_X^2 \geq 5$.

Proposition 3.7. *The following assertions hold.*

(1) *Fix an integer m satisfying $1 \leq m \leq 5$. Let $P_1, \ldots, P_m$ be distinct points on $\mathbb{P}^2$ such that the blowup X of $\mathbb{P}^2$ along $\{P_1, \ldots, P_m\}$ is a weak del Pezzo surface. Then X is F-split.*
(2) *Fix an integer n satisfying $1 \leq n \leq 4$. Let $Q_1, \ldots, Q_n$ be distinct points on $\mathbb{P}^1 \times \mathbb{P}^1$ such that the blowup X of $\mathbb{P}^1 \times \mathbb{P}^1$ along $\{Q_1, \ldots, Q_n\}$ is a weak del Pezzo surface. Then X is F-split.*

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Proof. Let us show (1). In what follows, we only treat the case when $m = 5$, as otherwise the problem is easier. Let L (resp. L') be the line on $\mathbb{P}^2$ passing through P_1 and P_2 (resp. P_3 and P_4). Since X is weak del Pezzo, we obtain $L \neq L'$. Pick a general line L'' on $\mathbb{P}^2$ passing through P_5. Then $L + L' + L''$ is simple normal crossing. Therefore, $(\mathbb{P}^2, L + L' + L'')$ is F-split (Example 2.7(2)), which implies that so is X (Proposition 2.4). Thus, (1) holds. The proof of (2) is similar to that of (1). Indeed, for each projection $\mathrm{pr}_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, it is enough to take two fibers F_i and F_i' such that $F_1 \cup F_1' \cup F_2 \cup F_2'$ contains $\{Q_1, \ldots, Q_n\}$ (Example 2.7(3)). □

Proposition 3.8. *Let X be a smooth weak del Pezzo surface satisfying $K_X^2 \geq 5$. Then X is F-split.*

Proof. By [Dol12, Theorem 8.1.15], we may assume that there is a birational morphism $f \colon X \to \mathbb{P}^2$. In what follows, we only treat the case when $K_X^2 = 5$, as the other cases are simpler. There are the following five cases [Dol12, Section 8.5].

(i) P, Q, R, S.
(ii) $P' \succ P, Q, R$.
(iii) $P' \succ P, Q' \succ Q$.
(iv) $P'' \succ P' \succ P, Q$.
(v) $P'' \succ P' \succ P \succ Q$.

For the definition of $P' \succ P$, we refer to [Dol12, Section 7.3.2]. For example, in the case (iii), we have $X = Y'' \to Y' \to Y = \mathbb{P}^2$, where $Y' = \mathrm{Bl}_{P \amalg Q} Y$, $Y'' = \mathrm{Bl}_{P' \amalg Q'}$, and P' and Q' are points on Y' lying over P and Q, respectively.

The case (i) has been settled in Proposition 3.7, because we have $X = \mathrm{Bl}_{P \amalg Q \amalg R \amalg S} \mathbb{P}^2$ for distinct points $P, Q, R, S \in \mathbb{P}^2$ in this case. In the case (ii), P, Q, R are distinct points on $\mathbb{P}^2$, and we have $X = \mathrm{Bl}_{P'} Y'$ for $Y' := \mathrm{Bl}_{P \amalg Q \amalg R} \mathbb{P}^2$ and a closed point P' lying over P. Take the line $L_1 := \overline{PQ}$ passing through P and Q. Let L_2 and L_3 be general lines passing through P and R, respectively. Then $(\mathbb{P}^2, L_1 + L_2 + L_3)$ is F-split (Example 2.7(2)). Since Δ is effective for the divisor Δ defined by $K_X + \Delta = f^*(K_{\mathbb{P}^2} + L_1 + L_2 + L_3)$, it follows that X is F-split (Proposition 2.4). Similarly, (iii) is settled by taking the line $L_1 := \overline{PQ}$ and general lines L_2 and L_3 passing through P and Q, respectively.

Let us treat the case (iv). In this case, we have $X = Y''' \to Y'' \to Y' \to Y = \mathbb{P}^2$, where $Y' := \mathrm{Bl}_{P \amalg Q} Y, Y'' := \mathrm{Bl}_{P'} Y', Y''' := \mathrm{Bl}_{P''} Y''$, and P' (resp. P'') is lying over P (resp. P'). Let L_1 be the line on $Y = \mathbb{P}^2$ such that $P \in L_1$ and $P' \in L_1'$ for the proper transform L_1' of L_1 on Y'. Let L_2 and L_3 be general lines passing through P and Q, respectively. Then we can check that the divisor Δ defined by $K_X + \Delta = f^*(K_{\mathbb{P}^2} + L_1 + L_2 + L_3)$ is effective. Since $(\mathbb{P}^2, L_1 + L_2 + L_3)$ is F-split (Example 2.7(2)), so is X. This completes the proof for the case (iv).

Let us consider the case (v). In this case, we apply a similar method to that of (iv) after replacing $\mathbb{P}^2$ by $\mathbb{F}_1$. We have a sequence of one-point blowups:

$$f \colon X = Y''' \to Y'' \to Y' \to Y = \mathbb{F}_1 \to Z = \mathbb{P}^2,$$

where $Y := \mathrm{Bl}_Q \mathbb{P}^2 = \mathbb{F}_1$, $Y' := \mathrm{Bl}_P Y, Y'' := \mathrm{Bl}_{P'} Y', Y''' := \mathrm{Bl}_{P''} Y''$, and P, P', P'' are lying over Q, P, P', respectively. For the (-1)-curve C on Y, we have $P \in C$. It is well known that there is another section $\widetilde{C}$ of the $\mathbb{P}^1$-bundle $\pi \colon Y = \mathbb{F}_1 \to B = \mathbb{P}^1$ such that $C \cap \widetilde{C} = \emptyset$ and $\widetilde{C}^2 = 1$. Let F is a fiber of π. Since $(K_Y + C + \widetilde{C}) \cdot F = 0$, there exists $n \in \mathbb{Z}$ such that $K_Y + C + \widetilde{C} \sim nF$. Then $n = C \cdot nF = C \cdot (K_Y + C + \widetilde{C}) = -2$. Since the proper transform C' of C on Y' satisfies $C'2 = -2$, we obtain

(1) $P' \notin C'$,

as otherwise, the proper transform C'' of C' on Y'' would satisfy $C''2 = -3$, which contradicts the fact that Y'' is weak del Pezzo.

We now treat the case when $P' \in F_P'$, where F_P' denotes the proper transform of the fiber F_P of $\pi \colon Y = \mathbb{F}_1 \to \mathbb{P}^1$ passing through P. Let $\widetilde{F}$ be a general fiber of π. As we have seen above,

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$K_Y + \widetilde{F} + C + F_P + \widetilde{C} \sim 0$. Since $\widetilde{F}$ is nef and $\mathbb{F}_1$ is toric, we obtain $H^1(\mathbb{F}_1, \mathcal{O}_{F_1}(\widetilde{F})) = 0$ by [Tot24, Proposition 1.3]. Moreover, $(\widetilde{F}, (C + F_P + \widetilde{C})|_{\widetilde{F}}) = (\widetilde{F}, C|_{\widetilde{F}} + \widetilde{C}|_{\widetilde{F}})$ is F-split (Example 2.7 (1)). Thus, $(Y, C + F_P + \widetilde{C} + \widetilde{F})$ is F-split (Proposition 2.6), which implies that X is F-split (Proposition 2.4). In what follows, we assume that

(2) $P' \notin F'_P$ for the proper transform F'_P of the fiber F_P of $\pi : Y = \mathbb{F}_1 \to \mathbb{P}^1$ passing through P.

Claim. There is a section D of $\pi\colon Y = \mathbb{F}_1 \to \mathbb{P}^1$ such that

(a) $D \sim \widetilde{C} + F$,
(b) $P \in D$, and
(c) $P' \in D'$ for the proper transform D' of D on Y'.

Proof of Claim. Since $\widetilde{C} + F$ is an ample Cartier divisor on $Y = \mathbb{F}_1$, it follows that $|\widetilde{C} + F|$ is very ample [Har77, Ch. V, Corollary 2.18]. Then there is an effective Cartier divisor D on $Y = \mathbb{F}_1$ satisfying (a)–(d).

(d) D is smooth at P.

In fact, since $|\widetilde{C} + F|$ is very ample, the elements of $H^0(Y, \mathcal{O}_Y(\widetilde{C} + F))$ separate tangent vectors. Let $s_{P'} \in \mathfrak{m}_P/\mathfrak{m}_P^2$ is an element that corresponds to P'. We take D as a divisor of zeros of a global section $s \in H^0(Y, \mathcal{O}_Y(\widetilde{C} + F))$ that maps to $s_{P'} \in \mathfrak{m}_P/\mathfrak{m}_P^2$. Then (a)–(c) are satisfied. Since $s \in \mathfrak{m}_P/\mathfrak{m}_P^2$ is non-zero, the divisor D is smooth at P, i.e., (d) is satisfied. Since $D \cdot F = (\widetilde{C} + F) \cdot F = 1$, we can write $D = D_0 + F_1 + \cdots + F_r$, where $r \geq 0$, D_0 is a section of $\pi\colon Y = \mathbb{F}_1 \to \mathbb{P}^1$, and each F_i is a fiber of π. It suffices to prove $r = 0$. Suppose $r > 0$. The following holds:

$$D_0 \cdot C + r = (D_0 + F_1 + \cdots + F_r) \cdot C = D \cdot C = (\widetilde{C} + F) \cdot C = 1. \tag{3.8.1}$$

We now treat the case when $D_0 \neq C$. In this case, $D_0 \cdot C \geq 0$ and (3.8.1) imply $r = 1$ and $D_0 \cdot C = 0$. Hence, we get $D_0 \cap C = \emptyset$. Since $P \in C$, we have $P \notin D_0$. By (b), we obtain $P \in D = D_0 + F_1$, and thus $P \in F_1$. Hence, $D = D_0 + F_P$, where F_P denotes the fiber passing through P. Since $P' \notin C'$, we obtain $P' \in D' \setminus C' \subset F'_P$. This contradicts (2).

Hence, we may assume that $D_0 = C$. We then get $D = C + F_1 + \cdots + F_r$. Since $P \in C$, we obtain $P \notin F_1 \cup \cdots \cup F_r$ by (d). Then $P' \notin F'_1 \cup \cdots \cup F'_r$, where $F'_1, F'_2, \ldots, F'_r$ are proper transforms of $F_1, F_2, \ldots, F_r$ on Y', and thus we obtain $P' \in D' \setminus \{F'_1 \cup \cdots \cup F'_r\} \subset C'$ by (c). This contradicts (1). This completes the proof of the claim. □

We have $C \cdot D = 1$. Hence, $C \cap D = P$ and $C + D$ is a simple normal crossing divisor. Since both C and D are sections of $\pi\colon Y = \mathbb{F}_1 \to \mathbb{P}^1$, it follows that $C + D + \widetilde{F}$ is still simple normal crossing for a general fiber $\widetilde{F}$ of π. Then we see that $(Y, C + D + \widetilde{F})$ is F-split (Proposition 2.6, Example 2.7(1)). Since the divisor Δ defined by $K_X + \Delta = f^*(K_Y + C + D + \widetilde{F})$ is effective, X is F-split (Proposition 2.4). This completes the proof of Proposition 3.8. □

Proof of Theorem B. If $K_X^2 \leq 4$ (resp. $K_X^2 \geq 5$), then the assertion follows from Proposition 3.5 (resp. Proposition 3.8). □

3.3. Global F-regularity: Proof of Theorem A

In this subsection, we deduce Theorem A from Theorem B.

Lemma 3.9. *Let X be a canonical del Pezzo surface. Suppose that one of the following holds.*

(1) $p > 5$
(2) $K_X^2 \geq 2$ *and* $p > 3$.
(3) $K_X^2 \geq 4$ *and* $p > 2$.
(4) $K_X^2 \geq 5$.

Then X is strongly F-regular.

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Remark 3.10. Combining [KN23, Table 1] and [KT24a, Table 1], we can see that the assumption of p is optimal for each degree.

Proof. Strongly F-regular surface singularities are completely classified by Hara [Har98b, Theorem 1.1]. In what follows, we confirm the singularities on X satisfying one of (1)–(4) are all strongly F-regular.

(1) follows from [Har98b, Theorem 1.1]. In what follows, let $f : Y \to X$ be the minimal resolution. Then Y is a smooth weak del Pezzo surface (Subsection 2.1.2), and Y is obtained by a blowup of $\mathbb{P}^2$ at some points [Dol12, Theorem 8.1.15]. We have holds.

We prove (2). Since $K_Y^2 = K_X^2 \geq 2$, we have $\rho(Y) = 10-K_Y^2 \leq 8$. Thus, the number of the (-2)-curves contracted by f is at most $8 - \rho(X) \leq 7$. Therefore, X does not have canonical singularities of E_8-type. Then X is strongly F-regular by [Har98b, Theorem 1.1] since $p > 3$ (see also [KT24a, Table 1]).

Next, we prove (3). Since $K_Y^2 = K_X^2 \geq 4$, we have $\rho(Y) = 10-K_Y^2 \leq 6$. Thus, the number of the (-2)-curves contracted by f is at most $6 - \rho(X) \leq 5$. Therefore, X does not have canonical singularities of E-type. Then X is strongly F-regular by [Har98b, Theorem 1.1] since $p > 2$ (see also [KT24a, Table 1]).

Finally, we prove (4). Since $K_Y^2 = K_X^2 \geq 5$, we have $\rho(Y) = 10 - K_Y^2 \leq 5$. Thus, the number of the (-2)-curves contracted by f is at most $5 - \rho(X) \leq 4$. If $\rho(X) \geq 2$, then X has only A-type singularities, which are strongly F-regular [Har98b, Theorem 1.1]. If $\rho(X) = 1$, then X has only A-type singularities by [KN23, Theorem 1.1]. □

Proof of Theorem A. Let X be as in the statement of Theorem A. Taking the anticanonical model of X, we may assume that $-K_X$ is ample (Corollary 2.5). Then, by Theorem 1.5, it is enough to prove that X is strongly F-regular and F-split, which follow from Lemmas 3.9 and Theorem B, respectively. □

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