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EVOLUTION OF A PAIR OF RANDOM INHOMOGENEOUS WAVE SYSTEMS OVER INFINITE-DEPTH WATER

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Abstract

A system of two coupled nonlinear spectral transport equations is derived for two obliquely interacting narrowband Gaussian random surface wavetrains, slowly varying in space and time. Using these two equations, stability analysis is performed for two initially homogeneous wave spectra, subject to unidirectional perturbations. We observe that the effect of randomness produces a decrease in the growth rate of instability, but it is higher than the growth for a single wavetrain. The growth rate of instability is observed to decrease with the increase in spectral width.

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1. Introduction

One approach to examining the stability properties of a weakly nonlinear random wavetrain is through the use of the spectral transport equation derived by Alber [2] and Crawford et al. [5]. Alber obtained a spectral transport equation for narrowband Gaussian random surface wavetrains, starting from the Davey–Stewartson equations [6] for deep water. He showed that a random deep-water wavetrain becomes unstable if the normalized spectral bandwidth is less than twice the root mean square wave slope, multiplied by a function of the perturbation wave angle. Crawford et al. also studied the evolution of a random inhomogeneous field of deep-water waves. They derived a transport equation following Zakharov's [19] approach and used the Lorentz shape of spectrum [17]. They also studied the stability of an initially homogeneous wave spectrum, subject to small oblique wave perturbations. Following Alber [2] and Crawford et al. [5], a number of authors, including Dhar and Das [7] and Senapati et al. [15], investigated the effects of randomness on the stability properties of surface wavetrains in different contexts. It is also interesting to see the effect of randomness in a situation of crossing sea states when two wave systems

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interact obliquely. Onorato et al. [12] considered such a crossing sea states situation. They obtained evolution equations for weakly nonlinear interaction of two obliquely propagating wave packets. Using these evolution equations which are coupled nonlinear Schrödinger equations, Onorato et al. [12] performed stability analysis of two uniform wavetrains meeting in crossing seas under unidirectional perturbations. They found that the growth rate of instability of either wavetrain is much higher than that for the case of a single wavetrain. This led the authors to conclude that freak waves can be generated due to weakly nonlinear interaction in crossing sea states. The analysis of Onorato et al. [12] is further extended by Shukla et al. who performed stability analysis in a situation of crossing seas for bidirectional perturbations. Later, many authors, including Laine-Pearson [10], Gramstad and Trulsen [8] and Ruban [14], made investigations of the situation of crossing sea states.

The approach presented in all the above-mentioned papers is from a deterministic point of view. As an alternative approach, we consider here the effect of randomness in a situation of crossing sea states over infinite-depth water. Onorato et al. [11] carried out numerical simulations of a cubic nonlinear Schrödinger equation to show the formation of freak waves in a random sea characterized by the Joint North Sea Wave Project spectrum [9]. Onorato et al. [13] conducted experiments in two different wave basins to study the statistical properties of the ocean water surface elevation for different degrees of directional energy distribution. Toffoli et al. [18] also conducted an experiment in a large wave basin to study the statistical properties of the water surface elevation in crossing sea conditions. They reported that the number of extreme events depends on the angle between two interacting wave systems. This experimental finding was also supported by numerical simulations, which they carried out using a higher-order method for solving Euler equations.

In the present study, we consider a crossing sea states situation formed by two obliquely interacting initially homogeneous Gaussian wave systems [17]. We derive a pair of spectral transport equations for the two wave systems. Using these two equations, we perform a stability analysis of two obliquely interacting initially homogeneous Gaussian wave spectra for a range of spectral bandwidths. We obtain a nonlinear dispersion relation in the form of a nonlinear integral equation. By solving this integral equation numerically, we show the growth rate of instability in Figures 1 to 5. We find that the growth rate of instability is slightly less than that for the corresponding deterministic situation. We also find that the growth rate of instability decreases as the spectral bandwidth increases. We observe that the growth rate of instability of one wave system increases as the mean square wave steepness of the second wave system increases.

This paper is organized as follows. In Section 2 we derive the spectral transport equations. In Section 3 we discuss the results of the stability analysis of a pair of obliquely interacting random inhomogeneous wave systems subject to unidirectional perturbations. In Section 4 we report the main results obtained in the stability analysis.

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2. Transport equations for the spectral functions

We start with the following two coupled nonlinear Schrödinger equations obtained by Onorato et al. [12], which are correct up to third order in wave steepness:

$$i\frac{\partial A_1}{\partial \tau} + i\beta_1\frac{\partial A_1}{\partial \tilde{x}} + i\beta_2\frac{\partial A_1}{\partial \tilde{y}} + \beta_3\frac{\partial^2 A_1}{\partial \tilde{x}^2} + \beta_4\frac{\partial^2 A_1}{\partial \tilde{x}\partial \tilde{y}} + \beta_5\frac{\partial^2 A_1}{\partial \tilde{y}^2} = \lambda_1 A_1^2 A_1^* + \mu_1 A_1 A_2 A_2^*, \tag{2.1}$$

$$i\frac{\partial A_2}{\partial \tau} + i\beta_1\frac{\partial A_2}{\partial \tilde{x}} - i\beta_2\frac{\partial A_2}{\partial \tilde{y}} + \beta_3\frac{\partial^2 A_2}{\partial \tilde{x}^2} - \beta_4\frac{\partial^2 A_2}{\partial \tilde{x}\partial \tilde{y}} + \beta_5\frac{\partial^2 A_2}{\partial \tilde{y}^2} = \lambda_1 A_2^2 A_2^* + \mu_1 A_2 A_1 A_1^*, \tag{2.2}$$

with $i = \sqrt{-1}$. In equations (2.1) and (2.2), $\tilde{x}$, $\tilde{y}$, τ are slow space and time variables defined by

$$\tilde{x} = \epsilon x, \quad \tilde{y} = \epsilon y, \quad \tau = \epsilon t,$$

where ϵ is a small ordering parameter, and A_1 and A_2 denote the complex amplitudes of the two wave envelopes

$$\zeta_1(\tilde{x}, \tilde{y}, \tau) = \tfrac{1}{2}[A_1(\tilde{x}, \tilde{y}, \tau)\exp\{i(kx + ly - \omega t)\} + \text{c.c.}],$$
$$\zeta_2(\tilde{x}, \tilde{y}, \tau) = \tfrac{1}{2}[A_2(\tilde{x}, \tilde{y}, \tau)\exp\{i(kx - ly - \omega t)\} + \text{c.c.}],$$

where (k, l) and $(k, -l)$ are the carrier wavenumbers of the two wave packets and c.c. denotes the complex conjugate of the previous term. The wave frequency ω is determined by the linear dispersion relation

$$\omega^2 = g\sqrt{k^2 + l^2},$$

g being the gravitational acceleration. The coefficients β_i, λ_1 and μ_1 are given by Onorato et al. [12] and also by Shukla et al. [16].

We assume that the above two obliquely interacting wave packets are random in nature. Thus, $A_1(\tilde{\xi}, \tau)$ and $A_2(\tilde{\xi}, \tau)$ are random functions of $\tilde{\xi} = (\tilde{x}, \tilde{y})$. For the two wave packets, we define the two-point space correlation functions as follows:

$$\rho_1(\vec{\xi}_1, \vec{\xi}_2, \tau) = \langle A_1(\vec{\xi}_1, \tau) A_1^*(\vec{\xi}_2, \tau)\rangle,$$
$$\rho_2(\vec{\xi}_1, \vec{\xi}_2, \tau) = \langle A_2(\vec{\xi}_1, \tau) A_2^*(\vec{\xi}_2, \tau)\rangle,$$

where $\vec{\xi}_1 = (x_1, y_1)$, $\vec{\xi}_2 = (x_2, y_2)$ are two points in space, and the angle brackets denote the ensemble average.

We now find equations governing the slow variation of ρ_1 and ρ_2. First of all, we find one equation by considering equation (2.1) at the point $\vec{\xi}_1$ and multiplying both sides of it by $A_1^*(\vec{\xi}_2, \tau)$. We obtain another equation from the complex conjugate of equation (2.1) at the point $\vec{\xi}_2$, by multiplying both sides of it by $A_1(\vec{\xi}_1, \tau)$. Subtracting the second equation from the first equation and then taking the ensemble average,

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we get

$$\begin{aligned}
&i\frac{\partial\rho_1}{\partial\tau} + i\beta_1\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\Big)\rho_1 + i\beta_2\Big(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}\Big)\rho_1 + \beta_3\Big(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\Big)\rho_1 \\
&\quad + \beta_4\Big(\frac{\partial^2}{\partial x_1\partial y_1} - \frac{\partial^2}{\partial x_2\partial y_2}\Big)\rho_1 + \beta_5\Big(\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2}\Big)\rho_1 \\
&= \lambda_1\langle A_1^2(\vec{\xi}_1)A_1^*(\vec{\xi}_1)A_1^*(\vec{\xi}_2)\rangle - \lambda_1\langle A_1^{*2}(\vec{\xi}_2)A_1(\vec{\xi}_2)A_1(\vec{\xi}_1)\rangle \\
&\quad + \mu_1\langle A_1(\vec{\xi}_1)A_2(\vec{\xi}_1)A_2^*(\vec{\xi}_1)A_1^*(\vec{\xi}_2)\rangle - \mu_1\langle A_1^*(\vec{\xi}_2)A_2^*(\vec{\xi}_2)A_2(\vec{\xi}_2)A_1(\vec{\xi}_1)\rangle. \qquad (2.3)
\end{aligned}$$

On the right-hand side of equation (2.3), we have written $A_1(\vec{\xi}_1,\tau)$, $A_1(\vec{\xi}_2,\tau)$, $A_2(\vec{\xi}_1,\tau)$ and $A_2(\vec{\xi}_2,\tau)$ simply as $A_1(\vec{\xi}_1)$, $A_1(\vec{\xi}_2)$, $A_2(\vec{\xi}_1)$ and $A_2(\vec{\xi}_2)$, respectively. Hereafter, we will follow this notation for the sake of simplicity. Equation (2.3) shows that the evolution of the second-order correlation function ρ_1 depends on the fourth-order correlation terms. We now rewrite equation (2.3) using the average coordinates

$$X = \tfrac{1}{2}(x_1 + x_2), \quad Y = \tfrac{1}{2}(y_1 + y_2),$$

and the spatial separation coordinates

$$r_x = (x_1 - x_2), \quad r_y = (y_1 - y_2).$$

Following Alber [2], we assume that $A_1(\vec{\xi},\tau)$ and $A_2(\vec{\xi},\tau)$ are initially Gaussian random processes, and that they follow the same Gaussian statistical properties as they undergo evolution. Sine the fourth-order cumulant of Gaussian statistics vanishes, we are able to write the fourth-order correlation terms on the the right-hand side of equation (2.3) in terms of the products of pairs of second-order correlations. Thus,

$$\langle A_1(\vec{\xi}_1)A_1^*(\vec{\xi}_2)A_1(\vec{\xi}_1)A_1^*(\vec{\xi}_1)\rangle = 2\langle A_1(\vec{\xi}_1)A_1^*(\vec{\xi}_2)\rangle\langle A_1(\vec{\xi}_1)A_1^*(\vec{\xi}_1)\rangle = 2\rho_1\bar{a}_1{}^2(\vec{\xi}_1),$$

where

$$\bar{a}_1{}^2(\vec{\xi}_1) = \langle A_1(\vec{\xi}_1)A_1^*(\vec{\xi}_1)\rangle$$

is the ensemble averaged mean square amplitude of the first wave packet. Similarly, the ensemble averaged mean square amplitude of the second wave packet is given by

$$\bar{a}_2{}^2(\vec{\xi}_1) = \langle A_2(\vec{\xi}_1)A_2^*(\vec{\xi}_1)\rangle.$$

Now, $\vec{\xi}_1$ and $\vec{\xi}_2$, in terms of the variables $\vec{\xi} = (X, Y)$ and $r = (r_x, r_y)$, can be written as

$$\begin{aligned}
\vec{\xi}_1 &= (X + \tfrac{1}{2}r_x, Y + \tfrac{1}{2}r_y) = \vec{\xi} + \tfrac{1}{2}\vec{r}, \\
\vec{\xi}_2 &= (X - \tfrac{1}{2}r_x, Y - \tfrac{1}{2}r_y) = \vec{\xi} - \tfrac{1}{2}\vec{r}.
\end{aligned}$$

By Taylor's expansion of $\bar{a}_1{}^2(\vec{\xi}_1)$ about the point $\vec{\xi}$, we get

$$\bar{a}_1{}^2(\vec{\xi}_1) = \exp\Big(\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\Big)\bar{a}_1{}^2(\vec{\xi}).$$

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Thus, the first term on the right-hand side of equation (2.3) becomes

$$2\lambda_1\rho_1 \exp\left(\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_1{}^2(\vec{\xi}).$$

Similarly, the second, third and fourth terms on the right-hand side of equation (2.3) become

$$-2\lambda_1\rho_1 \exp\left(-\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_1^2(\vec{\xi}), \quad \mu_1\rho_1 \exp\left(\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_2^2(\vec{\xi}), \quad -\mu_1\rho_1 \exp\left(-\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_2^2(\vec{\xi}),$$

respectively. Thus, in terms of the variables $\vec{\xi}$ and $\vec{r}$, equation (2.3) becomes

$$\begin{aligned} i\frac{\partial\rho_1}{\partial\tau} + i\beta_1\frac{\partial\rho_1}{\partial X} + i\beta_2\frac{\partial\rho_1}{\partial Y} + 2\beta_3\frac{\partial\rho_1}{\partial X\partial r_x} + \beta_4\left(\frac{\partial^2}{\partial X\partial r_y} + \frac{\partial^2}{\partial Y\partial r_x}\right)\rho_1 + 2\beta_5\frac{\partial^2\rho_1}{\partial Y\partial r_y} \\ = 4\lambda_1\rho_1 \sinh\left(\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_1^2 + 2\mu_1\rho_1 \sinh\left(\frac{1}{2}\vec{r}.\frac{\partial}{\partial\vec{\xi}}\right)\bar{a}_2^2. \end{aligned} \tag{2.4}$$

The wave-envelope power spectral density functions are defined by

$$\begin{aligned} F_1(\vec{P},\vec{\xi},\tau) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\rho_1\left(\vec{\xi}_1,\vec{\xi}_2,\tau\right)e^{-i(P_xr_x+P_yr_y)}\,dr_x\,dr_y \\ &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\rho_1\left(\vec{\xi}+\frac{\vec{r}}{2},\vec{\xi}-\frac{\vec{r}}{2},\tau\right)e^{-i\vec{P}\cdot\vec{r}}\,d\vec{r}, \\ F_2(\vec{P},\vec{\xi},\tau) &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\rho_2\left(\vec{\xi}_1,\vec{\xi}_2,\tau\right)e^{-i(P_xr_x+P_yr_y)}\,dr_x\,dr_y \\ &= \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\rho_2\left(\vec{\xi}+\frac{\vec{r}}{2},\vec{\xi}-\frac{\vec{r}}{2},\tau\right)e^{-i\vec{P}\cdot\vec{r}}\,d\vec{r}. \end{aligned} \tag{2.5}$$

Here $\vec{P} = (P_x, P_y)$ is the Fourier wavenumber conjugate to the spatial separation coordinates $\vec{r} = (r_x, r_y)$. The Fourier inversions of the system of equation (2.5) are

$$\begin{aligned} \rho_1\left(\vec{\xi}+\frac{\vec{r}}{2},\vec{\xi}-\frac{\vec{r}}{2},\tau\right) &= \iint_{-\infty}^{\infty} F_1(\vec{P},\vec{\xi},\tau)e^{i\vec{P}\cdot\vec{r}}\,d\vec{P}, \\ \rho_2\left(\vec{\xi}+\frac{\vec{r}}{2},\vec{\xi}-\frac{\vec{r}}{2},\tau\right) &= \iint_{-\infty}^{\infty} F_2(\vec{P},\vec{\xi},\tau)e^{i\vec{P}\cdot\vec{r}}\,d\vec{P}. \end{aligned} \tag{2.6}$$

Setting $\vec{r} = \vec{0}$ in (2.6) yields

$$\begin{aligned} \bar{a}_1{}^2(\vec{\xi},\tau) = \rho_1(\vec{\xi},\vec{\xi},\tau) &= \iint_{-\infty}^{\infty} F_1(\vec{P},\vec{\xi},\tau)\,d\vec{P}, \\ \bar{a}_2{}^2(\vec{\xi},\tau) = \rho_2(\vec{\xi},\vec{\xi},\tau) &= \iint_{-\infty}^{\infty} F_2(\vec{P},\vec{\xi},\tau)\,d\vec{P}. \end{aligned} \tag{2.7}$$

Then, taking the Fourier transform of equation (2.4) and using relations (2.5)–(2.7), we get the following spectral transport equation for the first wave packet:

$$\begin{aligned} \frac{\partial F_1}{\partial\tau} + (\beta_1 + 2\beta_3P_x + \beta_4P_y)\frac{\partial F_1}{\partial X} + (\beta_2 + \beta_4P_x + 2\beta_5P_y)\frac{\partial F_1}{\partial Y} \\ = 4\lambda_1 \sin\left(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\right)F_1\bar{a}_1{}^2 + 2\mu_1 \sin\left(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\right)F_1\bar{a}_2{}^2. \end{aligned} \tag{2.8}$$

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On the right-hand side of equation (2.8), the spatial derivatives operate on $\bar{a}_1^2$ or $\bar{a}_2^2$, and the wavenumber derivatives operate on $F_1(\vec{P}, \vec{\xi}, \tau)$. Thus,

$$\begin{aligned}\sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_1\bar{a}_1^2 &= \sin\Big(\frac{1}{2}\frac{\partial}{\partial X}\cdot\frac{\partial}{\partial P_x}+\frac{1}{2}\frac{\partial}{\partial Y}\cdot\frac{\partial}{\partial P_y}\Big)F_1\bar{a}_1^2\\ &= \frac{1}{2}\Big[\frac{\partial\bar{a}_1^2}{\partial X}\frac{\partial F_1}{\partial P_x}+\frac{\partial\bar{a}_1^2}{\partial Y}\frac{\partial F_1}{\partial P_y}\Big]-\frac{1}{3!}\Big(\frac{1}{2}\Big)^3\Big[\frac{\partial^3\bar{a}_1^2}{\partial X^3}\frac{\partial^3 F_1}{\partial P_x^3}\\ &\quad + 3\frac{\partial^3\bar{a}_1^2}{\partial X^2\partial Y}\frac{\partial^3 F_1}{\partial P_x^2\partial P_y}+3\frac{\partial^3\bar{a}_1^2}{\partial X\partial Y^2}\frac{\partial^3 F_1}{\partial P_x\partial P_y^2}+\frac{\partial^3\bar{a}_1^2}{\partial Y^3}\frac{\partial^3 F_1}{\partial P_y^3}\Big]-\cdots.\end{aligned}$$

Repeating the same procedure for the second evolution equation (2.2), we get the following spectral transport equation for the second wave packet:

$$\begin{aligned}&\frac{\partial F_2}{\partial\tau}+(\beta_1+2\beta_3P_x-\beta_4P_y)\frac{\partial F_2}{\partial X}-(\beta_2+\beta_4P_x-2\beta_5P_y)\frac{\partial F_2}{\partial Y}\\ &\quad = 4\lambda_1\sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)\bar{a}_2^2+2\mu_1\sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_2\bar{a}_1^2.\end{aligned}\tag{2.9}$$

The coupled nonlinear system of equations (2.8) and (2.9) governs the evolution of the spectral transport functions F_1 and F_2 in a situation of crossing sea states over infinite-depth water. Setting $\bar{a}_2^2 = 0$ in equation (2.8), we can recover the spectral transport equation for the propagation of a single wave packet.

3. Stability analysis

One basic solution ro the system of spectral transport equations (2.8) and (2.9) is given by

$$F_1 = F_1^{(0)}(\vec{P}),\quad F_2 = F_2^{(0)}(\vec{P}),\tag{3.1}$$

where $F_1^{(0)}(\vec{P})$ and $F_2^{(0)}(\vec{P})$ are independent of $\vec{\xi}$ and τ. The solution (3.1) is statistically uniform in space and time, and it is the random counterpart of uniform-amplitude Stokes wavetrains in the deterministic theory of crossing seas. We assume that $F_1^{(0)}(\vec{P})$ and $F_2^{(0)}(\vec{P})$ satisfy Gaussian properties. It is known that the evolving random statistical amplitude field retains the same Gaussian statistical properties [4].

We now study the stability of the homogeneous solution (3.1) subject to the infinitesimal perturbations

$$\begin{aligned}F_1 &= F_1^{(0)}(\vec{P})+\epsilon F_1^{(1)}(\vec{P},\vec{\xi},\tau),\\ F_2 &= F_2^{(0)}(\vec{P})+\epsilon F_2^{(1)}(\vec{P},\vec{\xi},\tau),\end{aligned}\tag{3.2}$$

where ϵ is a small ordering parameter.

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Making use of (3.2) in (2.7),

$$\bar{a}_1^2(\vec{\xi},\tau) = \iint_{-\infty}^{\infty} F_1^{(0)}(\vec{P})\,d\vec{P} + \epsilon \iint_{-\infty}^{\infty} F_1^{(1)}(\vec{P},\vec{\xi},\tau)\,d\vec{P} = \bar{a}_{10}^2 + \epsilon\bar{a}_{11}^2(\vec{\xi},\tau), \tag{3.3}$$

$$\bar{a}_2^2(\vec{\xi},\tau) = \iint_{-\infty}^{\infty} F_2^{(0)}(\vec{P})\,d\vec{P} + \epsilon \iint_{-\infty}^{\infty} F_2^{(1)}(\vec{P},\vec{\xi},\tau)\,d\vec{P} = \bar{a}_{20}^2 + \epsilon\bar{a}_{21}^2(\vec{\xi},\tau), \tag{3.4}$$

where $\bar{a}_{10}^2$ and $\bar{a}_{20}^2$ are the mean square wave steepness of the first and second wave packets, respectively. Substituting (3.2)–(3.4) in the spectral transport equations (2.8) and (2.9) and then linearizing those equations yields

$$\frac{\partial F_1^{(1)}}{\partial\tau} + (\beta_1 + 2\beta_3 P_x + \beta_4 P_y)\frac{\partial F_1^{(1)}}{\partial X} + (\beta_2 + \beta_4 P_x + 2\beta_5 P_y)\frac{\partial F_1^{(1)}}{\partial Y} = 4\lambda_1 \sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_1^{(0)}\bar{a}_{11}^2 + 2\mu_1 \sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_1^{(0)}\bar{a}_{21}^2, \tag{3.5}$$

$$\frac{\partial F_2^{(1)}}{\partial\tau} + (\beta_1 + 2\beta_3 P_x - \beta_4 P_y)\frac{\partial F_2^{(1)}}{\partial X} - (\beta_2 + \beta_4 P_x - 2\beta_5 P_y)\frac{\partial F_2^{(1)}}{\partial Y} = 4\lambda_1 \sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_2^{(0)}\bar{a}_{21}^2 + 2\mu_1 \sin\Big(\frac{1}{2}\frac{\partial}{\partial\vec{\xi}}\cdot\frac{\partial}{\partial\vec{P}}\Big)F_2^{(0)}\bar{a}_{11}^2. \tag{3.6}$$

We assume the space–time dependence of $F_1^{(1)}, F_2^{(1)}, \bar{a}_{11}^2, \bar{a}_{21}^2$ to take the form

$$\begin{aligned} F_1^{(1)}(\vec{P},\vec{\xi},\tau) &= f_1(\vec{P})\exp[i(L_xX + L_yY - \Omega\tau)],\\ F_2^{(1)}(\vec{P},\vec{\xi},\tau) &= f_2(\vec{P})\exp[i(L_xX + L_yY - \Omega\tau)],\\ \bar{a}_{11}^2(\vec{\xi},\tau) &= \alpha_1\exp[i(L_xX + L_yY - \Omega\tau)],\\ \bar{a}_{21}^2(\vec{\xi},\tau) &= \alpha_2\exp[i(L_xX + L_yY - \Omega\tau)], \end{aligned} \tag{3.7}$$

where (L_x, L_y) is the perturbation wavenumber vector, Ω is the perturbed frequency, and α_1 and α_2 are two constants. Substituting (3.7) in (3.5) and (3.6) and then equating the coefficient of $\exp[i(L_xX + L_yY - \Omega\tau)]$ on both sides of these equations, we get

$$[-\Omega + G_+(\vec{P})]f_1(p) = (2\lambda_1\alpha_1 + \mu_1\alpha_2)\Big[F_1^{(0)}\Big(\vec{P} + \frac{\vec{L}}{2}\Big) - F_1^{(0)}\Big(\vec{P} - \frac{\vec{L}}{2}\Big)\Big], \tag{3.8}$$

$$[-\Omega + G_-(\vec{P})]f_2(p) = (2\lambda_1\alpha_2 + \mu_1\alpha_1)\Big[F_2^{(0)}\Big(\vec{P} + \frac{\vec{L}}{2}\Big) - F_2^{(0)}\Big(\vec{P} - \frac{\vec{L}}{2}\Big)\Big], \tag{3.9}$$

where

$$G_\pm(\vec{P}) = (\beta_1 + 2\beta_3 P_x \pm \beta_4 P_y)L_x \pm (\beta_2 + \beta_4 P_x \pm 2\beta_5 P_y)L_y.$$

Using (3.7) in (3.3) and (3.4),

$$\alpha_1 = \iint_{-\infty}^{\infty} f_1(\vec{P})\,d\vec{P}, \quad \alpha_2 = \iint_{-\infty}^{\infty} f_2(\vec{P})\,d\vec{P}. \tag{3.10}$$

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Having determined $f_1(\vec{P})$ and $f_2(\vec{P})$ from equations (3.8) and (3.9) respectively, we substitute those in equations (3.10), yielding

$$\begin{aligned}(2\lambda_1 I_1 + 1)\alpha_1 + \mu_1 I_1 \alpha_2 &= 0,\\ \mu_1 I_2 \alpha_1 + (2\lambda_1 I_2 + 1)\alpha_2 &= 0,\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}I_1 &= \iint_{-\infty}^{\infty} \frac{F_1^{(0)}(\vec{P}+\vec{L}/2) - F_1^{(0)}(\vec{P}-\vec{L}/2)}{\Omega - G_+(\vec{P})}\, d\vec{P},\\ I_2 &= \iint_{-\infty}^{\infty} \frac{F_2^{(0)}(\vec{P}+\vec{L}/2) - F_2^{(0)}(\vec{P}-\vec{L}/2)}{\Omega - G_-(\vec{P})}\, d\vec{P}.\end{aligned} \tag{3.12}$$

The condition of non-trivial (non-zero) solution for the system of equations (3.11) produces

$$(4\lambda_1^2 - \mu_1^2) I_1 I_2 + 2\lambda_1 (I_1 + I_2) + 1 = 0. \tag{3.13}$$

Thus, the perturbation wavenumber and perturbed frequency satisfy the nonlinear integral equation (3.13). We assume that $F_1^{(0)}(\vec{P})$ and $F_2^{(0)}(\vec{P})$ are described by the two-dimensional normal spectra

$$\begin{aligned}F_1^{(0)}(\vec{P}) &= \frac{\bar{a}_{10}^2}{2\pi\sigma^2} \exp\Big[-\frac{P_x^2 + P_y^2}{2\sigma^2}\Big],\\ F_2^{(0)}(\vec{P}) &= \frac{\bar{a}_{20}^2}{2\pi\sigma^2} \exp\Big[-\frac{P_x^2 + P_y^2}{2\sigma^2}\Big],\end{aligned} \tag{3.14}$$

where σ is the bandwidth of an undisturbed spectrum; σ is, in fact, the degree of randomness. We now make the vector transformations

$$\begin{aligned}\vec{K}_1 &= (2\beta_3 L_x + \beta_4 L_y)\hat{x} + (\beta_4 L_x + 2\beta_5 L_y)\hat{y},\\ \vec{K}_2 &= (2\beta_3 L_x - \beta_4 L_y)\hat{x} - (\beta_4 L_x - 2\beta_5 L_y)\hat{y},\end{aligned} \tag{3.15}$$

where $\hat{x}$, $\hat{y}$, $\hat{z}$ are the unit vectors in the direction of increasing x, y and z, respectively. The transformation (3.15) helps us reduce the double integrals in (3.12) to single integrals. Using the vectors $\vec{K}_1$ and $\vec{K}_2$, we rewrite $G_\pm(\vec{P})$ as

$$\begin{aligned}G_+(\vec{P}) &= \beta_1 L_x + \beta_2 L_y + \vec{P}\cdot\vec{K}_1 = \beta_1 L_x + \beta_2 L_y + P_1|\vec{K}_1|,\\ G_-(\vec{P}) &= \beta_1 L_x - \beta_2 L_y + \vec{P}\cdot\vec{K}_2 = \beta_1 L_x - \beta_2 L_y + P_2|\vec{K}_2|,\end{aligned}$$

where P_1 and P_2 are the components of $\vec{P}$ in the directions of $\vec{K}_1$ and $\vec{K}_2$, respectively. Thus,

$$\begin{aligned}P_1 &= \frac{(2\beta_3 L_x + \beta_4 L_y)P_x + (\beta_4 L_x + 2\beta_5 L_y)P_y}{[(2\beta_3 L_x + \beta_4 L_y)^2 + (\beta_4 L_x + 2\beta_5 L_y)^2]^{1/2}},\\ P_2 &= \frac{(2\beta_3 L_x - \beta_4 L_y)P_x - (\beta_4 L_x - 2\beta_5 L_y)P_y}{[(2\beta_3 L_x - \beta_4 L_y)^2 + (\beta_4 L_x - 2\beta_5 L_y)^2]^{1/2}}.\end{aligned}$$

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If P_1' and P_2' are the components of $\vec{P}$ in the directions normal to $\vec{K}_1$ and $\vec{K}_2$ respectively, then

$$P_1' = \vec{P} \cdot \left(\hat{z} \times \frac{\vec{K}_1}{|\vec{K}_1|}\right) = \frac{1}{|\vec{K}_1|}[(2\beta_3 L_x + \beta_4 L_y)P_y - (\beta_4 L_x + 2\beta_5 L_y)P_x],$$

$$P_2' = \vec{P} \cdot \left(\hat{z} \times \frac{\vec{K}_2}{|\vec{K}_2|}\right) = \frac{1}{|\vec{K}_2|}[(2\beta_3 L_x - \beta_4 L_y)P_y - (-\beta_4 L_x + 2\beta_5 L_y)P_x].$$

Now, I_1 can be rewritten as

$$I_1 = \iint_{-\infty}^{\infty} \frac{F_1^{(0)}(\vec{P} + \vec{L}/2) - F_1^{(0)}(\vec{P} - \vec{L}/2)}{\Omega - G_+(\vec{P})}\, dP_1\, dP_1'.$$

Performing integration with respect to P_1',

$$I_1 = \int_{-\infty}^{\infty} \frac{\bar{F}_1^{(0)}(P_1 + \gamma_1/2) - \bar{F}_1^{(0)}(P_1 - \gamma_1/2)}{\Omega - (\beta_1 L_x + \beta_2 L_y + P_1|\vec{K}_1|)}\, dP_1, \tag{3.16}$$

where

$$\bar{F}_1^{(0)}(P_1) = \int_{-\infty}^{\infty} F_1^{(0)}(\vec{P})\, dP_1', \tag{3.17}$$

and

$$\gamma_1 = (L_x, L_y) \cdot \frac{\vec{K}_1}{|\vec{K}_1|}.$$

Similarly, I_2 can be rewritten as

$$I_2 = \int_{-\infty}^{\infty} \frac{\bar{F}_2^{(0)}(P_2 + \gamma_2/2) - \bar{F}_2^{(0)}(P_2 - \gamma_2/2)}{\Omega - (\beta_1 L_x - \beta_2 L_y + P_2|\vec{K}_2|)}\, dP_2, \tag{3.18}$$

where

$$\bar{F}_2^{(0)}(P_2) = \int_{-\infty}^{\infty} F_2^{(0)}(\vec{P})\, dP_2', \tag{3.19}$$

and

$$\gamma_2 = (L_x, L_y) \cdot \frac{\vec{K}_2}{|\vec{K}_2|}.$$

Substituting the form of $F_1^{(0)}(\vec{P})$ and $F_2^{(0)}(\vec{P})$ from (3.14) into equations (3.17) and (3.19) and keeping in mind that

$$P_x^2 + P_y^2 = P_1^2 + P_1'^2 = P_2^2 + P_2'^2,$$

we obtain

$$\bar{F}_1^{(0)}(P_1) = \frac{\bar{a_{10}}^2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{P_1^2}{2\sigma^2}\right],$$

$$\bar{F}_2^{(0)}(P_2) = \frac{\bar{a_{20}}^2}{\sqrt{2\pi}\sigma} \exp\left[-\frac{P_2^2}{2\sigma^2}\right].$$

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Finally, substituting $\bar{F}_1^{(0)}(P_1)$ in (3.16) and $\bar{F}_2^{(0)}(P_2)$ in (3.18), we rewrite I_1 and I_2 as

$$I_1 = -\frac{i\bar{a}_{10}^2}{\sigma|\vec{K}_1|}\sqrt{\frac{\pi}{2}}[w(\Omega_1^{(+)}) - w(\Omega_1^{(-)})], \tag{3.20}$$

$$I_2 = -\frac{i\bar{a}_{20}^2}{\sigma|\vec{K}_2|}\sqrt{\frac{\pi}{2}}[w(\Omega_2^{(+)}) - w(\Omega_2^{(-)})]. \tag{3.21}$$

Here $w(z)$, a complex integral function introduced by Abramowitz and Stegun [1], is defined as

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-u^2}}{z-u}\,du, \quad \mathrm{Im}(z) > 0;$$

$w(z)$ can also be expressed in terms of the complementary error function as

$$w(z) = e^{-z^2}\mathrm{erfc}(-iz).$$

The arguments $\Omega_1^{(\pm)}$, $\Omega_2^{(\pm)}$ appearing in equations (3.20) and (3.21) are given by

$$\begin{aligned}\Omega_1^{(\pm)} &= \frac{1}{\sqrt{2}\sigma|\vec{K}_1|}[\Omega - \beta_1 L_x - \beta_2 L_y \pm \beta_3 L_x^2 \pm \beta_4 L_x L_y \pm \beta_5 L_y^2],\\ \Omega_2^{(\pm)} &= \frac{1}{\sqrt{2}\sigma|\vec{K}_2|}[\Omega - \beta_1 L_x + \beta_2 L_y \pm \beta_3 L_x^2 \mp \beta_4 L_x L_y \pm \beta_5 L_y^2].\end{aligned} \tag{3.22}$$

Substituting the forms of I_1 and I_2 as given in (3.20) and (3.21) respectively, we can rewrite the nonlinear dispersion relation (3.13) as

$$AI_1'I_2' + B_1I_1' + B_2I_2' + 1 = 0, \tag{3.23}$$

where

$$A = \frac{(4\lambda_1^2 - \mu_1^2)\bar{a}_{10}^2\bar{a}_{20}^2}{2\pi\sigma^2|\vec{K}_1||\vec{K}_2|}, \quad B_1 = \frac{2\lambda_1\bar{a}_{10}^2}{\sqrt{2\pi}\sigma|\vec{K}_1|}, \quad B_2 = \frac{2\lambda_1\bar{a}_{20}^2}{\sqrt{2\pi}\sigma|\vec{K}_2|},$$

$$\begin{aligned}I_1' &= \int_{-\infty}^{\infty}\frac{e^{-u^2}}{\Omega_1^{(+)} - u}\,du - \int_{-\infty}^{\infty}\frac{e^{-u^2}}{\Omega_1^{(-)} - u}\,du,\\ I_2' &= \int_{-\infty}^{\infty}\frac{e^{-u^2}}{\Omega_2^{(+)} - u}\,du - \int_{-\infty}^{\infty}\frac{e^{-u^2}}{\Omega_2^{(-)} - u}\,du.\end{aligned}$$

When the bandwidth of both the wave packets becomes vanishingly small, we can recover the nonlinear dispersion relation for Benjamin–Feir [3] type instability for two obliquely interacting deterministic wave packets from the nonlinear integral equation (3.23). When $\sigma \to 0$, expressions in (3.22) show that $\Omega_1^{(\pm)}$ and $\Omega_2^{(\pm)}$ tend to infinity. As $z \to \infty$, $w(z)$ has the asymptotic behaviour

$$w(z) = \frac{i}{\sqrt{\pi}}z^{-1} + O(z^{-3}).$$

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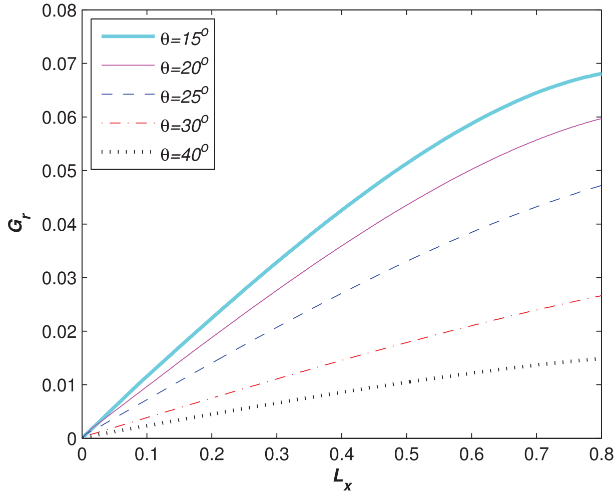

FIGURE 1. Growth rate of instability G_r against perturbation wavenumber L_x for different values of θ in the range $0^\circ < \theta < 45^\circ$.

Therefore, as $\sigma \to 0$, the nonlinear dispersion relation (3.23) becomes

$$(4\lambda_1^2 - \mu_1^2)\bar{a}_{10}^2\bar{a}_{20}^2\left(\frac{1}{\Omega_3^{(+)}} - \frac{1}{\Omega_3^{(-)}}\right)\left(\frac{1}{\Omega_4^{(+)}} - \frac{1}{\Omega_4^{(-)}}\right)$$
$$+ 2\lambda_1\bar{a}_{10}^2\left(\frac{1}{\Omega_3^{(+)}} - \frac{1}{\Omega_3^{(-)}}\right) + 2\lambda_1\bar{a}_{20}^2\left(\frac{1}{\Omega_4^{(+)}} - \frac{1}{\Omega_4^{(-)}}\right) + 1 = 0, \tag{3.24}$$

where

$$\Omega_3^{(\pm)} = \sqrt{2}\sigma|\vec{K}_1|\Omega_1^{(\pm)}, \quad \Omega_4^{(\pm)} = \sqrt{2}\sigma|\vec{K}_2|\Omega_2^{(\pm)}.$$

If in equation (3.24) $2\bar{a}_{10}^2$ and $2\bar{a}_{20}^2$ are replaced by their deterministic counterparts a_{10}^2 and a_{20}^2 respectively, then one can recover the nonlinear dispersion relation of Shukla et al. [16].

We consider long-crested perturbations in the $\tilde{x}$ direction for which $L_y = 0$. In this case, $\Omega_1^{(+)} = \Omega_2^{(+)}$ and $\Omega_1^{(-)} = \Omega_2^{(-)}$. Hence, it follows that $I_1' = I_2'$. The dispersion relation (3.23) can now be solved as

$$I_1' = \frac{-(B_1 + B_2) \pm \sqrt{(B_1 + B_2)^2 - 4A}}{2A}.$$

One can verify that I_1' is real. Substituting $\Omega = \beta_1 L_x + \Omega_r + i\Omega_i$, and Ω_r, Ω_i being real, we get

$$\frac{I_1'}{\alpha} = \int_{-\infty}^{\infty} \frac{(\Omega_r + \beta_3 L_x^2 - \alpha u - i\Omega_i)e^{-u^2}}{(\Omega_r + \beta_3 L_x^2 - \alpha u)^2 + \Omega_i^2}\,du - \int_{-\infty}^{\infty} \frac{(\Omega_r - \beta_3 L_x^2 - \alpha u - i\Omega_i)e^{-u^2}}{(\Omega_r - \beta_3 L_x^2 - \alpha u)^2 + \Omega_i^2}\,du, \tag{3.25}$$

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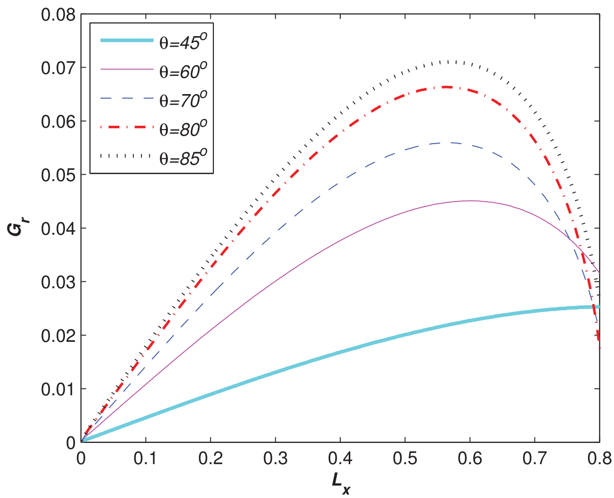

FIGURE 2. Growth rate of instability G_r against perturbation wavenumber L_x for different values of θ in the range $45° \leq \theta < 90°$.

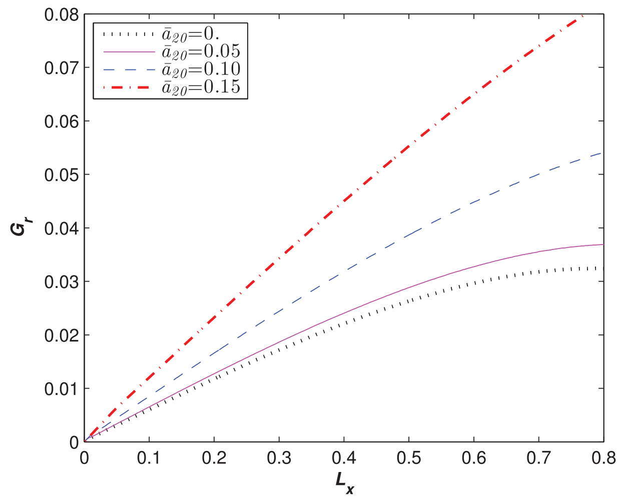

FIGURE 3. Growth rate of instability G_r against perturbation wavenumber L_x for different values of $\bar{a}_{20}$ taking $\bar{a}_{10} = 0.1$, $\sigma = 0.1$, $\theta = 22.5°$.

where

$$\alpha = \sigma L_x \sqrt{2(4\beta_3^2 + \beta_4^2)}.$$

Since I_1' is real, it follows from equation (3.25) for I_1' that $\Omega_r = 0$. Equation (3.25) then reduces to

$$I_1' = 2\int_{-\infty}^{\infty} \frac{(p-u)e^{-u^2}}{(p-u)^2 + q^2}\, du = 2\pi\, \mathrm{Im}[w(p+iq)], \tag{3.26}$$

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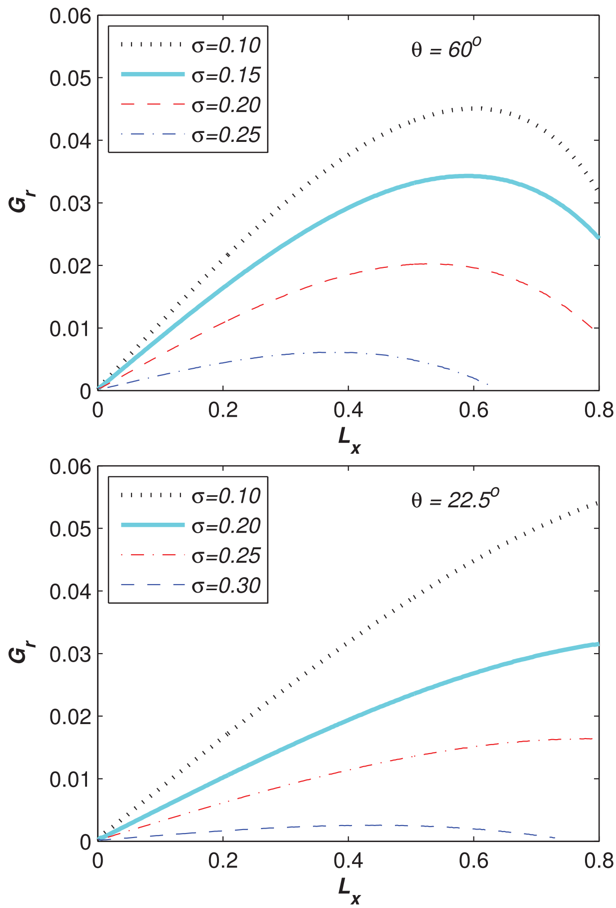

FIGURE 4. Growth rate of instability G_r against perturbation wavenumber L_x for different values of σ taking $\bar{a}_{10} = 0.1$, $\bar{a}_{20} = 0.1$.

where $p = \beta_3 L_x^2/\alpha$ and $q = \Omega_i/\alpha$. Using equation (3.26), we have plotted several figures which are limited to long-crested perturbations. In Figures 1 and 2 we have plotted the growth rate of instability G_r against perturbation wavenumber L_x for different values of θ, taking $\bar{a}_{10} = \bar{a}_{20} = 0.1$ and $\sigma = 0.1$. Here θ is the half-angle between the directions of propagation of the two wave systems so that $\theta = \arctan(l/k)$. We observe that G_r decreases with the increase in θ for $0^\circ < \theta < 45^\circ$, while G_r increases with the increase in θ for $45^\circ \leq \theta < 90^0$. In Figure 3 we have plotted the growth rate of instability G_r against perturbation wavenumber L_x for different values of $\bar{a}_{20}$, taking $\bar{a}_{10} = 0.1$ and $\theta = 22.5^\circ$. Figure 3 shows that G_r increases as the value of $\bar{a}_{20}$ increases. In Figure 4 we have shown the growth rate of instability G_r against perturbation wavenumber L_x for different values of σ, taking $\theta = 22.5^\circ$ and 60°. We observe that the growth rate of instability decreases as the value of σ increases. Figure 5 shows a comparison between the growth rate values for the deterministic situation and the

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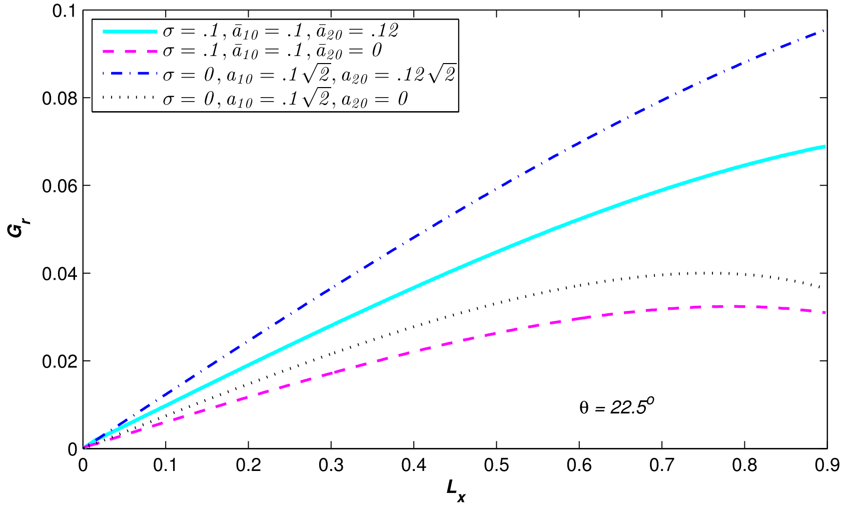

FIGURE 5. Growth rate of instability G_r against perturbation wavenumber L_x: comparison with deterministic growth rate values.

corresponding values when modified by the effect of randomness. We observe that the growth rate of instability decreases due to the effect of randomness, but it is greater than that for a single wave packet.

4. Conclusion

Making use of the evolution equations obtained by Onorato et al. [12] for crossing sea states, we have obtained a set of two coupled nonlinear transport equations for the spectral functions corresponding to two obliquely interacting random field of weakly nonlinear gravity wave packets. These two equations are useful to study the effect of inhomogeneity and the energy transfer mechanism associated with the homogeneous spectrum. Using the two spectral transport equations derived here, we have carried out stability analysis of a pair of obliquely interacting random wave packets following a Gaussian distribution. We observe that randomness reduces the growth rate of instability slightly, as in the case of a single wave system. Although the effect of randomness has a stabilizing influence, it is interesting to note that the growth rate of instability in a situation of crossing seas characterized by two random wave systems is higher than that for a single wave system. We have shown that the growth rate of instability decreases with the increase in half-angle θ between the directions of propagation of the two wave packets for $0° < \theta < 45°$. In the range $45° \leq \theta < 90°$, the growth rate of instability increases as θ increases. As the mean square wave steepness of one wave packet increases, the growth of instability of the second wave packet also increases. This observation is similar to the corresponding deterministic situation. The growth rate of instability is found to decrease with the increase of the bandwidth of spectral functions.

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