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LETTER TO THE EDITOR

Dear Editor,

On perturbed random walks

Let (X_n, M_n), $n \geq 1$, be an independent and identically distributed sequence of two-dimensional random vectors such that M_1 has finite mean and is long tailed (i.e. $\mathrm{P}(M_1 > x + y)/\mathrm{P}(M_1 > x) \to 1$ as $x \to \infty$ for y fixed), $\mathrm{E}[X_1] = -\mu < 0$, and there exists a $\zeta > 0$ such that $\mathrm{E}[\mathrm{e}^{\zeta X_1}] < \infty$, i.e. the right tail of X_1 is light tailed. It was shown in [1] that

$$\mathrm{P}\Big(\sup_{n\geq 1}[M_n - \mu n] > x\Big) \sim \frac{1}{\mu}\int_x^\infty \mathrm{P}(M_1 > u)\,\mathrm{d}u := G_\mu(x).$$

This still holds if $-\mu n$ is replaced by $S_{n-1} = \sum_1^{n-1} X_i$. This is the main result of Ha *et al.* [3] under the additional assumption that $\mathrm{E}[X_1^2] < \infty$; see their Theorem 1(i) in the case $\gamma = 0$. The asymptotic lower bound of this result was already covered in [4] without the assumption that $\mathrm{E}[X_1^2] < \infty$; see the top half of page 352 of [4]. Indeed, the assumption in [4] is that $\max\{M_1, X_1\}$ is long tailed, but

$$\mathrm{P}(M_1 > x) \leq \mathrm{P}(\max\{M_1, X_1\} > x) \leq \mathrm{P}(M_1 > x) + \mathrm{P}(X_1 > x) \sim \mathrm{P}(M_1 > x),$$

and the class of long-tailed distributions is closed under tail equivalence, so that $\max\{M_1, X_1\}$ is also long tailed.

The aim of this letter is to provide a proof of the corresponding asymptotic upper bound which is shorter and more general than the proof in [3], i.e. we will show that

$$\limsup_{x\to\infty} \frac{\mathrm{P}(\sup_{n\geq 1}[M_n + S_{n-1}] > x)}{G_\mu(x)} \leq 1. \tag{1}$$

Define, for $\varepsilon > 0$, the event $A_\varepsilon(x) = \bigcup_n \{M_n > x + (n-1)(\mu - \varepsilon)\}$. Also, define $\bar{S}_n = S_n + (\mu - \varepsilon)n$ and $\bar{M}_n = M_n - (\mu - \varepsilon)(n-1)$. Then

$$\mathrm{P}\Big(\sup_{n\geq 1}[S_{n-1} + M_n] > x\Big) \leq \mathrm{P}(A_\varepsilon(x)) + \sum_{n=1}^{\infty} \mathrm{P}(\bar{S}_{n-1} + \bar{M}_n > x;\, \bar{M}_n \leq x).$$

The first term behaves as $G_{\mu-\varepsilon}(x)$, so we focus on the second term. Since $\bar{S}_1$ has mean $-\varepsilon$ and has a light right tail (similarly as X_1), there exists $\theta > 0$ such that $r := \mathrm{E}[\mathrm{e}^{\theta \bar{S}_1}] < 1$. Thus,

$$\begin{aligned}
\mathrm{P}(\bar{S}_{n-1} + \bar{M}_n > x;\, \bar{M}_n \leq x) &\leq \mathrm{e}^{-\theta x}\,\mathrm{E}[\mathrm{e}^{\theta \bar{S}_{n-1} + \bar{M}_n};\, \bar{M}_n \leq x] \\
&= \mathrm{e}^{-\theta x} r^{n-1}\,\mathrm{E}[\mathrm{e}^{\theta \bar{M}_n};\, \bar{M}_n \leq x] \\
&= r^{n-1} \int_0^x \mathrm{e}^{-\theta(x-u)}\,\mathrm{d}\mathrm{P}(\bar{M}_n \leq u) \\
&= r^{n-1}\,\mathrm{P}(E_\theta + \bar{M}_n > x;\, \bar{M}_n \leq x) \\
&\leq r^{n-1}\,\mathrm{P}(E_\theta + \bar{M}_n > x),
\end{aligned}$$

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with E_θ exponential(θ) distributed. Let, in addition, N be geometric with rate r to conclude that

$$\begin{aligned}\sum_{n=1}^{\infty} \mathrm{P}(\bar{S}_{n-1} + \bar{M}_n > x;\, \bar{M}_n \le x) &\le \frac{1}{1-r} \mathrm{P}(E_\theta + N(\mu - \varepsilon) + M_1 > x) \\ &\sim \frac{1}{1-r} \mathrm{P}(M_1 > x) \\ &= o(G_\mu(x)),\end{aligned}$$

using basic properties of long-tailed distributions in the last two steps. In particular, the asymptotic equivalence follows from Breiman's [2] theorem (since the tail of e^{M_1} is slowly varying; see [2]), and the final equality follows from the elementary fact that $G_\mu(x) \ge u\,\mathrm{P}(M_1 > x + u)/\mu$ for every u. This implies (1) by letting ε become arbitrarily small.

This proof is shorter and more general than the proof in [3] since we do not assume that $\mathrm{E}[X_1^2] < \infty$, we only require the right tail of X_1 be light tailed.

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Yours sincerely,

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