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RELATIVE KLOOSTERMAN INTEGRALS FOR GL(3): II

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ABSTRACT. Let G' be a quasi-split reductive group over a local field F, f' the characteristic function of a maximal compact subgroup K' of G', N' a maximal unipotent subgroup of G'. We consider the orbits of maximal dimension for the action of $N' \times N'$ on G' and the weighted orbital integral of f' on such an orbit, the weight being a generic character. The resulting integral, we call a Kloosterman integral. A relative version of this construction is to consider a symmetric space S associated to a quasi-split group G, a maximal unipotent subgroup N of G, a maximal compact K of G and the orbits of maximal dimension for the action of N on S. The weighted orbital integral of the characteristic function f of $K \cap S$ on such an orbit is what we call a relative Kloosterman integral; the weight is an appropriate character of N. We conjecture that a relative Kloosterman integral is actually a Kloosterman integral for an appropriate group G'. We prove the conjecture in a simple case: E is an unramified quadratic extension of F, G is $\mathrm{GL}(3, E)$, S is the set of 3×3 matrices s such that $s\bar{s} = 1$; the group G' is then the quasi-split unitary group in three variables.

1. **Introduction.** Let F be a non-archimedean field of odd residual characteristic. We denote by R_F the ring of integers of F, by P_F the maximal ideal and we set $q_F = \#R_F/P_F$. Let G' be a quasi-split reductive group defined over F. Let B' be a Borel subgroup of G' defined over F, N' the unipotent radical of B' and A' a maximal torus of G' such that $B' = A'N$. We choose a representative w for the longest element of the Weyl group of A' and a generic character θ' of N'. Finally, we let f' be the characteristic function of a good maximal compact K' of G'. Then the integral

$$I(a) = \int_{N' \times N'} f'(n_1^{-1} w a n_2) \theta'(n_1) \theta'(n_2) \, dn_1 \, dn_2 \qquad (1)$$

is what we call a *Kloosterman integral*. It is the local analogue of a Kloosterman sum (see [Fr], [G] and the references therein). More generally, we can consider the action of $N' \times N'$ on G' given by:

$$g' \longmapsto n_1^{-1} g' n_2.$$

Then the normalizer of A' in G' is a system of representatives for the orbits. The orbits of maximal dimension are those which admit a representative of the form wa. We could consider also "orbital integrals" associated to the other orbits; we will not do so in this article.

Now, let G be another quasi-split group, σ an automorphism of order two of G defined over F and $B = AN$ a Borel subgroup of G defined over F; let S be the variety of $g \in G$

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such that $gg^\sigma = 1$. Then G operates on S by:

$$s \longmapsto g^{-\sigma} s g.$$

One expects the orbits of N on S to be fairly simple ([S]). To be definite let us assume that B and A are stable under σ. (In [J-Y], we considered a case where σ takes B to its opposite.) Let us further assume that any Weyl element has a representative w such that $w^\sigma = w$. Then it is fairly easy to see that each orbit of N on S has a representative of the form wa where $w^2 = 1$ and $waw = a^{-\sigma}$ ([S]). The orbits of maximal dimension have a representative of the form wa where w is the longest Weyl element and a belongs to a certain torus A'. Let us assume that G has a good maximal compact K invariant under σ. Let Φ be the characteristic function of $K \cap S$. Consider the integral:

$$J(a) = \int \Phi(n^{-\sigma} wan)\theta(n)\,dn. \tag{2}$$

where θ is a generic character of N. This integral we call a *relative Kloosterman integral*. Again, we do not discuss orbital integrals attached to other orbits.

We conjecture that the two types of integral are essentially the same. More precisely, given $G, \sigma, B, K, \theta, A'$ as above there should be another group G' with data B', A', K', θ' such that:

$$I(a) = J(a).$$

We emphasize the fact that A' is the same group for both sets of data: it is a maximal torus in G and the subtorus of $a \in A$ such that $waw = a^{-\sigma}$. Needless to say, our conjecture is much too vague. The exact relation between the two functions might be slightly more complicated; at any rate, it will depend on the choice of the characters. Moreover, we offer no rule to choose G'. Nonetheless, we expect a result of this form to be true for any quasi-split group G and any automorphism of order two of G. (see [J], [Y], [J-Y]). Our purpose in establishing our conjecture in a simple case is to provide evidence for it. Moreover, we hope that, even though our method of proof is rudimentary, some feature of the proof will generalize or that the proof will give some clue to the precise formulation of the conjecture.

The motivation for our conjecture comes from representation theory but is best discussed in the special case at hand. We consider an unramified quadratic extension E of F with Galois conjugation $z \longmapsto \sigma(z)$ or $z \longmapsto \bar{z}$. We let ψ_F be an additive character of F of conductor R_F and ψ_E the additive character of E defined by $\psi_E(z) = \psi_F(z + \bar{z})$. We often write ψ for ψ_E, q for q_F; thus $q_E = q^2$. The group G is then the group $\mathrm{GL}(3, E)$, regarded as an algebraic group over F. We regard σ as an automorphism of $\mathrm{GL}(3, E)$. The variety S is the set of $s \in \mathrm{GL}(3, E)$ such that $s\bar{s} = 1$. The fixator of e is $H = \mathrm{GL}(3, F)$ and G operates transitively on S. We denote by B the group of upper triangular matrices, by A the group of diagonal matrices, by N the group of upper triangular matrices with unit diagonal. We set:

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3}$$

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The orbits of N on S of maximal dimension are those with a representative of the form:

$$wa$$

where

(4) $$waw^{-1} = \bar{a}^{-1}.$$

We denote by A' the torus of $a \in A$ satisfying this condition. Thus A' is the set of matrices of the form:

(5) $$\begin{pmatrix} a & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix}, \quad u\bar{u} = 1.$$

We define a character θ of N by:

(6) $$\theta \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \psi_E(x - y).$$

We set $K = \mathrm{GL}(3, R_E)$ and let Φ be the characteristic function of $K \cap S$. Note that $K \cap S$ is the orbit of 1_3 under K. The integral we want to consider is:

(7) $$I(a) = \int \Phi(\bar{n}^{-1} wan)\theta(n)\, dn,$$

where a is in A'. Here $\mathrm{meas}(N \cap K) = 1$.

Here the group G' is the group of matrices $g \in G$ such that:

(8) $${}^t\bar{g}wg = w.$$

Thus G' is a quasi-split unitary group in three variables. We denote by K', B' and N' the intersections of K, B and N with G'. The group A' is then the intersection of A and G', in particular, a maximal torus in G'. The group N' is the group of matrices of the form:

(9) $$n' = \begin{pmatrix} 1 & x & t - \frac{x\bar{x}}{2} \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix}, \quad t + \bar{t} = 0.$$

We define a character θ' of N' by:

(10) $$\theta'(n') = \psi_E(x).$$

We set $K' = K \cap G'$ and we let f' be the characteristic function of K'. The integral we want to consider takes the form:

(11) $$J(a) = \int_{N' \times N'} f'(n_1^{-1} wan_2)\theta'(n_1 n_2)\, dn_1\, dn_2,$$

where a is in A'. Here $\mathrm{meas}(K' \cap N') = 1$. Our main result is:

THEOREM 1. *With the above notations:*

$$I(a) = J(a).$$

We now briefly explain the relation of this result with representation theory and the principle of functoriality. To that end, we go to a global situation. We let F be a number field. Then there is a functorial map (stable unitary base change) from the automorphic representations of G' to those of G([F2], [R]). A representation π of G should be in the image of the correspondence if and only if it is distinguished with respect to H, that is, there is ϕ in the space of π such that the integral

$$\int_{H(F)\backslash(F_A)} \phi(h)\, dh$$

is non zero. This notion takes its origin in [H-L-R]. In [J-Y] we discussed the dual case where, roughly speaking, the role of H and G' are exchanged. For GL(2) the two cases get entangled because of the isogenies between H and G'; we refer to the work of Flicker (see [F3] and the references therein) for more details and the relation with the poles of the appropriate L—function ([F1]).

To prove this conjecture one can try to establish an identity of the form:

$$\iint K(h, n)\, dh\theta(n)\, dn = \iint K'(n_1', n_2')\theta(n_1'n_2')\, dn_1'\, dn_2'$$

where K and K' are the kernels representing the action of functions f and f' on the discrete spectrum of G and G'. The integral on the left depends only on the integral

$$\int_{H(F_A)} f(hg)\, dh.$$

This may be viewed as function Φ on S:

$$\Phi(g^{-\sigma}g) = \int_{H(F_A)} f(hg)\, dh.$$

Associated to Φ and f', we can define two integrals I and J and the above trace formula should be true if and only if $I(a) = J(a)$. In establishing this identity, the first step is the above theorem ("fundamental lemma for the unit element of the Hecke algebra"). We refer to [Y] and [J] for examples of this kind of trace formula.

The paper is arranged as follows. Both I and J may be viewed as functions on $E^\times$. In Section 2, we compute the integral J; in Section 3 we compute its formal Mellin transform. In Section 4, we compute the integral I; in Section 5, we compute its formal Mellin transform. Roughly speaking, the Mellin transforms turn out to be the square of a Gaussian sum (or integral), the same for both integrals, times an elementary factor. The difficulty is to prove the elementary factors are the same in both cases. In contrast to the previous cases ([J] and [Y-J]), the proof does not involve an identity between Gaussian sums: the same Gaussian sum appears for both I and J. For us, this is another motivation for this paper.

2. **Computation of J.** We note that the center of G' consists of all scalar matrices $u1_3$ with $u\bar{u} = 1$. Both $I(a)$ and $J(a)$ are invariant under multiplication by such matrices. Thus it will suffice to compare the values of I and J on a diagonal matrix α of the form:

$$\alpha = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix}. \tag{12}$$

We then write $I(a)$ and $J(a)$ for $I(\alpha)$ and $J(\alpha)$. After changing n'_1 to its inverse, the integral for J takes the form:

$$J(a) = \iint \theta'^{-1}(n'_1)\theta'(n'_2)\, dn'_1\, dn'_2$$

the integral over the set

$$n'_1 w\alpha n'_2 \in K'.$$

In particular, the integrand is zero unless $n'_1 w\alpha$ is in $K'N'$. Thus we may write the integral as follows:

$$J(a) = \int \theta'^{-1}(n'_1 n')\, dn'_1, \tag{13}$$

where

$$n'_1 w\alpha = k'n'$$

with $k' \in K'$ and $n' \in N'$. Now, for n_1 as in (9), we have:

$$n'_1 w\alpha = \begin{pmatrix} (t - \frac{x\bar{x}}{2})a & x & \bar{a}^{-1} \\ -\bar{x}a & 1 & 0 \\ a & 0 & 0 \end{pmatrix}. \tag{14}$$

Recall that $t + \bar{t} = 0$. The Haar measure on N' is given by:

$$dn' = dx\, dt$$

where

$$\int_{|x|=1} dx = 1, \int_{|t|=1} dt = 1.$$

Let A_i, $1 \le i \le 3$, be the entries in the three first columns and D_i, $1 \le i \le 3$, the minors formed with the two first columns of (14). The matrix (14) is in $K'N'$ if and only if it is KN; in turn, this is equivalent to the relations

$$\sup(|A_i|) = 1, \ \sup(|D_i|) = 1.$$

On the other hand, we have the relation:

$$A_1 D_1 - A_2 D_2 + A_3 D_3 = 0.$$

We thank S. Friedberg for pointing out to us the importance of this relation. Explicitly, we obtain the conditions:

$$\sup\left(\left|\left(t - \frac{x\bar{x}}{2}\right)a\right|, |xa|, |a|\right) = 1 \tag{15}$$

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$$\sup\left(\left|\left(t+\frac{x\bar{x}}{2}\right)a\right|, |xa|, |a|\right) = 1 \tag{16}$$

and

$$-\left(t-\frac{x\bar{x}}{2}\right)aa - xa\bar{x}a + \left(t+\frac{x\bar{x}}{2}\right)aa = 0. \tag{17}$$

If $|a| = 1$, then the other relations imply $|x| \leq 1$ and $|t| \leq 1$ and in turn $J(a) = 1$. From now on we assume $|a| < 1$. Then the above relations simplify to:

$$|ax| < 1, \ \left|a\left(t+\frac{x\bar{x}}{2}\right)\right| = 1.$$

After an easy but lengthy computation we find that

$$n_1'w\alpha = k\begin{pmatrix} 1 & b & c \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

with $k \in K'$ and

$$b = -\frac{2x}{a(x\bar{x}-2t)}. \tag{19}$$

Note that the value of c is actually irrelevant. We conclude that

$$J(a) = \iint \psi_E\left(-x + \frac{2x}{a(x\bar{x}-2t)}\right) dx\, dt. \tag{20}$$

It will be convenient to change t to $tx\bar{x}/2$ to arrive at:

$$J(a) = \iint \psi_E\left(-x + \frac{2}{a\bar{x}(1-t)}\right) |x|_E\, dx\, dt, \tag{21}$$

with the following domain of integration:

$$|a| < 1, \ |ax| < 1, \ |ax\bar{x}(1+t)| = 1.$$

Recall that the first condition is actually the domain on which the integration formula is valid. These conditions can also be written in the form:

$$|ax\bar{x}(1+t)| = 1, \ 1 < |x\bar{x}(1+t)|, \ 1 < |x(1+t)|. \tag{22}$$

3. **Mellin transform of J.** In this section, we compute the Mellin transform of J viewed as a function on $F^\times$. More precisely, we compute the Mellin transform of the function J_0 equal to J for $|a| < 1$ and to 0 otherwise.

Let χ be a character of $E^\times$; we write

$$\chi(z) = \chi_0(z)|z|_E^s$$

where χ_0 has module one and is trivial if χ is not ramified. We also set $X = q_E^{-s}$. The Mellin transform is a formal Laurent series in X:

$$\hat{J}_0(\chi) = \int J_0(a)\chi(a)\, d^\times a = \iiint \psi_E\left(-x + \frac{2}{a\bar{x}(1-t)}\right) |x|\, dx\, dt \chi(a)\, d^\times a. \tag{23}$$

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We compute formally at first and then justify our steps. In the above integral, we change x to $-x$, a to

$$\frac{-2a}{(1-t)\bar{x}}$$

and then a to a^{-1}. The above integral takes the form:

$$\hat{J}_0(\chi)\int\psi(x)\chi^{-1}(\bar{x})|x|_E\psi(a)\chi^{-1}(a)\chi\Big(\frac{-2}{1-t}\Big)\,dx\,d^\times a\,dt. \tag{24}$$

We write ψ for ψ_E. The range of integration is now:

$$|a|=|x|,\ 1<|x\bar{x}(1+t)|,\ 1<|x(1+t)|. \tag{25}$$

Now suppose χ is ramified of conductor P_E^m. Then nothing is changed if in (24) we restrict x and a to the domain:

$$|a|=|x|=q_E^m.$$

After converting the multiplicative Haar measure to an additive one, we find:

$$\hat{J}_0(\chi)=(1-q^{-2})^{-1}\int\psi(x)\chi^{-1}(\bar{x})\,dx\int\psi(a)\chi^{-1}(a)\,da\times\int\chi\Big(\frac{-2}{1-t}\Big)\,dt. \tag{26}$$

By (25) the variable t is restricted to

$$|1+t|_E>q^{-2m}.$$

Since $t=-\bar{t}$, this condition is vacuous. We may also observe that the integral in a and x are actually equal. We arrive then at our final form for the Mellin transform of J_0:

$$\hat{J}_0(\chi)=(1-q^{-2})^{-1}\Big(\int_{|x|=q_E^m}\psi(x)\chi^{-1}(x)\,dx\Big)^2\times\int\chi\Big(\frac{-2}{1-t}\Big)\,dt. \tag{27}$$

We now assume χ unramified. In (24) the integral in a and x can be restricted to the domain:

$$|a|=|x|\le q_E.$$

The contribution of the set $|a|=|x|=q_E$ will be noted $\hat{J}_t$ ("top term"). It is given by (27) with $m=1$. Next, we sum the contributions of the sets $|x|=|a|=q^{-2m}$, with $m\ge 0$. This sum we call $\hat{J}_b$ ("bottom term"). It is given by:

$$\hat{J}_b=\sum_{m\ge 0}X^{-2m}q^{-4m}(1-q^{-2})\int_{|1+t|_E>q_E^{2m}}\chi\Big(\frac{-2}{1-t}\Big)\,dt. \tag{28}$$

The integral in t can be written as:

$$\int_{t\in F,|t|_F>q^{2m}}\chi^{-1}(t)\,dt=\sum_{r\ge 2m+1}X^rq^r(1-q^{-1}).$$

Altogether then we obtain the following expression for $\hat{J}_b$:

$$(1-q^{-1})(1-q^{-2})\sum_{r\ge 2m+1,m\ge 0}X^{r-2m}q^{r-4m}.$$

If we set $s = r - 2m$, we arrive at:

$$(1-q^{-1})(1-q^{-2})\sum_{s\geq 1} X^s q^s \sum_{m\geq 0} q^{-2m} = (1-q^{-1})\frac{qX}{1-qX}.$$

Finally, we arrive at the following expression for the Mellin transform of J_0:

$$\hat{J}_0(\chi) = \frac{X(q-1)}{1-qX} + \hat{J}_t \tag{29}$$

where the "top term" $\hat{J}_t$ is given by:

$$\hat{J}_t = (1-q^{-2})^{-1}\left(\int_{|x|=q_E} \psi(x)\chi^{-1}(x)\,dx\right)^2 \int \chi\left(\frac{-2}{1-t}\right) dt. \tag{30}$$

To justify our steps, we let ϕ be the characteristic function of the set

$$q_E^{-A} \leq |a| \leq q_E^{A}.$$

Then $\hat{J}_0(\chi)$ is the limit, as A tends to infinity, of the Laurent polynomial

$$\int J_0(a)\phi(a)\chi(a)\,d^\times a.$$

The topology on the space of formal Laurent series is given by the convergence of the coefficients. Say χ is unramified. We must show this Laurent polynomial tends to (29) as A tends to infinity. We apply to this integral our sequence of formal manipulations. They are justified and we find this Laurent polynomial is the sum of two terms. The first term is given by the integral (30) with the range of t restricted by:

$$q^{-2A-4} \leq |1-t| \leq q^{2A-4}.$$

As A tends to infinity, this tends to (30). The second term is equal to

$$(1-q^{-1})(1-q^{-2})\sum X^{r-2m}q^{r-4m},$$

the sum over the pairs (r, m) such that:

$$r \geq 2m+1,\ m \geq 0,\ -A \leq r-2m \leq A.$$

As before, we set $s = r - 2m$. then the above expression becomes:

$$(1-q^{-1})(1-q^{-2})\sum_{1\leq s\leq A} X^s q^s \sum_{m\geq 0} q^{-2m}.$$

As A tends to infinity, this tends to

$$(1-q^{-1})\frac{qX}{1-qX}.$$

We have thus justified our formal computations.

4. **Computation of I.** We now evalutate the integral I:

$$I(a) = \int \Phi(\overline{n}^{-1} w\alpha n)\theta(n)\, dn.$$

The integrand is zero unless $\overline{n}^{-1} w\alpha$ is in KN. Writing

$$\overline{n}^{-1} w\alpha = kn'$$

we find the integrand is zero, unless $n'n \in K$. Thus

$$I(a) = \int \theta(n)\, dn \text{ where } \overline{n}^{-1} w\alpha = kn', \quad n'n \in K. \tag{31}$$

We set:

$$n = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{32}$$

Then

$$dn = dx\, dy\, dz$$

the Haar measure on E being the one for which R_E has measure one. We have:

$$\overline{n}^{-1} w\alpha = \begin{pmatrix} a(\overline{xy} - \overline{z}) & -\overline{x} & \overline{a}^{-1} \\ -a\overline{y} & 1 & 0 \\ a & 0 & 0 \end{pmatrix}. \tag{33}$$

If A_i are the entries in the first column and D_i the minors formed with the two first columns of (33), then this matrix is in KN if and only if:

$$\sup(|A_i|) = 1, \ \sup(|D_i|) = 1.$$

We have also:

$$A_1 D_1 - A_2 D_2 + A_3 D_3 = 0.$$

Explicitly, this gives:

$$\sup\big(|a(xy - z)|, |ay|, |a|\big) = 1, \tag{34}$$

$$\sup(|az|, |ax|, |a|) = 1, \tag{35}$$

and the trivial relation:

$$a(\overline{xy} - \overline{z})(-a) + a\overline{y}a\overline{x} - aa\overline{z} = 0. \tag{36}$$

If $|a| = 1$ we find that $|x| \leq 1, |y| \leq 1, |z| \leq 1$ and then $I(a) = 1$.

From now on, we assume that $|a| < 1$. We easily find that

$$\overline{n}^{-1} w\alpha$$

is in KN if and only if one of the three following sets of conditions is satisfied:

$$|a| < 1,\ |ay| = 1,\ |az| = 1,\ |a(xy - z)| \leq 1 \tag{37}$$

$$|a| < 1,\ |ax| = 1,\ |a(xy - z)| = 1,\ |az| \leq 1 \tag{38}$$

$$|a| < 1,\ |ax| < 1,\ |ay| < 1,\ |a(xy - z)| = 1,\ |az| = 1 \tag{39}$$

For instance, assume that (34) and (35) are satisfied. Assume further that

$$|ay| = 1.$$

Then we get from (36) $|ax| < 1$ and then from (35) $|az| = 1$. This leads to (37).

The matrices n' corresponding to cases (37) to (39) can be easily computed:

$$\begin{pmatrix} 1 & 0 & -(aa^\sigma z^\sigma)^{-1} \\ 0 & 1 & -y^\sigma(a^\sigma z^\sigma)^{-1} \\ 0 & 0 & 1 \end{pmatrix} \tag{40}$$

$$\begin{pmatrix} 1 & -x^\sigma\big(a(x^\sigma y^\sigma - z^\sigma)\big)^{-1} & \big(aa^\sigma(x^\sigma y^\sigma - z^\sigma)\big)^{-1} \\ 0 & 1 & -(a^\sigma x^\sigma)^{-1} \\ 0 & 0 & 1 \end{pmatrix} \tag{41}$$

$$\begin{pmatrix} 1 & -x^\sigma\big(a(x^\sigma y^\sigma - z^\sigma)\big)^{-1} & \big(aa^\sigma(x^\sigma y^\sigma - z^\sigma)\big)^{-1} \\ 0 & 1 & -y^\sigma(a^\sigma z^\sigma)^{-1} \\ 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

Accordingly, I is the sum of three integrals I_i, $1 \leq i \leq 3$, with the same integrand:

$$I_i(a) = \int \psi(x - y)\, dx\, dy\, dz,$$

but three different domains of integration defined by the conditions 37 to 39 and the condition

$$n'n \in K.$$

4.1. *Computation of* I_1. The domain of integration for I_1 is defined by:

$$|a| < 1,\ |ay| = 1,\ |az| = 1,\ |a(xy - z)| \leq 1; \tag{43}$$

$$|x| \leq 1,\ \left|y - \frac{\bar{y}}{\bar{a}\bar{z}}\right| \leq 1,\ \left|z - \frac{1}{a\bar{a}\bar{z}}\right| \leq 1. \tag{44}$$

Since $|x| \leq 1$, the integral in x disappears leaving us with the expression:

$$I_1 = \iint \psi(y)\, dy\, dz,$$

integrated over the set:

$$|a| < 1,\ |ay| = 1,\ |az| = 1; \tag{45}$$

$$\left|y - \frac{\bar{y}}{\bar{a}\bar{z}}\right| \leq 1,\ |a\bar{a}z\bar{z} - 1| \leq |a|. \tag{46}$$

Let us change z into zy in this integral. We arrive at:

$$I_1 = |a|^{-1} \iint \psi(y)\, dy\, dz.$$

The domain of integration is now:

$$|a| < 1, \ |ay| = 1, \ |z| = 1; \tag{47}$$

$$\left| y - \frac{1}{\overline{az}} \right| \leq 1, \ |a\bar{a}y\bar{y}z\bar{z} - 1| \leq |a|. \tag{48}$$

We now remark that the above relations imply:

$$\psi(y) = \psi\left(\frac{1}{\overline{az}}\right).$$

Taking this relation into account and changing y to

$$\frac{y}{\overline{az}}$$

we get

$$I_1 = |a|^{-2} \iint \psi\left(\frac{1}{\overline{az}}\right) dz\, dy.$$

The domain of integration is now:

$$|a| < 1, \ |z| = 1, \ |y - 1| \leq |a|, \ |y\bar{y} - 1| \leq |a|.$$

After integrating over y we get

$$I_1 = |a|^{-1} \int_{|z|=1} \psi\left(\frac{1}{\overline{az}}\right) dz,$$

or simply:

$$I_1 = \int_{|az|=1} \psi(z)\, dz.$$

Finally, we obtain the following formula for I_1:

$$I_1(a) = -1 \text{ if } |a| = q_E^{-1} \tag{49}$$

$$I_1(a) = 0 \text{ if } |a| \neq q_E^{-1} \tag{50}$$

4.2. *Computation of I_2.* The domain of integration for I_2 is given by:

$$|a| < 1, \ |a(xy - z)| = 1, \ |ax| = 1, \ |az| \leq 1, \tag{51}$$

$$\left| x - \frac{\bar{x}}{a(\overline{xy} - \bar{z})} \right| \leq 1, \ \left| y - \frac{1}{\overline{ax}} \right| \leq 1, \ \left| z + \frac{1}{a\bar{a}(\overline{xy} - \bar{z})} - \frac{\bar{x}y}{a(\overline{xy} - \bar{z})} \right| \leq 1. \tag{52}$$

We first remark that the inequality $|az| \leq 1$ can be replaced by:

$$|y| \leq 1.$$

Now we change z to

$$z + xy.$$

The integrand does not change but the domain of integration is now:

$$|a| < 1,\ |az| = 1,\ |ax| = 1,\ |y| \leq 1;$$
$$\left|x + \frac{\overline{x}}{a\overline{z}}\right| \leq 1,\ \left|z - \frac{1}{a\overline{a}\overline{z}} + \left(x + \frac{\overline{x}}{a\overline{z}}\right)y\right| \leq 1.$$

This simplifies to

$$|a| < 1,\ |az| = 1,\ |ax| = 1;$$
$$|y| \leq 1,\ \left|x + \frac{\overline{x}}{a\overline{z}}\right| \leq 1,\ \left|z - \frac{1}{a\overline{a}\overline{z}}\right| \leq 1.$$

If we change z to $-z$ and compare with the expression for I_1 we see that $I_1 = I_2$.

We conclude that

$$I_1(a) + I_2(a) = -2 \text{ if } |a| = q_E^{-1}, \tag{53}$$
$$I_1(a) + I_2(a) = 0 \text{ if } |a| \neq q_E^{-1}. \tag{54}$$

4.3. *Computation of* I_3. It will be convenient to express the domain of definition of I_3 as follows. We will set

$$z' = xy - z.$$

The domain of integration is then defined by the following conditions:

$$|a| < 1,\ |ax| < 1,\ |ay| < 1,\ |az| = 1,\ |az'| = 1;$$
$$\left|x - \frac{\overline{x}}{a\overline{z}'}\right| \leq 1,\ \left|y - \frac{\overline{y}}{a\overline{z}}\right| \leq 1,\ \left|z + \frac{1}{a\overline{a}\overline{z}'} - \frac{\overline{x}y}{a\overline{z}'}\right| \leq 1.$$

The last condition can be written as:

$$|a\overline{a}z\overline{z}' + 1 - \overline{a}\overline{x}y| \leq |a|.$$

We change z to zxy (and z' to $z'xy$). We obtain then the following expression:

$$I_3(a) = \iiint \psi(x - y)|xy|\, dx\, dy\, dz \tag{55}$$

with domain of integration

$$z + z' = 1,\ |zxy| > 1,\ |zx| > 1,\ |zy| > 1; \tag{56}$$
$$|azxy| = 1,\ |z| = |z'|;$$
$$\left|x - \frac{1}{a\overline{z}'\overline{y}}\right| \leq 1,\ \left|y - \frac{1}{a\overline{z}\overline{x}}\right| \leq 1,\ |a\overline{a}x\overline{x}y\overline{y}z\overline{z}' + 1 - \overline{a}\overline{x}y| \leq |a|.$$

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5. **Mellin tranform of I.** We now compute the formal Mellin transform of I, more precisely, of I_0, the function equal to I for $|a| < 1$ and to 0 otherwise. We have

$$I_0 = I_1 + I_2 + I_3.$$

From (53), we get at once:

$$\hat{I}_1(\chi) + \hat{I}_2(\chi) = 0 \text{ if } \chi \text{ is ramified} \tag{57}$$

$$\hat{I}_1(\chi) + \hat{I}_2(\chi) = -2X \text{ if } \chi \text{ is unramified} \tag{58}$$

We pass to the computation of the Mellin transform of I_3. We go back to (55). We obtain

$$\hat{I}_3(\chi) = \iiiint \psi(x-y)|xy|\chi(a)\,dx\,dy\,dz\,d^\times a,$$

with domain of integration defined by (56). We change a to

$$\frac{a}{x\bar{y}}$$

We get then the following expression

$$\hat{I}_3(\chi) = \int \psi(x-y)|xy|\chi^{-1}(x\bar{y})\chi(a)\,dx\,dy\,dz\,d^\times a, \tag{59}$$

with domain of integration:

$$\begin{gathered} z + z' = 1,\ |az| = 1,\ |z| = |z'|; \\ |zxy| > 1,\ |zx| > 1,\ |zy| > 1; \\ |1 - a\bar{z}'| \leq |x|^{-1},\ |1 - az| \leq |y|^{-1}; \\ |a\bar{a}z\bar{z}' - \bar{a} + 1| \leq |ax^{-1}y^{-1}|. \end{gathered} \tag{60}$$

Multiplying the last inequality by $|z|$ or $|z'|$, we observe that it can be put into one of the following equivalent forms:

$$|a\bar{a}z\bar{z}\bar{z}' - \overline{az} + \bar{z}| \leq |x^{-1}y^{-1}|,$$
$$|a\bar{a}z'\bar{z}'z - \bar{a}z' + z'| \leq |x^{-1}y^{-1}|.$$

5.1. χ *ramified.* Suppose that χ is ramified of conductor P_E^m. Then nothing is changed if in (59) we restrict x and y to the set:

$$|x| = |y| = q_E^m.$$

Our integral decomposes into a product:

$$\chi^{(-2)} \int_{|x|=q_E^m} \psi(x)\chi^{-1}(x)\,dx \int_{|y|=q_E^m} \psi(y)\chi^{-1}(\bar{y})\,dy \times A$$

where

$$A = q^{4m} \int \chi(a)\,d^\times a\,dz. \tag{61}$$

The domain of integration for A is determined by:

(62) $$z+z'=1,\ |az|=1,\ |z|=|z'|$$

(63) $$|1-az|\le q_E^{-m},\ |1-a\bar{z}'|\le q_E^{-m},$$

(64) $$|a\bar{a}z\bar{z}\bar{z}'-\bar{a}\bar{z}+\bar{z}|\le q_E^{-2m}.$$

We first remark that

$$\chi(a)=\chi^{-1}(z).$$

Next, we remark that the second equality in (63) can be written:

$$|z-az\bar{z}'|\le q_E^{-m}|z|.$$

Taking in account the other conditions, we see this is equivalent to:

$$|z+\bar{z}-1|\le q_E^{-m}|z|.$$

Thus the domain of integration for A is also defined by the previous condition, the equalities

$$|az|=1,\ |z|=|1-z|$$

and the conditions

$$|1-az|\le q_E^{-m},\ |(az-1)\bar{a}\bar{z}+\bar{z}(1-a\bar{a}z\bar{z})|\le q_E^{-2m}.$$

After changing a to az^{-1}, we get:

(65) $$A=q^{4m}\int\chi^{-1}(z)\,dz\,d^{\times}a,$$

where the domain of integration is defined by:

(66) $$|z+\bar{z}-1|\le q_E^{-m}|z|,\ |z|=|1-z|,$$

(67) $$|1-a|\le q_E^{-m},\ |(a-1)\bar{a}+\bar{z}(1-a\bar{a})|\le q_E^{-2m}.$$

Let us set $E=F(\sqrt{\tau})$ with $|\tau|=1$ and

$$z=\alpha+\beta\sqrt{\tau}$$

with α and β in F. Then (66) is satisfied if and only if one the two following conditions is satisfied:

(68) $$|\beta|_F>1,\ \left|\alpha-\frac{1}{2}\right|_F\le q^{-m}|\beta|_F$$

(69) $$|\beta|_F\le 1,\ \left|\alpha-\frac{1}{2}\right|_F\le q^{-m}$$

In any case, we have

$$\left|\alpha-\frac{1}{2}\right|_E\le q_E^{-m}\left|\frac{1}{2}+\beta\sqrt{\tau}\right|_E$$

and thus

$$\chi^{-1}(z) = \chi^{-1}\Big(\frac{1}{2} + \beta\sqrt{\tau}\Big).$$

On the other hand, we will set

$$a = 1 + \varpi^m(x + y\sqrt{\tau}),$$

where ϖ is a uniformizer and x, y are in F. At this point, our integral takes the form;

$$A = (1 - q^{-2})^{-1} q^{2m} \int \chi\Big(\frac{1}{2} + \beta\sqrt{\tau}\Big)^{-1} \, d\alpha \, d\beta \, dx \, dy. \tag{70}$$

with domain of integration defined by (68) and (69) and the conditions:

$$\begin{gathered} \big|-x + \alpha\big(2x + \varpi^m(x^2 - y^2\tau)\big)\big| \leq q^{-m}, \\ \big|y + \beta\big(2x + \varpi^m(x^2 - y^2\tau)\big)\big| \leq q^{-m}, \\ |x| \leq 1, \ |y| \leq 1. \end{gathered}$$

We change variables once more and use

$$2x + \varpi^m(x^2 - y^2\tau)$$

and y for variables of integration. Our integral becomes:

$$A = q^{2m}(1 - q^{-2})^{-1} \int \chi^{-1}\Big(\frac{1}{2} + \beta\sqrt{\tau}\Big) \, d\alpha \, d\beta \, dx \, dy \tag{71}$$

with domain of integration defined by (68) and (69) and the conditions

$$\Big|\Big(\alpha - \frac{1}{2}\Big)x\Big| \leq q^{-m}, \ |y + \beta x| \leq q^{-m}, \ |x| \leq 1, \ |y| \leq 1 \tag{72}$$

At this point we write

$$A = A_1 + A_2$$

where A_1 (resp. A_2) is defined by the same integrand as A but domain of integration defined by (68) (resp. (69)) and (72).

We compute A_2. Since $|\alpha - \frac{1}{2}| \leq q^{-m}$ the first relation in (72) is a consequence of the others and we find that in A_2 the integration is over the domain defined by (69) and the conditions

$$|x| \leq 1, \ |y| \leq 1, \ |y + \beta x| \leq q^{-m}.$$

Since $|x| \leq 1$ we may change y to $y - \beta x$. We find for A_2 the same integrand as before but the domain of integration is now:

$$|\beta| \leq 1, \ \Big|\alpha - \frac{1}{2}\Big| \leq q^{-m}, \ |x| \leq 1, \ |y| \leq q^{-m}.$$

After evaluating the integrals in α, x, y we find:

$$A_2 = (1 - q^{-2})^{-1} \int_{|\beta| \leq 1} \chi^{-1}(1 + \beta\sqrt{\tau})\, d\beta$$

Next, we evalute A_1. To that end, we change x to

$$x\beta^{-1}$$

and arrive at the following expression:

$$A_2 = q^{2m}(1 - q^{-2})^{-1} \int |\beta|^{-1}\chi^{-1}\left(\frac{1}{2} + \beta\sqrt{\tau}\right) d\beta\, d\alpha\, dx\, dy \tag{73}$$

with domain of integration defined by (68) and

$$|x| \leq |\beta|,\ \left|\alpha - \frac{1}{2}\right||x| \leq q^{-m}|\beta|,\ |y + x| \leq q^{-m},\ |y| \leq 1.$$

The two first inequalities above are actually a consequence of the others and (68). If we change x to $x - y$ the integrand does not change but the domain of integration is now:

$$|\beta| > 1,\ \left|\alpha - \frac{1}{2}\right| \leq q^{-m}|\beta|,\ |x| \leq q^{-m},\ |y| \leq 1.$$

Upon integrating in x, y we obtain

$$A_1 = (1 - q^{-2})^{-1} \int_{|\beta| > 1} \chi^{-1}\left(\frac{1}{2} + \beta\sqrt{\tau}\right) d\beta. \tag{74}$$

Altogether, we obtain

$$A = A_1 + A_2 = (1 - q^{-2})^{-1} \int \chi^{-1}\left(\frac{1}{2} + \beta\sqrt{\tau}\right) d\beta.$$

This can also be written as

$$A = (1 - q^{-2})^{-1} \int_{t + \bar{t} = 0} \chi\left(\frac{2}{1 + t}\right) dt.$$

Finally, we obtain:

$$\hat{I}_3(\chi) = (1 - q^{-2})^{-1}\left(\int_{|x| = q_E^m} \chi^{-1}(x)\psi(x)\, dx\right)^2 \times \int_{t + \bar{t} = 0} \chi\left(\frac{-2}{1 + t}\right) dt. \tag{75}$$

5.2. χ *unramified.* We now assume χ unramified. In the integral for $\hat{I}_3$ we can impose the conditions $|x| \leq q_E$ and $|y| \leq q_E$. We separate the domain of integration into four subregions defined by:

$$|x| = q_E, \quad |y| = q_E \tag{76}$$

$$|x| = q_E, \quad |y| \leq 1 \tag{77}$$

$$|x| \leq 1, \quad |y| = q_E \tag{78}$$

$$|x| \leq 1, \quad |y| \leq 1 \tag{79}$$

The contribution of (76) we denote by $\hat{I}_t$ ("top term"). Just as before, it is given by:

$$\hat{I}_t = (1 - q^{-2})^{-1}\left(\int_{|x|=q_E} \chi^{-1}(x)\psi(x)\,dx\right)^2 \times \int_{t+\bar{t}=0} \chi\left(\frac{-2}{1+t}\right) dt. \tag{80}$$

The contribution of (77) (resp. (78)) we denote by $\hat{I}_{m1}$ (resp. $\hat{I}_{m2}$). We have:

$$\hat{I}_{m1} = \int \psi(x-y)|xy|\chi^{-1}(xy)\,dx\,dy\chi(a)\,d^\times a\,dz.$$

The domain of integration is defined by (60) and (77). To compute this, we must sum the contributions of the sets:

$$|x| = q_E,\ |y| = q_E^{-r}, \quad r \geq 0.$$

For such an (x, y) the other conditions simplify to:

$$|z| > q_E^r,\ |1 - a\bar{z}'| \leq q_E^{-1};$$
$$|a\bar{z}'\bar{a}z'z - \bar{a}z' + z'| \leq q_E^{r-1}.$$

Thus $\hat{I}_{m1}$ is equal to the expression

$$\sum_{r\geq 0} -q^2(1-q^{-2})X^{1-r}q^{-4r}\int \chi(a)\,d^\times a\,dz',$$

where the integral is over the domain defined by the above conditions. Next, we change a to $a\bar{z}'^{-1}$. We find:

$$\sum_{r\geq 0} -q^2(1-q^{-2})X^{1-r}q^{-4r}\int \chi(a)\chi^{-1}(z')\,d^\times a\,dz',$$

where, for given r, the integral in (a, z') is over the set:

$$|1-a| \leq q_E^{-1},\ |z| > q_E^r,$$
$$|\bar{a}(a-1) + z'(1-a\bar{a})| \leq q_E^{r-1}.$$

The last inequality simplifies to:

$$|z'(1-a\bar{a})| \leq q_E^{r-1}.$$

To continue, we sum the contribution of the sets:

$$|z'| = q_E^s, \quad s > r.$$

We obtain the following expression:

$$\sum_{r\geq 0, s>r} -X^{1-r+s}q^{2-4r+2s}(1-q^{-2})^2\int d^\times a,$$

where the last integral is over the set:

$$|1-a| \leq q_E^{-1},\ |(1-a\bar{a})|_F \leq q^{r-s-1}.$$

At this point, we need a lemma:

LEMMA 1. *For $b \geq 1$, the integral:*

$$\int d^\times a$$

taken over the set

$$|1 - a|_E \leq q_E^{-1},\ |1 - a\bar{a}|_F \leq q^{-b}$$

is equal to

$$(1 - q^{-2})^{-1} q^{-1-b}.$$

To prove the Lemma, we write:

$$a = 1 + u + v\sqrt{\tau}$$

with u, v in R_F. Then:

$$a\bar{a} = 1 + 2u + u^2 - v^2\tau$$

and we can use v and $2u + u^2 - v^2\tau$ for variables of integration.

Applying the lemma with $b = s + 1 - r$, we get for our integral:

$$\hat{I}_{m1} = \sum_{r \geq 0, s > r} -X^{1-r+s} q^{-3r+s}(1 - q^{-2}). \tag{81}$$

If we set $t = s - r$, this becomes:

$$\sum_{t > 1} -X^{1+t} q^t \sum_{r \geq 0} q^{-2r}(1 - q^{-2}).$$

or

$$\hat{I}_{m1} = -\frac{X^2 q}{1 - Xq}.$$

Similarly, we find:

$$\hat{I}_{m2} = -\frac{X^2 q}{1 - Xq}. \tag{82}$$

So altogether

$$\hat{I}_{m1} + \hat{I}_{m2} = -2\frac{X^2 q}{1 - Xq}. \tag{83}$$

Finally, we compute the contribution of (79). It will be noted $\hat{I}_b$ ("bottom term") and is equal to:

$$\hat{I}_b = \int |xy| \chi^{-1}(xy) \chi(a)\, dx\, dy\, d^\times a\, dz \tag{84}$$

The domain of integration is defined by:

$$|x| \leq 1,\ |y| \leq 1,\ |z| > |xy|^{-1},\ |az| = 1; \tag{85}$$

$$|az\overline{az}z' - \overline{az} + \bar{z}| \leq |xy|^{-1} \tag{86}$$

We change a to az^{-1}. Then the integral becomes:

$$\hat{I}_b = \int |xy| \chi^{-1}(xy) \chi(a) \chi^{-1}(z)\, dx\, dy\, d^\times a\, dz. \tag{87}$$

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with domain of integration

$$|x| \leq 1,\ |y| \leq 1,\ |z| > |xy|^{-1},\ |a| = 1; \tag{88}$$

$$|a\overline{az}' - \overline{a} + \overline{z}| \leq |xy|^{-1} \tag{89}$$

The last condition simplifies to

$$|1 - a\overline{a}| \leq |xyz|^{-1} \tag{90}$$

At this point, we appeal to another lemma:

LEMMA 2. *For $t \in E$ with $|t|_E \leq 1$, the integral*

$$\int d^\times a$$

taken over the set

$$|a|_E = 1, |1 - a\overline{a}|_E \leq |t|_E^{-1}$$

is equal to:

$$(1 - q^{-1})^{-1}|t|_E^{-1/2}.$$

We leave the proof to the reader. If we set

$$|x| = q_E^{-r}, |y| = q_E^{-s}, |z| = q_E^{t},$$

and apply the lemma our integral becomes a sum:

$$\hat{I}_b = \sum_{r \geq 0, s \geq 0, t > r+s} X^{t-r-s} q^{t-3r-3s}(1 - q^{-2})^3(1 - q^{-1})^{-1}. \tag{91}$$

We set $u = t - r - s$. Then this sum becomes:

$$\sum_{u > 1, r \geq 0, s \geq 0} X^u q^u q^{-2r-2s}(1 - q^{-2})^3(1 - q^{-1})^{-1}$$

or

$$\hat{I}_b = (1 + q^{-1})\frac{qX}{1 - Xq}. \tag{92}$$

Finally, we sum our results:

$$\hat{I}_0(\chi) = \hat{I}_t + \hat{I}_1 + \hat{I}_2 + \hat{I}_{m1} + \hat{I}_{m2} + \hat{I}_b = \hat{I}_t - 2X - 2\frac{X^2 q}{1 - Xq} + \frac{(q+1)X}{1 - Xq}.$$

or

$$\hat{I}_b = \hat{I}_t + \frac{X(q-1)}{1 - Xq}. \tag{93}$$

It remains to justify our formal computations. As before, we let ϕ be the characteristic function of the set

$$q^{-A} \leq |a| \leq q^A$$

and consider the integral

$$\int I_0(a)\phi(a)\chi(a)\,d^{\times}a.$$

As A tend to infinity, this tends to $\hat{I}_0$. Now we perform our sequence of manipulations on this integral. They are justified. For instance, let us assume χ is unramified and consider the integral which replaces $\hat{I}_b$: it is still given by (87) with the extra condition

$$q^{-A}|xy| \leq |a| \leq q^{A}|xy|.$$

In (87) we must impose the extra condition

$$-A \leq t - r - s \leq A.$$

If we set, as before,

$$u = t - r - s$$

the expression for $\hat{I}_b$ is now replaced by:

$$\sum_{A\geq u>1, r\geq 0, s\geq 0} X^u q^u q^{-2r-2s}(1-q^{-2})^3(1-q^{-1})^{-1} = \sum_{A\geq u>1} X^u q^u q^{-2r-2s}(1+q^{-1}).$$

As A tends to infinity, this does approach

$$\hat{I}_b = (1+q^{-1})\frac{qX}{1-Xq}.$$

One can treat the other terms the same way. So our formal computations are justifed.

6. **Conclusion.** Now to prove our theorem. For $|a| = 1$, we have $I(a) = J(a) = 1$. Thus it suffices to show that $I_0(a) = J_0(a)$ or

$$\hat{I}_0(\chi) = \hat{J}_0(\chi).$$

This follows from (27) and (75) if χ is ramified. If χ is unramified, the top terms coincide and the remaining terms coincide as follows from (29) and (93).

So we are done.

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