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The Parabolic Littlewood–Paley Operator with Hardy Space Kernels

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Abstract. In this paper, we give the L^p boundedness for a class of parabolic Littlewood–Paley g-function with its kernel function Ω is in the Hardy space $H^1(S^{n-1})$.

1 Introduction

Let $\mathbb{R}^n$ be the Euclidean space with the routine norm $|x|$ for each $x \in \mathbb{R}^n$. Denote by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere on $\mathbb{R}^n$ equipped with the Lebesgue measure $\sigma(x')$. Let $\alpha_1, \dots, \alpha_n$ be fixed real numbers with $\alpha_i \geq 1$. It is easy to see that for fixed $x \in \mathbb{R}^n$, the function

$$F(x, \rho) = \sum_{i=1}^{n} \frac{{x_i}^2}{\rho^{2\alpha_i}}$$

is a strictly decreasing function of $\rho > 0$. Therefore, there exists a unique $\rho(x)$ such that $F(x, \rho) = 1$. It was proved in [7] that $\rho(x)$ is a metric on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, set

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos\varphi_1 \cdots \cos\varphi_{n-2} \cos\varphi_{n-1} \\ x_2 &= \rho^{\alpha_2} \cos\varphi_1 \cdots \cos\varphi_{n-2} \sin\varphi_{n-1} \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2 \\ x_n &= \rho^{\alpha_n} \sin\varphi_1. \end{aligned}$$

Then $dx = \rho^{\alpha-1} J(x') d\rho d\sigma(x')$, and $\rho^{\alpha-1} J(x')$ is the Jacobian of the above transform, where $\alpha = \sum_{i=1}^{n} \alpha_i$ and $J(x') = \alpha_1 x_1'^2 + \cdots + \alpha_n x_n'^2$. It is easy to see that $J(x') \in C^\infty(S^{n-1})$ with $1 \leq J(x') \leq M$ for some $M \geq 1$. Without loss of generality, we may assume $\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq 1$.

For $t > 0$, let $A_t = \operatorname{diag}[t^{\alpha_1}, \dots, t^{\alpha_n}]$. Suppose that $\Omega(x)$ is a real valued and measurable function defined on $\mathbb{R}^n$. We say $\Omega(x)$ is homogeneous of degree zero with respect to A_t, if for any $t > 0$ and $x \in \mathbb{R}^n$

$$\Omega(A_t x) = \Omega(x). \tag{1.1}$$

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Moreover, we also assume that $\Omega(x)$ satisfies the following cancellation condition:

$$\int_{S^{n-1}} \Omega(x')J(x')d\sigma(x') = 0. \tag{1.2}$$

In 1966, Fabes and Rivière [7] proved that if $\Omega \in C^1(S^{n-1})$ satisfies (1.1) and (1.2), then the parabolic singular integral operator T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y)}{\rho(y)^\alpha} f(x-y)\, dy.$$

In 1976, Nagel, Rivière and Wainger [9] improved the above result. They showed T_Ω is still bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if replacing $\Omega \in C^1(S^{n-1})$ by a weaker condition $\Omega \in L\log^+ L(S^{n-1})$.

On the other hand, in 1974, Madych considered the L^p boundedness with respect to the transform A_t of the Littlewood–Paley operator. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\hat{\psi}(0) = 0$, where and below, $\hat{\psi}$ denotes the Fourier transform of ψ. Let $\psi_t(x) = t^{-\alpha}\psi(A_{t^{-1}}x)$ for $t > 0$. Then the Littlewood–Paley operator related to A_t is defined by

$$g_\psi(f)(x) = \Big(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t} \Big)^{1/2}.$$

Theorem A [8] *The Littlewood–Paley operator g_ψ is of type (p,p) for $1 < p < \infty$.*

Inspired by the works in [7–9], recently Ding, Xue and Yabuta [5] improved the above result. More precisely, the authors in [5] proved that the parabolic Littlewood–Paley operator is still bounded on L^p if $\psi(x)$is replaced by a kernel function $\phi(x) = \Omega(x)\rho(x)^{-\alpha+1}\chi_{\{\rho(x)\le 1\}}(x)$ with $\Omega \in L^q(S^{n-1})$ $(q > 1)$ satisfying (1.1) and (1.2).

Theorem B [5] *If $\Omega \in L^q(S^{n-1})(q > 1)$ satisfies* (1.1) *and* (1.2), *then g_ϕ is of type (p,p) for $1 < p < \infty$.*

Notice that on the unit sphere S^{n-1}, there are the following containing relationships:

$$C^\infty \subsetneqq L^q\ (q > 1) \subsetneqq L\log^+ L \subsetneqq H^1 \subsetneqq L^1,$$

where H^1 denotes the Hardy space on S^{n-1} (see §2 for its definition). Hence, a natural question is whether the size condition assumed on Ω can be weakened further. The purpose of this paper is to give a positive answer to this question.

Theorem 1.1 *If $\Omega \in H^1(S^{n-1})$ satisfies* (1.1) *and* (1.2), *then g_ϕ is of type (p,p) for $1 < p < \infty$.*

Remark. If $\alpha_1 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$ and $\alpha = n$. In this case, $g_\phi = \mu_\Omega$ and the latter is just the classical Marcinkiewicz integral, which was studied by many authors. (See [1, 4, 10], for example.) Moreover, note also that the Ω in Theorem 1.1 (also Theorem B) has no any smoothness on S^{n-1}.

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2 Definitions and Lemmas

Let us begin with the definition of Hardy space $H^1(S^{n-1})$. For $f \in L^1(S^{n-1})$ and $x' \in S^{n-1}$, we denote

$$P^+ f(x') = \sup_{0<t<1} \Big| \int_{S^{n-1}} f(y') P_{tx'}(y')\, d\sigma(y') \Big|,$$

where $P_{tx'}(y') = \frac{1-t^2}{|y'-tx'|^n}$ for $y' \in S^{n-1}$. Then

$$H^1(S^{n-1}) = \{ f \in L^1(S^{n-1}) : \|P^+ f\|_{L^1(S^{n-1})} < \infty \},$$

and we define $\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})}$ if $f \in H^1(S^{n-1})$.

A very useful characterization of the space $H^1(S^{n-1})$ is its atomic decomposition. Let us first recall the definition of atoms. A regular $H^1(S^{n-1})$ atom is a function $a(x')$ on $L^\infty(S^{n-1})$ satisfying the following conditions:

$$\text{(2.1)} \quad \operatorname{supp}(a) \subset S^{n-1} \cap \{ y \in \mathbb{R}^n : |y - \xi'| < r \text{ for some } \xi' \in S^{n-1} \text{ and } r \in (0,2] \};$$

$$\text{(2.2)} \qquad \int_{S^{n-1}} a(x') Y_m(x')\, d\sigma(x') = 0$$

for any spherical harmonic polynomial Y_m with degree $m \le N$, where N is any fixed integer;

$$\text{(2.3)} \qquad \|a\|_{L^\infty(S^{n-1})} \le r^{1-n}.$$

An exceptional $H^1(S^{n-1})$ atom $u(x')$ is an $L^\infty(S^{n-1})$ function bounded by 1.

From [3], we find that any $\Omega \in H^1(S^{n-1})$ has an atomic decomposition

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{i=1}^{\infty} \delta_i u_i,$$

where each a_j is a regular $H^1(S^{n-1})$ atom and each u_i is an exceptional atom. Moreover,

$$\sum_{j=1}^{\infty} |\lambda_j| + \sum_{i=1}^{\infty} |\delta_i| \le C \|\Omega\|_{H^1(S^{n-1})}.$$

We note that for any $x' \in S^{n-1}$,

$$\Big| \sum_{i=1}^{\infty} \delta_i u_i(x') \Big| \le \sum_{i=1}^{\infty} |\delta_i|.$$

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Without loss of generality, we can assume

$$\Big|\sum_{i=1}^{\infty}\delta_i u_i(x')\Big| \le \|\Omega\|_{H^1(S^{n-1})}.$$

Thus we write

$$\sum_{i=1}^{\infty}\delta_i u_i(x') = \|\Omega\|_{H^1(S^{n-1})}\omega(x'),$$

with $\omega(x') = \sum_{i=1}^{\infty}\delta_i u_i(x')/\|\Omega\|_{H^1(S^{n-1})}$. In this new definition, for $x' \in S^{n-1}$,

$$\Omega(x') = \sum_{j=1}^{\infty}\lambda_j a_j(x') + \|\Omega\|_{H^1(S^{n-1})}\omega(x') \quad\text{and}\quad \|\omega\|_{L^\infty(S^{n-1})} \le 1. \tag{2.4}$$

The following Lemmas 2.1 and 2.2 can be found in [6].

Lemma 2.1 [6] *Suppose that $n \ge 3$ and b satisfies* (2.1), (2.3), *and*

$$\int_{S^{n-1}} b(y')\,d\sigma(y') = 0. \tag{2.5}$$

Let

$$F_b(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} b(s,(1-s^2)^{1/2}\widetilde{y})d\sigma(\widetilde{y}),$$

$$G_b(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} |b(s,(1-s^2)^{1/2}\widetilde{y})|d\sigma(\widetilde{y}).$$

Then there exists a constant C, independent of b, such that

$$\operatorname{supp}(F_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')), \tag{2.6}$$

$$\operatorname{supp}(G_b) \subset (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')), \tag{2.7}$$

$$\|F_b\|_\infty \le C/r(\xi'), \quad \|G_b\|_\infty \le C/r(\xi'), \tag{2.8}$$

$$\int_{\mathbb{R}} F_b(s)\,ds = 0, \tag{2.9}$$

where $r(\xi') = |\xi|^{-1}|L_r\xi|$ and $L_r\xi = (r^2\xi_1, r\xi_2, \dots, r\xi_n)$.

Lemma 2.2 [6] *Suppose that $n = 2$ and b satisfies* (2.1), (2.3) *and* (2.5). *Let*

$$F_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\big(b(s,(1-s^2)^{1/2}) + b(s,-(1-s^2)^{1/2})\big),$$

$$G_b(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\big(|b(s,(1-s^2)^{1/2})| + |b(s,-(1-s^2)^{1/2})|\big).$$

Then $F_b(s)$ satisfies (2.6) *and* (2.9), *and $\|F_b\|_q \le C|L_r(\xi')|^{-1+1/q}$. And $G_b(s)$ satisfies* (2.7) *and $\|G_b\|_q \le C|L_r(\xi')|^{-1+1/q}$ for some $q \in (1,2)$.*

Lemma 2.3 [5] *For $\Omega \in L^1(S^{n-1})$, denote*

$$\sigma_{2^t}(x) = 2^{-t}\Omega(x)\rho(x)^{-\alpha+1}\chi_{\{\rho(x)\leq 2^t\}}(x),$$

and $\sigma^*(f)(x) = \sup_{t\in\mathbb{R}} ||\sigma_{2^t}| * f(x)|$. Then $\|\sigma_{2^t}\|_1 \leq C$ and $\|\sigma^*(f)\|_p \leq C\|f\|_p$ for $1 < p < \infty$, where the constant C is independent of f and t.

Lemma 2.4 [5] *Suppose that m denotes the distinct numbers of $\{\alpha_j\}$. Then for any $x, y \in \mathbb{R}^n$, $0 \leq \beta \leq 1$*

$$\Big|\int_1^2 e^{-iA_\lambda x\cdot y}\frac{d\lambda}{\lambda}\Big| \leq C|x\cdot y|^{-\frac{\beta}{m}},$$

where $C > 0$ is independent of x and y.

3 Proof of Theorem 1.1

Since $\Omega \in H^1(S^{n-1})$ satisfies the cancellation condition (1.2), by (2.4) we can write

$$\Omega(x') = \sum_{j=1}^{\infty}\lambda_j a_j(x') + \|\Omega\|_{H^1(S^{n-1})}\omega(x'),$$

where each a_j is a regular $H^1(S^{n-1})$ atom and $\|\omega\|_{L^\infty(S^{n-1})} \leq 1$. Moreover,

$$\sum_{j=1}^{\infty}|\lambda_j| \leq C\|\Omega\|_{H^1(S^{n-1})}.$$

For $y \in \mathbb{R}^n$ ($y \neq 0$), we write

$$\Omega(y) = \sum_{j=1}^{\infty}\lambda_j\tilde{a}_j(y) + \|\Omega\|_{H^1(S^{n-1})}\tilde{\omega}(y),$$

where $\tilde{a}_j(y) = a_j(A_{\rho(y)^{-1}}y)$ and $\tilde{\omega}(y) = \omega(A_{\rho(y)^{-1}}y)$. It is easy to check that $\tilde{\omega}(y') = \omega(y')$, $\tilde{a}_j(y') = a_j(y')$ for $y' \in S^{n-1}$ and $\tilde{\omega}$ and $\tilde{a}_j$ satisfy (1.1) for any $t > 0$ and $y \in \mathbb{R}^n$.

Noticing that $J(\frac{x}{|x|})|x|^2$ is a homogeneous polynomial of degree 2 on $\mathbb{R}^n$ by [11, Theorem 2.1], we can write

$$J\Big(\frac{x}{|x|}\Big)|x|^2 = P_2(x) + |x|^2P_0(x),$$

where $P_k(x)$ is a harmonic polynomial of degree k ($k = 0, 2$). Then $J(x') = P_2(x') + P_0(x')$, where $P_k(x')$ is a spherical harmonic polynomial of degree k ($k = 0, 2$). So by (2.2), we have

$$\text{(3.1)}\quad \int_{S^{n-1}} a_j(y')J(y')\,d\sigma(y') = \int_{S^{n-1}} a_j(y')P_2(y')\,d\sigma(y') + \int_{S^{n-1}} a_j(y')P_0(y')\,d\sigma(y') = 0,$$

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for all $j = 1, 2, \ldots$. Thus by (2.4) and (3.1), we know

$$\int_{S^{n-1}} \omega(y')J(y')\,d\sigma(y') = 0. \tag{3.2}$$

Therefore,

$$\|g_\phi(f)\|_p \le \sum_{j=1}^{\infty} |\lambda_j| \|g_{a_j}(f)\|_p + \|\Omega\|_{H^1(S^{n-1})} \|g_\omega(f)\|_p, \tag{3.3}$$

where

$$g_{a_j}(f)(x) = \Big(\int_0^\infty \Big| \int_{\rho(y)\le t} \frac{\tilde{a}_j(y)}{\rho(y)^{\alpha-1}} f(x-y)\,dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2},$$

$$g_\omega(f)(x) = \Big(\int_0^\infty \Big| \int_{\rho(y)\le t} \frac{\tilde{\omega}(y)}{\rho(y)^{\alpha-1}} f(x-y)\,dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2}.$$

Since $\omega(x') \in L^\infty(S^{n-1})$ and satisfies the cancellation condition (3.2), by Theorem B we get

$$\|g_\omega(f)\|_p \le C\|f\|_p, \tag{3.4}$$

where C is independent of ω and f. Thus, to prove Theorem 1.1, by (3.3) and (3.4) it suffices to show that there exists $C > 0$, independent of the atoms a_j and f, such that for $j = 1, 2, \ldots,$

$$\|g_{a_j}(f)\|_p \le C\|f\|_p. \tag{3.5}$$

We only prove (3.5) for the case $n > 2$. The case for $n = 2$ can be dealt with using the same method and Lemma 2.2. From now we denote simply a_j, $\tilde{a}_j$ and g_{a_j} by a, $\tilde{a}$, and g_a, respectively. Without loss of generality, we may also assume that $\operatorname{supp}(a)$ is contained in $B(\mathbf{1}, r) \cap S^{n-1}$, where $B(\mathbf{1}, r) = \{y : |y - \mathbf{1}| < r\}$ and $\mathbf{1} = (1, 0, \ldots, 0)$.

Choose a $C_0^\infty(\mathbb{R}^n)$ function φ such that $\varphi(x) = \varphi(\rho(x))$, $0 \le \varphi \le 1$ satisfying $\operatorname{supp}(\varphi) \subset \{y : 1/2 \le \rho(y) \le 2\}$ and $\int_0^\infty \varphi(t)/t\,dt = 1$. Define functions Φ and Δ by $\widehat{\Phi}(\xi) = \varphi(\rho(L_r\xi))$ and $\widehat{\Delta}(\xi) = \varphi(\rho(\xi))$, respectively, where $L_r\xi$ is defined in Lemma 2.1. If we denote $\Phi_t(x) = t^{-\alpha}\Phi(A_{t^{-1}}x)$ and $\Delta_t(x) = t^{-\alpha}\Delta(A_{t^{-1}}x)$, then it is easy to check that $\widehat{\Phi_t}(\xi) = \varphi(t\rho(L_r\xi))$, $\widehat{\Delta_t}(\xi) = \varphi(t\rho(\xi))$, and $\Phi_t(x) = \frac{1}{r^{n+1}} t^{-\alpha}\Delta(L_{r^{-1}}A_{t^{-1}}x)$, where

$$L_{r^{-1}}A_{t^{-1}}x = (r^{-2}t^{-\alpha_1}x_1, r^{-1}t^{-\alpha_2}x_2, \ldots, r^{-1}t^{-\alpha_n}x_n).$$

For any $f \in \mathcal{S}(\mathbb{R}^n)$, by taking Fourier transform we have

$$f(x) = \int_{-\infty}^{\infty} \Phi_{2^t} * f(x)dt \sim \int_0^\infty \Phi_t * f(x)\frac{dt}{t}. \tag{3.6}$$

Define

$$g_\Phi(f)(x) = \Big(\int_0^\infty |\Phi_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2} \sim \Big(\int_{-\infty}^\infty |\Phi_{2^t} * f(x)|^2\, dt\Big)^{1/2}.$$

Now we claim that

$$\|g_\Phi(f)\|_p \le C\|f\|_p, \tag{3.7}$$

with C independent of $r > 0$. In fact, by the definition of Φ_t, we have

$$\begin{aligned}\Phi_t * f(x) &= \frac{1}{r^{n+1}} t^{-\alpha} \int_{\mathbb{R}^n} \Delta(L_{r^{-1}} A_{t^{-1}} y) f(x-y)\, dy\\ &= t^{-\alpha} \int_{\mathbb{R}^n} \Delta(A_{t^{-1}} y) f(L_r(L_{r^{-1}} x - y))\, dy\\ &= \Delta_t * h(L_{r^{-1}} x),\end{aligned}$$

where $h(x) = f(L_r x)$. Since $\int_{\mathbb{R}^n} \Delta(x)\, dx = \widehat{\Delta}(0) = \varphi(0) = 0$, by Theorem A we get

$$\begin{aligned}\|g_\Phi(f)\|_p &= \Big\| \Big(\int_0^\infty |\Phi_t * f(\cdot)|^2 \frac{dt}{t}\Big)^{1/2} \Big\|_p\\ &= \Big\{ \int_{\mathbb{R}^n} \Big(\int_0^\infty |\Delta_t * h(L_{r^{-1}} x)|^2 \frac{dt}{t}\Big)^{p/2} dx \Big\}^{1/p}\\ &= \Big\{ r^{n+1} \int_{\mathbb{R}^n} \Big(\int_0^\infty |\Delta_t * h(x)|^2 \frac{dt}{t}\Big)^{p/2} dx \Big\}^{1/p}\\ &\le C r^{\frac{n+1}{p}} \|h\|_p\\ &= C\Big(r^{n+1} \int_{\mathbb{R}^n} |f(L_r x)|^p\, dx\Big)^{1/p} = C\|f\|_p.\end{aligned}$$

This is (3.7). Now we denote $\sigma_{2^t}(y) = 2^{-t}\tilde{a}(y)\rho(y)^{-\alpha+1}\chi_{\{\rho(y)\le 2^t\}}(y)$. Then

$$\begin{aligned}g_a(f)(x) &= \Big(\int_0^\infty \Big| \int_{\rho(y)\le t} \frac{\tilde{a}(y)}{\rho(y)^{\alpha-1}} f(x-y) dy \Big|^2 \frac{dt}{t^3}\Big)^{1/2}\\ &\sim \Big(\int_{-\infty}^\infty |\sigma_{2^t} * f(x)|^2\, dt\Big)^{1/2}.\end{aligned}$$

By (3.6) and the Minkowski inequality, we obtain

$$\begin{aligned}g_a(f)(x) &\sim \Big(\int_{-\infty}^\infty \Big| \int_{-\infty}^\infty \Phi_{2^{s+t}} * \sigma_{2^t} * f(x) ds \Big|^2 dt\Big)^{1/2}\\ &\le \int_{-\infty}^\infty \Big(\int_{-\infty}^\infty |\sigma_{2^t} * \Phi_{2^{s+t}} * f(x)|^2 dt\Big)^{1/2} ds\\ &=: \int_{-\infty}^\infty Q_s(f)(x)\, ds.\end{aligned}$$

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Using Minkowski's inequality again yields

$$\|g_a(f)\|_p \le C\Big(\int_0^\infty \|Q_s(f)\|_p\,ds + \int_{-\infty}^0 \|Q_s(f)\|_p\,ds\Big). \tag{3.8}$$

By (3.8), it is easy to see that the proof of (3.5) can be reduced to show the following estimates

$$\|Q_s(f)\|_p \le \begin{cases} C2^{-s\gamma}\|f\|_p & \text{for } s>0,\\ C2^{s\tau}\|f\|_p & \text{for } s<0,\end{cases} \tag{3.9}$$

where τ and γ are some positive constants, and C is independent s and f.

The proof of (3.9) will be completed in two steps.

Step 1: There exists $C>0$, independent of s and f, such that

$$\|Q_s(f)\|_p \le C\|f\|_p \quad \text{for } 1<p<\infty. \tag{3.10}$$

First we consider the case $1<p<2$. Denote $G_{s+t}(x) = \Phi_{2^{s+t}} * f(x)$. Since $a(x') \in L^1(S^{n-1})$, by Lemma 2.3, we know $\|\sigma_{2^t}\|_1 \le C$, then

$$\Big\|\int_{-\infty}^\infty \sigma_{2^t} * G_{s+t}(\cdot)dt\Big\|_1 \le C\Big\|\int_{-\infty}^\infty G_t(\cdot)dt\Big\|_1. \tag{3.11}$$

On the other hand, for $1<q<\infty$, also by Lemma 2.3, we get

$$\|\sup_{t\in\mathbb{R}}|\sigma_{2^t} * G_{s+t}|\|_q \le \|\sigma^*(\sup_{t\in\mathbb{R}}|G_t|)\|_q \le C\|\sup_{t\in\mathbb{R}}|G_t|\|_q. \tag{3.12}$$

If we define $TG_{s+t}(x) = \sigma_{2^t} * G_{s+t}(x)$, then (3.11) and (3.12) show that T is a bounded operator on $L^1(L^1(\mathbb{R}),\mathbb{R}^n)$ and $L^q(L^\infty(\mathbb{R}),\mathbb{R}^n)$, respectively. Since $1<p<2$, we can take $q>1$ such that $1/q = 2/p-1$. Then by using the operator interpolation theorem between (3.11) and (3.12), we know that the operator T is also bounded on $L^p(L^2(\mathbb{R}),\mathbb{R}^n)$. That is

$$\Big\|\Big(\int_{-\infty}^\infty |\sigma_{2^t} * G_{s+t}(\cdot)|^2 dt\Big)^{1/2}\Big\|_p \le C\Big\|\Big(\int_{-\infty}^\infty |G_t(\cdot)|^2 dt\Big)^{1/2}\Big\|_p.$$

From this and (3.7), we prove (3.10) for $1<p<2$. Moreover, by (3.7) and the L^2 boundedness of σ^*, (3.10) holds for the case $p=2$. Now let us deal with the case $p>2$. Let $q=(p/2)'$. Then

$$\|Q_s f\|_p^2 = \sup_\nu\Big|\int_{\mathbb{R}^n}\int_{-\infty}^\infty |\sigma_{2^t} * \Phi_{2^{s+t}} * f(x)|^2 \nu(x)\,dt\,dx\Big|,$$

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where the supremum is taken over all $\nu(x) \in L^q(\mathbb{R}^n)$ with $\|\nu\|_q \le 1$. Applying Hölder's inequality and noting the fact $\|\sigma_{2^t}\|_1 \le C$,

$$\begin{aligned}
\Big| \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} &|\sigma_{2^t} * \Phi_{2^{s+t}} * f(x)|^2 \nu(x)\, dt\, dx \Big| \\
&\le \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \Big\{ \Big(\int_{\mathbb{R}^n} |\Phi_{2^{s+t}} * f(y)|^2 |\sigma_{2^t}(x-y)|\, dy \Big)^{1/2} \\
&\qquad\qquad \times \Big(\int_{\mathbb{R}^n} |\sigma_{2^t}(x-y)|\, dy \Big)^{1/2} \Big\}^2 |\nu(x)|\, dx\, dt \\
&\le \|\sigma_{2^t}\|_1 \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\Phi_{2^{s+t}} * f(y)|^2 |\sigma_{2^t}(x-y)| |\nu(x)|\, dy\, dx\, dt \\
&\le C \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\Phi_{2^t} * f(y)|^2 \sigma^*(|\nu|)(y)\, dy\, dt \\
&= C \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} |\Phi_{2^t} * f(y)|^2\, dt\, \sigma^*(|\nu|)(y)\, dy,
\end{aligned}$$

where C is independent of s, f and ν. Using Hölder's inequality again and (3.7), Lemma 2.3, we obtain

$$\|Q_s f\|_p^2 \le C \sup_{\nu} \|g_\Phi(f)\|_p^2 \|\sigma^*(|\nu|)\|_q \le C\|f\|_p^2.$$

Thus we have (3.10) for $p > 2$. From the proof of (3.10) above, it is easy to check that the constant C is independent of s and f.

Step 2: There exists $C > 0$, independent of f and s, such that

$$\|Q_s(f)\|_2 \le \begin{cases} C2^{-s}\|f\|_2 & \text{for } s > 0, \\ C2^{\beta s/m}\|f\|_2 & \text{for } s < 0, \end{cases} \tag{3.13}$$

where $0 < \beta < \frac{1}{2\alpha_n}$ and m denotes the distinct numbers of $\{\alpha_j\}$.

By Plancherel's theorem,

$$\|Q_s f\|_2^2 \le \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\varphi(2^{s+t}\rho(L_r\xi))|^2 |\widehat{\sigma_{2^t}}(\xi)|^2\, d\xi dt, \tag{3.14}$$

where

$$\widehat{\sigma_{2^t}}(\xi) = 2^{-t} \int_0^{2^t} \int_{S^{n-1}} a(y')J(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')d\rho$$

and a is a regular $H^1(S^{n-1})$ atom supported in $B(\mathbf{1}, r) \cap S^{n-1}$,where $\mathbf{1} = (1, 0, \dots, 0)$. We first give the estimate of $|\widehat{\sigma_{2^t}}(\xi)|$. Let $\eta(y') = a(y')J(y')/\|J\|_{L^\infty(S^{n-1})}$. By (3.1) and $J(y') \in C_0^\infty(S^{n-1})$, we know $\eta(y')$ satisfies (2.3) and (2.5), and $\operatorname{supp}(\eta) \subset B(\mathbf{1}, r) \cap S^{n-1}$. Then

$$\widehat{\sigma_{2^t}}(\xi) = \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t} \int_0^{2^t} \int_{S^{n-1}} \eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')d\rho. \tag{3.15}$$

In the following, we want to prove $|\widehat{\sigma_{2^t}}(\xi)| \le C\min\{|L_rA_{2^t}\xi|, |L_rA_{2^t}\xi|^{-\beta/m}\}$, where $0 < \beta < \frac{1}{2\alpha_n}$ and m denotes the distinct numbers of $\{\alpha_j\}$. For any $\xi \ne 0$, denote $\frac{A_\rho\xi}{|A_\rho\xi|} =: \zeta := (\zeta_1', \zeta_*) \in S^{n-1}$, where $\zeta_* \in \mathbb{R}^{n-1}$. We choose a rotation $\mathcal{O}$ in $\mathbb{R}^n$ such that $\mathcal{O}(\zeta) = \mathbf{1}$. Since $\mathcal{O}^{-1} = \mathcal{O}^t$, where $\mathcal{O}^{-1}$ and $\mathcal{O}^t$ denote the inverse and transpose of $\mathcal{O}$, respectively, it is easy to check that ζ is the first row vector of $\mathcal{O}$. Thus, we have $\mathcal{O}^2(\zeta) = (\zeta_1', \gamma_*)$, where $\gamma_* \in \mathbb{R}^{n-1}$. Now, we take a rotation $\mathcal{Q}_{n-1}$ in $\mathbb{R}^{n-1}$ such that $\mathcal{Q}_{n-1}(\zeta_*) = \gamma_*$. Set $\mathcal{R} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{Q}_{n-1}\end{pmatrix}$; then $\mathcal{R}$ is a rotation in $\mathbb{R}^n$, such that for any $y' := (\ell, y_2', \dots, y_n')$ in S^{n-1}, $\langle \mathbf{1}, \mathcal{R}y'\rangle = \ell$. Thus

$$\widehat{\sigma_{2^t}}(\xi) = \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t}\int_0^{2^t}\int_{S^{n-1}} \eta(\mathcal{O}^{-1}(\mathcal{R}y'))e^{-2\pi i|A_\rho\xi|\langle \mathbf{1},\mathcal{R}y'\rangle}\,d\sigma(y')d\rho.$$

Now $\eta(\mathcal{O}^{-1}(\mathcal{R}y'))$ also satisfies (2.3) and (2.5), and is supported in $B(\zeta, r)\cap S^{n-1}$. Thus we have

$$\widehat{\sigma_{2^t}}(\xi) = \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t}\int_0^{2^t}\int_{\mathbb{R}} F_\eta(\ell)e^{-2\pi i|A_\rho\xi|\ell}\,d\ell d\rho,$$

where $F_\eta(\ell)$ is the function defined in Lemma 2.1. By Lemma 2.1, we know that F_η is supported in $(-2r(\zeta)+\delta_1, 2r(\zeta)+\delta_1)$, where $r(\zeta) = \frac{|L_rA_\rho\xi|}{|A_\rho\xi|}$ and $\delta_1 = \frac{\rho^{\alpha_1}\xi_1}{|A_\rho\xi|}$. Thus $N(\ell) = r(\zeta)F_\eta(r(\zeta)\ell)$ is a function with support in the interval $(-2+\frac{\delta_1}{r(\zeta)}, 2+\frac{\delta_1}{r(\zeta)})$, and $\|N\|_\infty < C$ (C is independent of η and ρ) and $\int_{\mathbb{R}} N(\ell)\,d\ell = 0$. After changing a variable we have

$$\widehat{\sigma_{2^t}}(\xi) = \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t}\int_0^{2^t}\int_{\mathbb{R}} N(\ell)e^{-2\pi i\ell|L_rA_\rho\xi|}\,d\ell d\rho.$$

So by the cancellation property of N, we obtain that

$$\begin{aligned}(3.16)\quad |\widehat{\sigma_{2^t}}(\xi)| &= \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t}\Big|\int_0^{2^t}\int_{\mathbb{R}} N(\ell)[e^{-2\pi i|L_rA_\rho\xi|\ell} - e^{-2\pi i\rho^{\alpha_1}\xi_1}]\,d\ell d\rho\Big|\\ &\le C2^{-t}\int_0^{2^t}\int_{|\ell-\frac{\zeta_1}{r(\zeta)}|}\\ &\le 2|N(\ell)||L_rA_\rho\xi|\Big|\ell - \frac{\zeta_1}{r(\zeta)}\Big|\,d\ell\,d\rho \le C\int_0^1 |L_rA_{2^t\rho}\xi|\,d\rho\\ &\le C|L_rA_{2^t}\xi|.\end{aligned}$$

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On the other hand, using Hölder's inequality and (3.15), we have

$$(3.17)\quad |\widehat{\sigma_{2^t}}(\xi)|^2 = \Big| \frac{\|J\|_{L^\infty(S^{n-1})}}{2^t} \int_0^{2^t}\int_{S^{n-1}} \eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')\, d\rho\Big|^2$$

$$\le C\frac{1}{2^t}\int_0^{2^t}\Big|\int_{S^{n-1}}\eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')\Big|^2 d\rho$$

$$= C\sum_{j=-\infty}^{0} 2^{j-1}\frac{1}{2^{t+j-1}}\int_{2^{t+j-1}}^{2^{t+j}}\Big|\int_{S^{n-1}}\eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')\Big|^2 d\rho$$

$$\le C\sum_{j=-\infty}^{0} 2^{j-1}\int_{2^{t+j-1}}^{2^{t+j}}\Big|\int_{S^{n-1}}\eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')\Big|^2 \frac{d\rho}{\rho}$$

$$= C\sum_{j=-\infty}^{0} 2^{j-1}B_{t,j}(\xi),$$

where

$$B_{t,j}(\xi) = \int_{2^{t+j-1}}^{2^{t+j}}\Big|\int_{S^{n-1}}\eta(y')e^{-2\pi i\xi\cdot A_\rho y'}\, d\sigma(y')\Big|^2 \frac{d\rho}{\rho}.$$

Then we get

$$B_{t,j}(\xi) = \int_{2^{t+j-1}}^{2^{t+j}}\iint_{S^{n-1}\times S^{n-1}}\eta(y')\overline{\eta(x')}e^{-2\pi i A_\rho(y'-x')\cdot\xi}\, d\sigma(y')d\sigma(x')\frac{d\rho}{\rho}$$

$$\le C\iint_{S^{n-1}\times S^{n-1}}|\eta(y')||\eta(x')|\Big|\int_{2^{t+j-1}}^{2^{t+j}} e^{-2\pi i A_\rho(y'-x')\cdot\xi}\frac{d\rho}{\rho}\Big|\, d\sigma(y')d\sigma(x').$$

By Lemma 2.4, we know

$$\Big|\int_{2^{t+j-1}}^{2^{t+j}} e^{-2\pi i A_\rho(y'-x')\cdot\xi}\frac{d\rho}{\rho}\Big| = \Big|\int_1^2 e^{-2\pi i A_{2^{t+j-1}\rho}(y'-x')\cdot\xi}\frac{d\rho}{\rho}\Big|$$

$$\le C\big(|(y'-x')\cdot A_{2^{t+j-1}}\xi|\big)^{-2\beta/m},$$

where $0<\beta<\frac{1}{2\alpha_n}$ and m denotes the distinct numbers of $\{\alpha_j\}$. Then by the above inequality we get

$$(3.18)\quad B_{t,j}(\xi) \le C\iint_{S^{n-1}\times S^{n-1}}|\eta(y')||\eta(x')| \times\big(|(y'-x')\cdot A_{2^{t+j-1}}\xi|\big)^{-2\beta/m}\, d\sigma(y')d\sigma(x') = C\mathrm{I}_1(\xi),$$

where

$$\mathrm{I}_1(\xi) = \iint_{S^{n-1}\times S^{n-1}}|\eta(y')||\eta(x')|\big(|(y'-x')\cdot A_{2^{t+j-1}}\xi|\big)^{-2\beta/m}\, d\sigma(y')d\sigma(x').$$

As was done above, for any $\xi \neq 0$, we choose a rotation $\mathcal{O}$ in $\mathbb{R}^n$ such that

$$\mathcal{O}(A_{2^{t+j-1}}\xi) = |A_{2^{t+j-1}}\xi|\mathbf{1} = |A_{2^{t+j-1}}\xi|(1, 0, \dots, 0).$$

Thus, we may take another rotation $\mathcal{R}$ in $\mathbb{R}^n$ such that for any

$$y' = (y_1', y_2', \dots, y_n') \in S^{n-1},$$

$\langle \mathbf{1}, \mathcal{R}y' \rangle = y_1' = \langle \mathbf{1}, y' \rangle$. Now, let $y' = (s, y_2', y_3', \dots, y_n')$, $x' = (\delta, x_2', x_3', \dots, x_n')$. Then it is easy to see that

$$\mathrm{I}_1(\xi) = \iint_{S^{n-1}\times S^{n-1}} |\eta(\mathcal{O}^{-1}(\mathcal{R}y'))||\eta(\mathcal{O}^{-1}(\mathcal{R}x'))| \times \big(|(y'-x')\cdot |A_{2^{t+j-1}}\xi|\mathbf{1}|\big)^{-2\beta/m} d\sigma(y')d\sigma(x'),$$

where $\mathcal{O}^{-1}$ is the inverse of $\mathcal{O}$. Now $\eta(\mathcal{O}^{-1}(\mathcal{R}y'))$ satisfies (2.3) and (2.5), and is supported in $B(\vartheta, r) \cap S^{n-1}$ where $\vartheta = \frac{A_{2^{t+j-1}}\xi}{|A_{2^{t+j-1}}\xi|}$. Thus we have

$$\mathrm{I}_1(\xi) = \iint_{\mathbb{R}\times\mathbb{R}} G_\eta(s)G_\eta(\delta)\big(|A_{2^{t+j-1}}\xi||s-\delta|\big)^{-2\beta/m} dsd\delta,$$

where $G_\eta(s)$ is the function defined in Lemma 2.1. By Lemma 2.1, we know $\mathrm{supp}(G_\eta) \subset (-2r(\vartheta)+\vartheta_1, 2r(\vartheta)+\vartheta_1)$, where $r(\vartheta) = \frac{|L_r A_{2^{t+j-1}}\xi|}{|A_{2^{t+j-1}}\xi|}$ and $\vartheta_1 = \frac{2^{(t+j-1)\alpha_1}\xi_1}{|A_{2^{t+j-1}}\xi|}$. Thus $\varphi(s) = r(\vartheta)G_\eta\big(r(\vartheta)(s - \frac{\vartheta_1}{r(\vartheta)})\big)$ is a function supported in the interval $(-2, 2)$, and $\|\varphi\|_\infty < C$ (C is independent of r, t, j and ϑ). Since $2\beta/m < 1$, we get

$$\begin{aligned}\mathrm{I}_1(\xi) &= \int_{-2}^{2}\int_{-2}^{2} \varphi(s)\varphi(\delta)\big(|L_r A_{2^{t+j-1}}\xi||s-\delta|\big)^{-2\beta/m} dsd\delta \\ &\le C|L_r A_{2^{t+j-1}}\xi|^{-2\beta/m}\int_{-2}^{2}\int_{-2}^{2}|s-\delta|^{-2\beta/m} dsd\delta \\ &\le C|L_r A_{2^{t+j}}\xi|^{-2\beta/m}.\end{aligned}$$

This together with (3.18) gives

$$B_{t,j}(\xi) \le C|L_r A_{2^{t+j}}\xi|^{-2\beta/m}. \tag{3.19}$$

Since $0 < \beta < \frac{1}{2\alpha_n}$ and $m \ge 1$, then by (3.17) and (3.19), we get

$$\begin{aligned}|\widehat{\sigma_{2^t}}(\xi)|^2 &\le C\sum_{j=-\infty}^{0} 2^{j-1}|L_r A_{2^{t+j}}\xi|^{-2\beta/m} \\ &\le C\sum_{j=-\infty}^{0} 2^{j(1-2\beta\alpha_n/m)}|L_r A_{2^t}\xi|^{-2\beta/m} \\ &\le C\sum_{j=-\infty}^{0} 2^{j(1-2\beta\alpha_n/m)}|L_r A_{2^t}\xi|^{-2\beta/m} \\ &\le C|L_r A_{2^t}\xi|^{-2\beta/m}.\end{aligned} \tag{3.20}$$

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By (3.16) and (3.20), we have

$$|\widehat{\sigma_{2^t}}(\xi)| \leq C \min\{|L_r A_{2^t}\xi|, |L_r A_{2^t}\xi|^{-\beta/m}\}.$$

Now we give the estimates $\|Q_s(f)\|_2$. For $s > 0$, by (3.14) and the properties of φ, using the estimate $|\widehat{\sigma_{2^t}}(\xi)| \leq C|L_r A_{2^t}\xi|$ and the Plancherel theorem, we get

$$\|Q_s(f)\|_2^2 \leq C \int_{-\infty}^{\infty} \int_{2^{-s-1} \leq \rho(L_r A_{2^t}\xi) \leq 2^{-s+1}} |\widehat{f}(\xi)|^2 |L_r A_{2^t}\xi|^2 d\xi dt$$

$$= C \frac{1}{r^{n+1}} \int_{-\infty}^{\infty} \int_{2^{-s-1} \leq 2^t \rho \leq 2^{-s+1}} \int_{S^{n-1}} J(\xi') |\widehat{f}(L_{r^{-1}} A_\rho \xi')|^2 |A_{2^t} A_\rho \xi'|^2 \rho^{\alpha-1} d\sigma(\xi') d\rho dt$$

$$\leq C \frac{1}{r^{n+1}} \int_{-\infty}^{\infty} \int_{S^{n-1}} J(\xi') |\widehat{f}(L_{r^{-1}} A_\rho \xi')|^2 \Big((2^{-s+1})^{2\alpha_1} + \cdots + (2^{-s+1})^{2\alpha_n} \Big)$$

$$\times \Big(\int_{-s-1-\frac{\log \rho}{\log 2}}^{-s+1-\frac{\log \rho}{\log 2}} dt \Big) \rho^{\alpha-1} d\sigma(\xi') d\rho$$

$$\leq C 2^{-2s\alpha_1} \frac{1}{r^{n+1}} \int_{-\infty}^{\infty} \int_{S^{n-1}} J(\xi') |\widehat{f}(L_{r^{-1}} A_\rho \xi')|^2 \rho^{\alpha-1} d\sigma(\xi') d\rho$$

$$\leq C 2^{-2s} \frac{1}{r^{n+1}} \int_{\mathbb{R}^n} |\widehat{f}(L_{r^{-1}}\xi)|^2 \, d\xi$$

$$\leq C 2^{-2s} \|f\|_2^2.$$

So we have $\|Q_s(f)\|_2 \leq C 2^{-s} \|f\|_2$ for $s > 0$. Using the estimate

$$|\widehat{\sigma_{2^t}}(\xi)| \leq C|L_r A_{2^t}\xi|^{-\beta/m}$$

and the same idea, we have $\|Q_s(f)\|_2 \leq C 2^{\beta s/m} \|f\|_2$ for $s < 0$. Thus we get (3.13), and obviously, the constant C is independent of s and f.

Applying the Riesz–Thorin interpolation theorem of sub-linear operators [2] between (3.10) and (3.13), we know that there exist two constants $\gamma, \tau > 0$ such that

$$\|Q_s(f)\|_p \leq C 2^{-\gamma s} \|f\|_p \qquad \text{for } s > 0, 1 < p < \infty,$$
$$\|Q_s(f)\|_p \leq C 2^{\tau s} \|f\|_p \qquad \text{for } s < 0, 1 < p < \infty.$$

Thus, we obtain (3.9) and (3.5) follows.

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