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HASSE PRINCIPLES FOR ÉTALE MOTIVIC COHOMOLOGY

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Dedicated to Shuji Saito on his 60th birthday

Abstract. We discuss the kernel of the localization map from étale motivic cohomology of a variety over a number field to étale motivic cohomology of the base change to its completions. This generalizes the Hasse principle for the Brauer group, and is related to Tate–Shafarevich groups of abelian varieties.

§1. Introduction

Let K be a global field, and X a smooth, proper and geometrically connected variety over K. Many classical invariants of X, like the Brauer group or the group of principal homogeneous spaces $H^1(K, A)$ of an abelian variety A over K, are large, but the subgroups of those elements which become trivial over every completion K_v of K are finite. In case of the Brauer group and $X = \operatorname{Spec} K$, this is the classical Hasse principle, and in case of $H^1(K, A)$ it is the conjectural finiteness of the Tate–Shafarevich group. We propose a generalization of these statements and conjectures in terms of étale motivic cohomology. We consider the groups

$$Ш^{i,n}(X) = \ker\left(\tau_n^i : H^i_{\mathrm{et}}(X, \mathbb{Z}(n)) \to \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n))\right),$$

where $\mathbb{Z}(n)$ is the motivic complex, and X_v is the base extension to the henselization of K at v.

Some of our results are that $Ш^{i,n}(X)$ is a torsion group, that $Ш^{i,n}(K)$ is finite for all i and n, and that $Ш^{i,0}(X)$ and $Ш^{i,1}(X)$ are finite for $i \leqslant 2$. We show that a conjecture of Lichtenbaum on the structure of motivic cohomology implies the finiteness of $Ш^{i,n}(X)$ for $i \neq 2n+2$ in the function field case, and for $i \leqslant 2n+1$ in the number field case.

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The groups $\text{Ш}^{2n+1,n}(X)$ appear to be the most interesting ones. For example,

$$\text{Ш}^{3,1}(X) = \ker \Big(\mathrm{Br}(X) \to \prod_v \mathrm{Br}(X_v)\Big),$$

so that the vanishing of $\text{Ш}^{3,1}(K)$ is equivalent to the Hasse principle for the Brauer group. For a curve X, $\text{Ш}^{3,1}(X)$ is isomorphic to the Brauer group $\mathrm{Br}(\mathcal{X})$ of a regular and proper model of X, hence finiteness of $\text{Ш}^{3,1}(X)$ is equivalent to the finiteness of the classical Tate–Shafarevich group $\text{Ш}(\mathrm{Pic}^0_X)$ and to the Tate conjecture for divisors on $\mathcal{X}$. For X of arbitrary dimension, finiteness of $\text{Ш}^{3,1}(X)$ implies finiteness of $\text{Ш}(\mathrm{Pic}^0_X)$, and for almost all l, finiteness of the l-primary part of $\text{Ш}(\mathrm{Pic}^0_X)\{l\}$ implies finiteness of $\text{Ш}^{3,1}(X)\{l\}$.

Our work is related to Schneider's [27] examination of the kernels of the maps

$$H^i(K, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to \prod H^i(K_v, \mathbb{Q}_l/\mathbb{Z}_l(n))$$

in the number field case. These kernels are easy to understand if $i \neq 1$, and Schneider conjectured that they are finite and related to the value of $\zeta_K(s)$ at $s = n$ for $i = 1$. This conjecture is still open if $n > 1$. Jannsen [15] extended Schneider's work by considering the groups $H^i(K, H^j_{\mathrm{et}}(\bar{X}, \mathbb{Q}/\mathbb{Z}(n)))$. Our expectation is that the kernel of the localization map has better properties if one considers integral coefficients and the abutment instead of the E_2-terms in the spectral sequence

$$E_2^{i,j} = H^i(K, H^j_{\mathrm{et}}(\bar{X}, \mathbb{Z}(n))) \Rightarrow H^{i+j}_{\mathrm{et}}(X, \mathbb{Z}(n)). \tag{1}$$

Notation: Let K be a global field of characteristic $p \geqslant 0$, and C the regular proper curve over the prime field with function field K and the ring of integers in the function field and number field case, respectively. For a finite place of K, we let $\mathcal{O}_v$ be the henselization of C at v with field of quotients K_v and residue field k_v. For an infinite place we let K_v be the completion of K at v. We fix a separable closure $\bar{K}$ of K with Galois group G_K. Then, for a finite place, $\bar{K}$ is a separable closure of K_v with Galois group $G_v \subseteq G_K$ the decomposition group of v. We let X be a smooth, proper, and geometrically connected scheme over K, $X_v = X \times_K K_v$, and $\bar{X} = X \times_K \bar{K}$.

For an abelian group A we denote by A_{tor} its torsion group, by $A\{l\}$ its subgroup of l-power torsion elements, by $A^{\wedge l} = \lim_r A/l^r$ its l-adic completion, and by $T_l A = \lim_r {}_{l^r}A$ its l-adic Tate-module.

§2. Motivic Tate–Shafarevich groups

We denote the étale hypercohomology of Bloch's cycle complex $\mathbb{Z}(n): V \mapsto z^n(V, * - 2n)$ by $H^i_{\mathrm{et}}(X, \mathbb{Z}(n))$. We frequently use that $H^i_{\mathrm{et}}(X, \mathbb{Q}(n))$ is isomorphic to the higher Chow group $CH^n(X, 2n-i)_{\mathbb{Q}}$ and that this vanishes if $i > 2n$.

Etale motivic cohomology is contravariant for flat maps and covariant for finite maps. Indeed, given a finite map $f : X' \to X$, covariant functoriality of the cycle complex induces a map $f_*\mathbb{Z}(n)_{X'} \to \mathbb{Z}(n)_X$. Finiteness of f implies $f_* \cong Rf_*$, hence we obtain the push-forward by applying hypercohomology on X.

In the function field case, we can spread out X to a smooth scheme $\mathcal{X}$ over an open set of C and see that $\mathbb{Z}/p^r(n)$ is isomorphic to the logarithmic de Rham–Witt sheaf $\nu^n_r[-n]$ [9]. The same holds for X_v because it is the limit of étale schemes over X (at this point it is more convenient to work with the henselization instead of the completion). In particular, $\mathbb{Z}/p^r(n) = 0$ for $n > \dim X + 1$. For $l \neq p$, $\mathbb{Z}/l^r(n)$ is the usual twisted root of unity sheaf $\mu_{l^r}^{\otimes n}$ [10]. This suggests our definition $\mathbb{Z}(n) = \operatorname{colim}_{p \nmid m} \mu_m^{\otimes n}[-1]$ for $n < 0$.

Definition 2.1. We define

$$Ш^{i,n}(X) = \ker \tau^i_n : H^i_{\mathrm{et}}(X, \mathbb{Z}(n)) \to \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n))$$

and

$$S^{i,n}(X) = \ker \sigma^i_n : H^i_{\mathrm{et}}(X, \mathbb{Q}/\mathbb{Z}(n)) \to \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Q}/\mathbb{Z}(n)),$$

where v runs through all places of K.

By our convention, $Ш^{i,n}(X) = S^{i-1,n}(X)$ for $n < 0$ and for $i > 2n+1$. We do not discuss the motivic cohomology version, that is, the kernel of

$$H^i_{\mathcal{M}}(X, \mathbb{Z}(n)) \to \prod_v H^i_{\mathcal{M}}(X_v, \mathbb{Z}(n)),$$

because the étale version has more interesting arithmetic properties and is more accessible to calculations due to the finite coefficient calculations in [25] and [11].

Conjecture 2.2. *The groups $Ш^{i,n}(X)$ are finite for all i and n.*

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We expect a "Weil-etale" version of motivic cohomology, and hence of $Ш^{i,n}(X)$, with better properties for $i > 2n+1$, and finiteness might only hold for the Weil-etale version. The groups $S^{i,n}(X)$ are not finite as the following example shows.

EXAMPLE 2.3. Let E be an elliptic curve over $\mathbb{Q}$, and consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(E)\otimes \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\subset} & H^2_{\mathrm{et}}(E,\mathbb{Q}_l/\mathbb{Z}_l(1)) \\ \downarrow & & \downarrow \\ \prod_v \mathrm{Pic}^0(E_v)\otimes \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\subset} & \prod_v H^2_{\mathrm{et}}(E_v,\mathbb{Q}_l/\mathbb{Z}_l(1)) \end{array}$$

The kernel of the left vertical map injects into the kernel of the right vertical map. The group $\mathrm{Pic}^0(E_v)\otimes \mathbb{Q}_l/\mathbb{Z}_l$ is isomorphic to $\mathbb{Q}_l/\mathbb{Z}_l$ for v the l-adic place and vanishes otherwise, so that the kernel of the left hand map contains a copy of $\mathbb{Q}_l/\mathbb{Z}_l$ if E has rank larger than one. Hence $S^{2,1}(E)$ is not finite in general.

REMARK 2.4. (Completion versus henselization) Let $\widehat{K}_v$ be the completion of K_v. We expect that the canonical map $H^i_{\mathrm{et}}(X_v,\mathbb{Z}(n)) \to H^i(X_{\widehat{K}_v},\mathbb{Z}(n))$ is injective with a $\mathbb{Z}_{(p)}$-module as cokernel. In particular, the injection $Ш^{i,n}(X) \to \ker\big(H^i_{\mathrm{et}}(X,\mathbb{Z}(n)) \to \prod_v H^i(X_{\widehat{K}_v},\mathbb{Z}(n))\big)$ would be an isomorphism.

We next show that $Ш^{i,n}(X)$ is a torsion group.

LEMMA 2.5. *Let $\mathcal{F}$ be a covariant functor from the category of extension fields of a fixed field k to the category of $\mathbb{Q}$-vector spaces. Assume that $\mathcal{F}$ commutes with filtered colimits, that $\mathcal{F}$ has trace maps $\mathcal{F}(L') \to \mathcal{F}(L)$ for finite field extensions L'/L such that the composition $\mathcal{F}(L) \to \mathcal{F}(L') \to \mathcal{F}(L)$ is multiplication by the degree $[L':L]$, and that $\mathcal{F}(L) \to \mathcal{F}(L(t))$ is injective for any extension L of k. Then $\mathcal{F}(L) \to \mathcal{F}(L')$ is injective for any field extension $k \subseteq L \subseteq L'$.*

Proof. Since $\mathcal{F}$ commutes with filtered colimits, we can assume that either L'/L is finite or purely transcendental of transcendence degree 1. In the first case, the map $\mathcal{F}(L) \to \mathcal{F}(L')$ is injective because its kernel is torsion, and in the second case by hypothesis. □

PROPOSITION 2.6. *Let X be separated and of finite type over K. Then $H^i_{\mathrm{et}}(X \times_K L, \mathbb{Q}(n)) \to H^i_{\mathrm{et}}(X \times_K L', \mathbb{Q}(n))$ is injective for any field extension $K \subseteq L \subseteq L'$.*

Proof. We show that the assignment $L \mapsto H^i_{\mathrm{et}}(X \times_K L, \mathbb{Q}(n)) \cong CH^n(X \times_K L, 2n-i)_{\mathbb{Q}}$ satisfies the hypothesis of the lemma. Functoriality is due to contravariance of higher Chow groups for flat maps, and the existence of the trace map follows from covariant functoriality for proper maps. Injectivity for transcendental extensions $L(t)/L$ follows by writing $L(t)$ as the filtered colimit of finitely generated localizations R of $L[t]$. Such an R admits a map to a finite extension L' of L, so that the composition, hence first map, in

$$H^i_{\mathrm{et}}(X \times_K L, \mathbb{Q}(n)) \to H^i_{\mathrm{et}}(X \times_K R, \mathbb{Q}(n)) \to H^i_{\mathrm{et}}(X \times_K L', \mathbb{Q}(n))$$

is injective. Here the second map is given by contravariant functoriality for maps between smooth schemes over L. ∎

COROLLARY 2.7. *The groups* $Ш^{i,n}(X)$ *agree with*

$$\ker l^i_n : H^i_{\mathrm{et}}(X, \mathbb{Z}(n))_{\mathrm{tor}} \to \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n))_{\mathrm{tor}}.$$

In particular, they are torsion groups.

Proof. By the Proposition, $H^i_{\mathrm{et}}(X, \mathbb{Q}(n)) \to H^i_{\mathrm{et}}(X_v, \mathbb{Q}(n))$ is injective for every v, and the Corollary follows by considering the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^i_{\mathrm{et}}(X, \mathbb{Z}(n))_{\mathrm{tor}} & \longrightarrow & H^i_{\mathrm{et}}(X, \mathbb{Z}(n)) & \longrightarrow & H^i_{\mathrm{et}}(X, \mathbb{Q}(n)) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n))_{\mathrm{tor}} & \longrightarrow & \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n)) & \longrightarrow & \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Q}(n))
\end{array}$$

∎

§3. Some Examples

The case $X = \operatorname{Spec} K$

Let $X = \operatorname{Spec} K$. Then the maps τ^1_0 and τ^2_1 are maps between zero groups, and

$$\begin{array}{lccc}
\tau^0_0 : & \mathbb{Z} & \to & \prod_v \mathbb{Z} \\
\tau^2_0 : & \mathrm{Gal}(K)^* & \to & \prod_v \mathrm{Gal}(K_v)^* \\
\tau^1_1 : & K^\times & \to & \prod_v K_v^\times \\
\tau^3_1 : & \mathrm{Br}(K) & \to & \prod_v \mathrm{Br}(K_v)
\end{array}$$

are injective. For τ^2_0 this follows from Chebotarev's density theorem, and for τ^3_1 it is the Hasse principle for Brauer groups. More generally, we have

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PROPOSITION 3.1. *The groups $\text{Ш}^{i,n}(K)$ are finite. They vanish for $i \neq 2$ or $n = 1$, and in the function field case $\text{Ш}^{2,n}(K) \cong H^2_{\text{et}}(C, \mathbb{Z}(n))$ for $n \neq 1$.*

Proof. Recall that from the calculation of higher K-theory of global fields due to Borel, Harder and Soulé we know that $H^i(K, \mathbb{Q}(n))$ vanishes unless $(i, n) = (0, 0), (1, 1)$, or K is a number field and $i = 1$. With p-coefficients we have $\mathbb{Z}/p^r(n) = 0$ hence $H^i(K, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$ for $n \geqslant 2$. If $i \geqslant 4$, then

$$\begin{aligned} H^i(K, \mathbb{Z}(n)) &\cong H^{i-1}(K, \mathbb{Q}/\mathbb{Z}(n)) \\ &\cong \prod_{v \text{ real}} H^{i-1}(K_v, \mathbb{Q}/\mathbb{Z}(n)) \cong \prod_{v \text{ real}} H^i(K_v, \mathbb{Z}(n)) \end{aligned}$$

by Tate–Poitou (the p-part in characteristic p for $n \leqslant 1$ follows because the cohomological p-dimension of K is one). If $i = 3$ and $n \neq 1$, then $H^i(K, \mathbb{Z}(n)) \cong H^{i-1}(K, \mathbb{Q}/\mathbb{Z}(n)) = 0$ by the argument of [27, Section 4 Satz 1]. The cases $i \leqslant 2$ and $n = 0, 1$ are covered in the example above, and it remains to consider the cases $i \leqslant 2$ and $n \geqslant 2$. If $i \leqslant 1$, we obtain the result from the injectivity of

$$\begin{aligned} H^i(K, \mathbb{Z}(n))\{l\} &\cong H^{i-1}(K, \mathbb{Q}_l/\mathbb{Z}_l(n)) \\ &\subseteq H^{i-1}(K_v, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^i(K_v, \mathbb{Z}(n))\{l\} \end{aligned}$$

for any v not dividing l (both sides are zero if $l = p = \text{char } K$).

Using the Rost–Voevodsky theorem [28] and the localization sequence for higher Chow groups [1] we obtain a sequence

$$0 \to H^2_{\text{et}}(C, \mathbb{Z}(n)) \to H^2(K, \mathbb{Z}(n)) \to \oplus_v H^1(k_v, \mathbb{Z}(n-1)) \to 0.$$

Comparing to the local situation we see that

$$\text{Ш}^{2,n}(K) = \ker H^2_{\text{et}}(C, \mathbb{Z}(n)) \to \prod_v H^2_{\text{et}}(\mathcal{O}_v, \mathbb{Z}(n)).$$

The groups $H^2_{\text{et}}(C, \mathbb{Z}(n))$ are finite, which shows the remaining statements in view of the following lemma. □

LEMMA 3.2. *If $\mathcal{O}_v$ has positive characteristic p, then $H^2_{\text{et}}(\mathcal{O}_v, \mathbb{Z}(n))$ is uniquely divisible for $n \geqslant 2$.*

Proof. It suffices to show that $H^1_{\text{et}}(\mathcal{O}_v, \mathbb{Q}_l/\mathbb{Z}_l(n)) = H^2_{\text{et}}(\mathcal{O}_v, \mathbb{Q}_l/\mathbb{Z}_l(n)) = 0$ for all l. If $l \neq p$, then $H^i_{\text{et}}(\mathcal{O}_v, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cong H^i_{\text{et}}(k_v, \mathbb{Q}_l/\mathbb{Z}_l(n))$ by the proper base-change theorem, and this vanishes for $i > 0$. On the other hand $\mathbb{Z}/p(n) = 0$ on $\mathcal{O}_v$ for $n > 1$ implies that $H^i_{\text{et}}(\mathcal{O}_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$ for all i. □

In contrast, if $\mathcal{O}_v$ has mixed characteristic $(0,p)$, then

$$\mathrm{Tor} H^2_{\mathrm{et}}(\mathcal{O}_v, \mathbb{Z}(n)) \cong H^1_{\mathrm{et}}(\mathcal{O}_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) \cong H^1(K_v, \mathbb{Q}_p/\mathbb{Z}_p(n))$$

because $H^i(k_v, \mathbb{Q}_p/\mathbb{Z}_p(n)) = 0$ for all i. But $H^1(K_v, \mathbb{Q}_p/\mathbb{Z}_p(n))$ is the sum of the dual of $\mathbb{Q}_p/\mathbb{Z}_p(1-n)^{G_v}$ and of $(\mathbb{Q}_p/\mathbb{Z}_p)^{[K_v:\mathbb{Q}_p]}$ by [27, Section 3 Satz 4].

The groups $S^{i,n}(K)$ have been studied by Schneider [27] in the number field case. They vanish for $i \neq 1$; this is clear for $i = 0$ and $i > 2$, and follows from [27, Satz 2] for $i = 2$. We also have $S^{1,0}(K) = 0$ by the Chebotarev density theorem. Schneider conjectures that the groups $S^{1,n}(K)$ are finite and that their order is related to the value of $\zeta_K(s)$ at $s = n$. For example, finiteness of $S^{1,1}(K)$ is equivalent to Leopoldt's conjecture for K [27, Section 5 Lemma 3, Section 7 Lemma 1]. Considering the coefficient sequence

$$0 \to H^1(K, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} \to H^1(K, \mathbb{Q}/\mathbb{Z}(n)) \to H^2(K, \mathbb{Z}(n)) \to 0,$$

we see that in addition to $Ш^{2,n}(K)$ the group $S^{1,n}(K)$ involves the term $H^1(K, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z}$.

The same argument shows that in the function field case $S^{i,n}(K)$ vanishes for $i \neq 1$ and for $(i,n) = (1,0)$. The coefficient sequence shows that $H^1(K, \mathbb{Q}/\mathbb{Z}(n)) \cong H^2(K, \mathbb{Z}(n))$, hence $S^{1,n}(K) \cong H^2_{\mathrm{et}}(C, \mathbb{Z}(n))$ which is related to the value of $\zeta_K(s)$ at $s = n$, see [5].

Other examples

PROPOSITION 3.3. *For $i \leqslant 2$ the groups $Ш^{i,0}(X)$ and $Ш^{i,1}(X)$ are finite.*

Proof. We have $H^0_{\mathrm{et}}(X, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1_{\mathrm{et}}(X, \mathbb{Z}) = 0$, so that $Ш^{0,0}(X) = Ш^{1,0}(X) = 0$. By [16, Theorem 1] the pull back along the structure map

$$H^1(K, \mathbb{Q}/\mathbb{Z}) \to H^2_{\mathrm{et}}(X, \mathbb{Z}) \cong H^1_{\mathrm{et}}(X, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(\pi_1^{ab}(X), \mathbb{Q}/\mathbb{Z}),$$

has finite cokernel, hence $Ш^{2,0}(X)$ is finite by Chebotarev's density theorem.

Clearly $Ш^{i,1}(X) = 0$ for $i \leqslant 0$, and $Ш^{1,1}(X) = \ker\left(K^\times \to \prod_v K_v^\times\right)$ and $Ш^{2,1}(X) = \ker\left(\mathrm{Pic}(X) \to \prod_v \mathrm{Pic}(X_v)\right)$ vanish as well. □

The case $n = d$ has been considered in [8], where it is shown that in the function field case $Ш^{2d+2,d}(X)$ is a p-group, and $Ш^{2d+1,d}(X)$ is related to the Tate–Shafarevich group $Ш(\mathrm{Alb}_X)$ of the Albanese variety of X and other invariants which are expected to be finite.

PROPOSITION 3.4. *Up to p-groups we have*

$$H^{2d+3}_{\mathrm{et}}(X,\mathbb{Z}(d+1)) \cong \mathrm{Br}(K), \qquad H^{2d+3}_{\mathrm{et}}(X_v,\mathbb{Z}(d+1)) \cong \mathrm{Br}(K_v),$$

and $H^i_{\mathrm{et}}(X,\mathbb{Z}(d+1)) = H^i_{\mathrm{et}}(X_v,\mathbb{Z}(d+1)) = 0$ *for* $i > 2d+3$. *In particular,* $Ш^{i,d+1}(X)$ *is a p-group for* $i \geqslant 2(d+1)+1$.

Proof. From the coefficient sequence and $H^i_{\mathrm{et}}(X,\mathbb{Q}(d+1)) = 0$ for $i \geqslant 2d+2$ we obtain $H^i_{\mathrm{et}}(X,\mathbb{Z}(d+1)) \cong H^{i-1}_{\mathrm{et}}(X,\mathbb{Q}/\mathbb{Z}(d+1))$ for $i \geqslant 2d+3$. By [22, VI Theorem 11.1] we have $H^{2d}_{\mathrm{et}}(\bar{X},\mathbb{Q}/\mathbb{Z}(d+1)) \cong \mathbb{Q}/\mathbb{Z}(1)$ up to p-groups. The Hochschild–Serre spectral sequence now shows that

$$H^{2d+2}_{\mathrm{et}}(X,\mathbb{Q}/\mathbb{Z}(d+1)) \cong H^2(K, H^{2d}_{\mathrm{et}}(\bar{X},\mathbb{Q}/\mathbb{Z}(d+1))) \cong \mathrm{Br}(K).$$

The final vanishing follows because $H^i_{\mathrm{et}}(\bar{X},\mathbb{Q}/\mathbb{Z}(d+1)) = 0$ for $i > 2d$. ∎

§4. Connections to motivic cohomology of a model

Let $\mathcal{X} \to C$ be a flat and proper map, with regular, connected $\mathcal{X}$ of dimension $d+1$ such that X is the generic fiber. If we disregard the p-torsion in characteristic p, we can assume that such a model exists in order to prove finiteness of $Ш^{i,n}(X)$. Indeed, by de Jong's theorem there is a finite extension K' of K of some degree e and a regular model of $X' = X \times_K K'$ which is projective and flat over the normalization C' of C in K'. It is easy to check that the composition $H^i_{\mathrm{et}}(X,\mathbb{Z}(n)) \to H^i_{\mathrm{et}}(X',\mathbb{Z}(n)) \to H^i_{\mathrm{et}}(X,\mathbb{Z}(n))$ is multiplication by e, hence $\ker Ш^{i,n}(X) \to Ш^{i,n}(X')$ is killed by e. Now applying Gabber's refinement of de Jong's theorem, we can find for every prime $l \neq p$ dividing e another X' such that $\ker Ш^{i,n}(X) \to Ш^{i,n}(X')$ is injective on l-torsion.

For finite places v of C, we let $\mathcal{X}_v = \mathcal{X} \times_C \mathcal{O}_v$ be the base change to the henselization $\mathcal{O}_v$ of C at v, with generic fiber X_v and closed fiber Y_v. For infinite places v we set $\mathcal{X}_v = X_v = X \times_K K_v$. Then for any open subset $U \subseteq C$ and set I of infinite places we have a map of long exact sequences

(2)

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_{\mathrm{et}}(\mathcal{X}_U,\mathbb{Z}(n)) & \longrightarrow & H^i_{\mathrm{et}}(X,\mathbb{Z}(n)) & \xrightarrow{\partial} & \bigoplus_{v\in U} H^{i+1}_{Y_v}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow \\
& \downarrow & & \downarrow{\scriptstyle \tau^i_n(U,I)} & & \downarrow & \\
\longrightarrow & \prod_{v\in U\cup I} H^i_{\mathrm{et}}(\mathcal{X}_v,\mathbb{Z}(n)) & \longrightarrow & \prod_{v\in U\cup I} H^i_{\mathrm{et}}(X_v,\mathbb{Z}(n)) & \xrightarrow{(\partial_v)} & \prod_{v\in U} H^{i+1}_{Y_v}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow
\end{array}$$

and similarly with finite coefficients.

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PROPOSITION 4.1. *The groups* $Ш^{i,n}(X)\{l\}$ *are of cofinite type if* $i > 2n$ *and* $l \neq p$. *In particular,* ${}_mШ^{i,n}(X)$ *is finite for any integer* m *not divisible by* p *and* $i > 2n$.

Proof. It follows from diagram (2) that for any U and I, the image of $H^i_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Z}(n))$ in $H^i_{\mathrm{et}}(X, \mathbb{Z}(n))$ contains $\ker \tau^i_n(U, I)$, which contains $Ш^{i,n}(X)$. But $H^{i-1}_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Q}/\mathbb{Z}(n))$ surjects onto $H^i_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Z}(n))$, because $H^i_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Z}(n))$ is torsion for $i > 2n$. Hence it suffices to show that the groups $H^{i-1}_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Q}_l/\mathbb{Z}_l(n))$ are of cofinite type, which is implied by the finiteness of $H^{i-1}_{\mathrm{et}}(\mathcal{X}_U, \mathbb{Z}/l(n))$. For a fixed l choose U to be C without the places above l and the places where $\mathcal{X}$ has bad reduction. Then $\mathbb{Z}/l(n) \cong \mu_l^{\otimes n}$ is locally constant [4], hence its cohomology groups are finite by [21, II Proposition 7.1]. ∎

LEMMA 4.2. *The map* τ^i_n *has image in the restricted direct product with respect to the subgroups* $\mathrm{im}(H^i_{\mathrm{et}}(\mathcal{X}_v, \mathbb{Z}(n)))$.

Proof. If we set $U = C$ in (2), we see that the image of any element under τ^i_n is contained in the direct sum on the lower right because ∂ has image in the direct sum. ∎

CONJECTURE 4.3. (Lichtenbaum [18]) *Let* $\mathcal{X}$ *be regular, and proper over* $\mathbb{Z}$. *Then the groups* $H^i_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(n))$ *are finitely generated for* $i \leqslant 2n$, *finite for* $i = 2n+1$, *and of cofinite type for* $i \geqslant 2n+2$. *If* $\mathcal{X}$ *is regular and proper over a finite field, they are finite for* $i \neq 2n, 2n+2$, *finitely generated for* $i = 2n$, *and of cofinite type for* $i = 2n+2$.

The conjecture includes the finiteness of the Brauer group of $\mathcal{X}$ conjectured by Artin. If $\mathrm{char}\, K = p$, finiteness of $H^{2n+1}_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(n))$ implies Tate's conjecture for $\mathcal{X}$ in weight n [6, Proposition 4.4].

The sequence (2) for $U = C$ together with Corollary 2.7 imply:

PROPOSITION 4.4. *Under Lichtenbaum's conjecture,* $Ш^{i,n}(X)$ *is finite for* $i \leqslant 2n+1$ *and for* $i \neq 2n+2$ *in the number field and function field case, respectively.*

Our next goal is to prove an unconditional result:

PROPOSITION 4.5. *Assume* K *is a function field. If* $n \notin [0, d+1]$ *or if* $i \neq 2n, 2n+1, 2n+2$, *then* $H^i_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(n))$ *is an extension of a uniquely divisible group by a finite group, and if* $i > 2n+2$ *it is a finite group.*

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Proof. By a weight argument and Gabber's theorem [3], we know that $H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}(n))$ is finite if $n \notin [0, d+1]$ or $i \neq 2n, 2n+1$. Hence $H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z} = 0$ for $n \notin [0, d+1]$ or $i \neq 2n, 2n+1$, and $H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(n))_{\text{tor}}$ is finite for $n \notin [0, d+1]$ or $i \neq 2n+1, 2n+2$. Now the first statement follows from the sequence

$$\begin{aligned} 0 &\to H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(n))_{\text{tor}} \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(n)) \\ &\to H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}(n)) \to H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z}, \end{aligned}$$

and the second statement follows because $H^i_{\text{et}}(\mathcal{X}, \mathbb{Q}(n)) = 0$ for $i > 2n$. ∎

Applying this to the diagram (2) for $U = C$ and using the argument in the beginning of the section we obtain:

COROLLARY 4.6. *Let K be a function field and $i > 2n+2$. Then $Ш^{i,n}(X)$ is finite if X has a regular proper model over C, and it is the sum of a finite group and a p-group in general.*

There should be a Weil-etale version of motivic cohomology such that all groups $H^i_W(\mathcal{X}, \mathbb{Z}(n))$ are finitely generated, and the above argument would imply that the Weil-etale version $Ш^{i,n}_W(X)$ is finite.

§5. The Brauer group

In this section we consider the group

$$Ш^{3,1}(X) = \ker\left(\operatorname{Br}(X) \to \prod_v \operatorname{Br}(X_v)\right).$$

If X has a regular proper model $\mathcal{X}$, the localization sequence (2) gives a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Br}(\mathcal{X}) & \longrightarrow & \operatorname{Br}(X) & \xrightarrow{\partial} & \bigoplus_{v \text{ finite}} H^3_{Y_v}(\mathcal{X}, \mathbb{G}_m) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \prod_v \operatorname{Br}(\mathcal{X}_v) & \longrightarrow & \prod_v \operatorname{Br}(X_v) & \xrightarrow{(\partial_v)} & \prod_{v \text{ finite}} H^3_{Y_v}(\mathcal{X}, \mathbb{G}_m) \end{array}$$

and we obtain a short exact sequence

$$0 \to Ш^{3,1}(X) \to \operatorname{Br}(\mathcal{X}) \to \prod_v \operatorname{Br}(\mathcal{X}_v).$$

In particular, finiteness of $\operatorname{Br}(\mathcal{X})$ implies finiteness of $Ш^{3,1}(X)$. We now consider the case that X is a curve.

PROPOSITION 5.1. *Let X be a curve with regular proper model $\mathcal{X}$. Then $Ш^{3,1}(X) \to \mathrm{Br}(\mathcal{X})$ is an isomorphism if K has no real places, and an injection with cokernel a finite 2-group in general.*

Proof. This follows from the vanishing of the groups $\mathrm{Br}(\mathcal{X}_v)$ for finite or complex places. Indeed, by Artin's theorem [13, Theorem 3.1], the Brauer group of a regular relative curve, proper and flat over a henselian discrete valuation ring agrees with the Brauer group of the special fiber, and the Brauer group of the special fiber vanishes by [13, Remark 2.5(b)]. If v is complex, then $\mathcal{X}_v$ is a curve over an algebraically closed field, hence its Brauer group vanishes by loc.cit. Corollary 1.2. If v is real, then $\mathrm{Br}(\mathcal{X}_v)$ is a finite 2-group by the Hochschild–Serre spectral sequence. □

Artin and Grothendieck showed in [13, Section 4] that $\mathrm{Br}(\mathcal{X})$ is finite if and only if the Tate–Shafarevich group of the Albanese $Ш(\mathrm{Alb}_X)$ is finite. We proved in [7] that if K has not real embeddings then (assuming finiteness of the groups) the orders are related by

$$|\mathrm{Br}(\mathcal{X})|\alpha^2\delta^2 = |Ш(\mathrm{Alb}_X)| \prod_{v\in V} \alpha_v\delta_v,$$

where δ and δ_v are the indices of X and X_v, and α and α_v are the orders of the cokernel of the inclusions $\mathrm{Pic}^0(X_K) \to H^0(K, \mathrm{Pic}^0_X)$ and $\mathrm{Pic}^0(X_v) \to H^0(K_v, \mathrm{Pic}^0_X)$, respectively. The argument of [19] then implies that the order of $\mathrm{Br}(\mathcal{X})$, hence of $Ш^{3,1}(X)$ is a square if it is finite.

Returning to X of arbitrary dimension, we relate $Ш^{3,1}(X)$ to the Tate–Shafarevich group $Ш(\mathrm{Pic}^0_X)$ of the Picard variety of X by comparing both terms to

$$Ш(\mathrm{Pic}_X) = \ker\left(H^1(K, \mathrm{Pic}_X) \to \prod_v H^1(K_v, \mathrm{Pic}_X)\right)$$

and using the Hochschild–Serre spectral sequence (1) for $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$. The following Proposition was explained to us by Colliot-Thélène:

PROPOSITION 5.2. *There exists an integer N such that the cokernel of $\mathrm{NS}(X_v) \to \mathrm{NS}(\bar{X})^{G_v}$ is killed by N for all v.*

Proof. For any field K we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{surj} & \mathrm{NS}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow u & & \downarrow s & & \downarrow \\
0 & \longrightarrow & \mathrm{Pic}^0(\bar{X})^{G_K} & \longrightarrow & \mathrm{Pic}(\bar{X})^{G_K} & \xrightarrow{v} & \mathrm{NS}(\bar{X})^{G_K} & \xrightarrow{\partial} & H^1(K, \mathrm{Pic}^0_X)
\end{array}$$

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where the lower row is the long exact sequence for Galois cohomology of group schemes. This gives an exact sequence $\operatorname{coker} u \to \operatorname{coker} s \to \operatorname{coker} v \to 0$ and it suffices to bound the outer terms. Let M be a finite Galois extension of K such that X has an M-rational point. Then the map of exact sequences

$$\begin{array}{ccccc} \operatorname{Pic}(X) & \xrightarrow{u} & \operatorname{Pic}(\bar{X})^{G_K} & \longrightarrow & \operatorname{Br}(K) \\ {\scriptstyle inj}\downarrow & & {\scriptstyle inj}\downarrow & & \downarrow \\ \operatorname{Pic}(X_M) & = & \operatorname{Pic}(\bar{X})^{G_M} & \xrightarrow{0} & \operatorname{Br}(M) \end{array}$$

coming from (1) shows that the cokernel of u injects into the kernel of $\operatorname{Br}(K) \to \operatorname{Br}(M)$ and hence is $[M : K]$-torsion by a trace argument.

We can write $\operatorname{NS}(\bar{X}) \cong R \oplus T$, with T a torsion group of some exponent t, and R free. Since $\operatorname{Pic}(\bar{X}) = \operatorname{colim}_{F/K \text{finite, separable}} \operatorname{Pic}(X_F)$, we can find a finite Galois extension L/K such that R lifts to $\operatorname{Pic}(X_L)$. After increasing L we can assume that G_L acts trivially on R. Then $\operatorname{coker} \operatorname{Pic}(\bar{X})^{G_L} \to \operatorname{NS}(\bar{X})^{G_L}$ is t-torsion and hence the cokernel of v is $t[L : K]$-torsion.

If K is a global field, then the above discussion applies to all K_v, hence the cokernel of $\operatorname{NS}(X_v) \to \operatorname{NS}(\bar{X})^{G_v}$ is N torsion for $N = t[L : K][M : K]$. □

LEMMA 5.3. *Consider the following commutative diagram of abelian groups with exact rows:*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\ A' & \xrightarrow{s} & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

If $\operatorname{coker} s$ *is zero, finite, or of exponent* n*, then so is the cohomology of* $\ker c \to \ker d \to \ker e$.

Proof. A diagram chase shows that the cohomology of $\ker c \to \ker d \to \ker e$ is isomorphic to the cohomology of $\operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} c$, which is a subgroup of $\operatorname{coker} b/ \operatorname{im} \operatorname{coker} a = B'/ \operatorname{im} b + \operatorname{im} s$. But $\operatorname{coker} s = B'/ \operatorname{im} s$ surjects onto the latter group. □

PROPOSITION 5.4. *The canonical map* $Ш(\operatorname{Pic}^0_X) \to Ш(\operatorname{Pic}_X)$ *has finite kernel, and its cokernel is of finite exponent.*

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Proof. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{NS}(\bar{X})^{G_K} & \xrightarrow{\partial} & H^1(K, \mathrm{Pic}^0_X) & \longrightarrow & H^1(K, \mathrm{Pic}_X) & \longrightarrow & H^1(K, \mathrm{NS}_X) \\
\downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \rho} \\
\prod_v \mathrm{NS}(\bar{X})^{G_v} & \xrightarrow[(\partial_v)]{} & \prod_v H^1(K_v, \mathrm{Pic}^0_X) & \longrightarrow & \prod_v H^1(K_v, \mathrm{Pic}_X) & \longrightarrow & \prod_v H^1(K_v, \mathrm{NS}_X)
\end{array}$$

The kernel of $Ш(\mathrm{Pic}^0_X) \to Ш(\mathrm{Pic}_X)$ is contained in the image of ∂, which is the finite cokernel of $\mathrm{NS}(X) \to \mathrm{NS}(\bar{X})^{G_K}$. On the other hand, by Lemma 5.3, the cohomology of $Ш(\mathrm{Pic}^0) \to Ш(\mathrm{Pic}) \to \ker \rho$ is killed by N of Proposition 5.2, and $\ker \rho$ is finite because NS_X is a finitely generated Galois module [21, Theorem I 4.20]. □

Lemma 5.5. *For almost all v, $\mathrm{Br}(K_v) \to \mathrm{Br}(X_v)$ has a section and $\mathrm{Pic}\, X_v \to \mathrm{Pic}_X(\bar{K})^{G_v}$ is an isomorphism.*

Proof. This follows because for any geometrically integral K-variety, X_v has a K_v-rational point for all almost all places v [23, Theorem 7.7.2]. For the statement on Pic we use the low terms of (1)

$$0 \to \mathrm{Pic}(X_v) \to \mathrm{Pic}_X(\bar{K})^{G_v} \to \mathrm{Br}(K_v) \to \mathrm{Br}(X_v). \qquad \square$$

Proposition 5.6. *There is a canonical map $Ш^{3,1}(X) \to Ш(\mathrm{Pic}_X)$ with finite kernel, and cyclic cokernel of order dividing the index of X. If X has a rational point, then the map is an isomorphism.*

Proof. Let $f : \mathcal{X} \to C$ be a normal, flat and proper model. By our hypothesis on X we have $f_*\mathbb{G}_m \cong \mathbb{G}_m$. Compare the spectral sequence (1) with its local analogue:

(3)

$$\begin{array}{ccccccc}
\mathrm{Pic}(\bar{X})^{G_K} & \xrightarrow{d_2} & E^{2,0} = \mathrm{Br}(K) & \xrightarrow{f^*} & \mathrm{Br}(X) & \xrightarrow{\xi} & \mathrm{Br}^0(X) \to 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\
\prod_v \mathrm{Pic}(\bar{X}_v)^{G_v} & \xrightarrow{(d_2^v)} & \bigoplus_v \mathrm{Br}(K_v) & \xrightarrow{(f_v^*)} & \prod'_v \mathrm{Br}(X_v) & \xrightarrow{(\xi_v)} & \prod_v \mathrm{Br}^0(X_v)
\end{array}$$

Here $\mathrm{Br}^0(X)$ and $\mathrm{Br}^0(X_v)$ are the cokernels of f^* and f_v^*, respectively. By Lemma 4.2, the image of β lies in the restricted direct product with respect to the subgroups $\mathrm{Br}(\mathcal{X}_v)$ and the lower row is exact because $\mathrm{Br}(\mathcal{O}_v) = 0$, and d_2^v is zero for almost all v by Lemma 5.5.

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Since α is injective, Lemma 5.3 implies that the kernel of $\Upsilon : Ш^{3,1}(X) \to \ker\gamma$ is finite because coker $\prod_v \mathrm{Pic}(X_v) \to \prod_v \mathrm{Pic}(\bar{X})^{G_v}$ is. On the other hand, the kernel of $u : \mathbb{Q}/\mathbb{Z} \cong \mathrm{coker}\,\alpha \to \mathrm{coker}\,\beta$ surjects onto coker Υ. But a zero-cycle of degree δ on X induces a compatible quasisplitting of (3), hence a map $v : \mathrm{coker}\,\beta \to \mathrm{coker}\,\alpha$ such that vu is multiplication by δ. Hence $\ker u \subseteq \mathbb{Z}/\delta$ surjects onto coker Υ. Finally the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1(K, \mathrm{Pic}_X) & \longrightarrow & \mathrm{Br}^0(X) & \longrightarrow & \mathrm{Br}(\bar{X})^{G_K} \\
 & & \downarrow & & \gamma\downarrow & & \downarrow \\
0 & \longrightarrow & \prod_v H^1(K_v, \mathrm{Pic}_X) & \longrightarrow & \prod_v \mathrm{Br}^0(X_v) & \longrightarrow & \prod \mathrm{Br}(\bar{X})^{G_v}
\end{array}$$

coming from (1) shows that $Ш(\mathrm{Pic}_X) \cong \ker\gamma$.

If X has a rational point, then the sequences in (3) are compatibly split short exact. □

Corollary 5.7. *Finiteness of* $Ш^{3,1}(X)$ *implies finiteness of* $Ш(\mathrm{Pic}^0_X)$. *Conversely, for almost all* l, *finiteness of* $Ш(\mathrm{Pic}^0_X)\{l\}$ *implies finiteness of* $Ш^{3,1}(X)\{l\}$.

Remark 5.8. We expect the existence of perfect pairings (for $n + u = d + 1$)

$$Ш^{2n+1,n}(X) \times Ш^{2u+1,u}(X) \to \mathbb{Q}/\mathbb{Z},$$

compatible with the finite coefficient versions arising from the Tate–Poitou sequences in [25] and [11]:

$$\begin{array}{ccccc}
Ш^{2n+1,n}(X) & \times\, Ш^{2u+1,u}(X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\downarrow & \partial\uparrow & & \| \\
Ш^{2n+1}(X, \mu_m^{\otimes n}) & \times\, Ш^{2u}(X, \mu_m^{\otimes u}) & \longrightarrow & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

If X is a curve with model $\mathcal{X}$, then under the identification $\mathrm{Br}(\mathcal{X}) \cong Ш^{3,1}(X)$ the pairing should agree with the Artin–Tate pairing (which is alternating [2])

$$\mathrm{Br}(\mathcal{X}) \times \mathrm{Br}(\mathcal{X}) \to \mathbb{Q}/\mathbb{Z}.$$

It is an interesting question if the order of $Ш^{2n+1,n}(X)$ is always a square for $2n = d + 1$.

If X has a rational point, and $n=1, u=d$, the pairing should be compatible with the classical pairing

$$\begin{array}{ccc} Ш^{3,1}(X)\times Ш^{2d+1,d}(X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \uparrow \qquad\qquad \downarrow & & \| \\ Ш(\mathrm{Pic}^0_X)\times\ Ш(\mathrm{Alb}_X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \end{array}$$

where the map $Ш^{2d+1,d}(X)\to Ш(\mathrm{Alb}_X)$ is induced by the map

$$\begin{aligned} H^{2d+1}_{\mathrm{et}}(X,\mathbb{Z}(d)) &\to H^1(K,H^{2d}_{\mathrm{et}}(\bar{X},\mathbb{Z}(d))) \\ &\cong H^1(K,CH_0(\bar{X})^0)\to H^1(K,\mathrm{Alb}_X) \end{aligned}$$

arising from the Hochschild–Serre sequence and the albanese map.

If X is an abelian variety with principal polarization, the classical Cassels–Tate pairing is not alternating [24]. However, the proof of Proposition 5.4 shows that this obstruction is contained in the kernel of $Ш(\mathrm{Pic}^0_X)\to Ш^{3,1}(X)$.

§6. Equivalence of conjectures

In this section, K is a function field and $\mathcal{X}\to C$ a flat and regular model. In [8] we proved the following theorem:

THEOREM 6.1. *Tate's conjecture for divisors on $\mathcal{X}$ is equivalent to Tate's conjecture for divisors on X and the finiteness of the Tate–Shafarevich group* $Ш(\mathrm{Alb}_X)$.

This was proven by considering weight d motivic cohomology. We give a shorter proof of a weaker result using weight 1 motivic cohomology. We have already seen that finiteness of $\mathrm{Br}(\mathcal{X})$ implies finiteness of $Ш^{3,1}(X)$ which implies finiteness of $Ш(\mathrm{Pic}^0_X)$.

THEOREM 6.2. *Finiteness of* $Ш(\mathrm{Pic}^0_X)$ *and Tate's conjecture for divisors on X imply the finiteness of* $\mathrm{Br}(\mathcal{X})$.

Proof. Fix $l\neq p$. Since the groups are finitely generated, completing the short exact sequence $0\to \mathrm{Pic}^0(X)\to \mathrm{Pic}(X)\to \mathrm{NS}(X)\to 0$ at l gives the exact upper row in the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Pic}^0(X)^{\wedge l} & \longrightarrow & \mathrm{Pic}(X)^{\wedge l} & \longrightarrow & \mathrm{NS}(X)^{\wedge l} & \to 0 \\ & a\downarrow & & b\downarrow & & c\downarrow & \\ 0 \longrightarrow & H^1(K,T_l\,\mathrm{Pic}^0_X) & \longrightarrow & H^2_{\mathrm{et}}(X,\mathbb{Z}_l(1))^0 & \longrightarrow & H^2_{\mathrm{et}}(\bar{X},\mathbb{Z}_l(1))^{G_K} & \end{array}$$

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The lower row comes from the Hochschild–Serre spectral sequence for continuous l-adic cohomology [14, Remark 3.5 b].

$$H^s(K, H^t_{\text{et}}(\bar{X}, \mathbb{Z}_l(1))) \Rightarrow H^t_{\text{et}}(X, \mathbb{Z}_l(1)),$$

where we set $H^2_{\text{et}}(X, \mathbb{Z}_l(1))^0 = \operatorname{coker} H^2(K, \mathbb{Z}_l(1)) \to H^2_{\text{et}}(X, \mathbb{Z}_l(1))$. For the left term we use that the injection

$$_{l^r}\operatorname{Pic}^0(\bar{X}) \to {}_{l^r}\operatorname{Pic}(\bar{X}) \cong H^1_{\text{et}}(\bar{X}, \mathbb{Z}/l^r(1))$$

induces isomorphisms $H^i(K, T_l \operatorname{Pic}^0) \cong H^i(K, H^1(\bar{X}, \mathbb{Z}_l(1)))$ in the limit as $\lim_r H^i(K, {}_{l^r}\operatorname{NS}_X) = 0$ for all i.

The map a is the composition of the inclusion $\operatorname{Pic}^0(X)^{\wedge l} \to H^0(K, \operatorname{Pic}^0_X)^{\wedge l}$ with finite cokernel with the left map in the short exact coefficient sequence

$$0 \to H^0(K, \operatorname{Pic}^0_X)^{\wedge l} \to H^1(K, T_l \operatorname{Pic}^0_X) \to T_l H^1(K, \operatorname{Pic}^0_X) \to 0.$$

The map b is the canonical map in the coefficient sequence with cokernel $T_l \operatorname{Br}^0(X)$. Finally, c is the cycle map, whose cokernel is finite if and only if Tate's conjecture for divisors holds on X. It is easy to see that the diagram commutes. Combining this with the following Proposition, we obtain that Tate's conjecture for divisors on X together with finiteness of $Ш(\operatorname{Pic}^0_X)\{l\}$ implies that $T_l \operatorname{Br}^0(X)$ vanishes. But the inclusion $\operatorname{Br}(\mathcal{X}) \to \operatorname{Br}^0(X)$ induces an inclusion $T_l \operatorname{Br}(\mathcal{X}) \to T_l \operatorname{Br}^0(X)$, hence $T_l \operatorname{Br}(\mathcal{X})$ vanishes as well. Since $\operatorname{Br}(\mathcal{X})\{l\}$ is of cofinite type, we conclude that $\operatorname{Br}(\mathcal{X})\{l\}$ is finite, which in turn implies that $\operatorname{Br}(\mathcal{X})$ is finite [20]. ∎

PROPOSITION 6.3. *Let A be an abelian variety over a global function field K. Then $Ш(A)$ is finite if and only if $T_l H^1(K, A)$ vanishes for some (all) $l \neq p$.*

Proof. Since $A(K_v)$ has a subgroup of finite index which is uniquely divisible by all integers prime to p, duality of $H^1(K_v, A)$ and $A(K_v)$ [26, Theorem 9.3] shows that $H^1(K_v, A)[1/p]$ is finite for all v. Hence taking Tate-modules in the short exact sequence

$$0 \to Ш(A) \to H^1(K, A) \to \prod_v H^1(K_v, A)$$

shows that $T_l Ш(A) \cong T_l H^1(K, A)$. Finally, $Ш(A)$ is finite if and only if $Ш(A)\{l\}$ is finite for some l, or equivalently if $T_l Ш(A)$ vanishes [21, I Remark 6.7]. ∎

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More generally, let A be an abelian variety over any global field, and consider the exact sequence dual to the main theorem of [12]

$$0 \to Ш(A) \to H^1(K, A) \to \bigoplus_v H^1(K_v, A) \to (T\,\mathrm{Sel}(A))^* \to 0.$$

Let l be different from the characteristic of K and τ be the corank of $Ш(A)\{l\}$. Then $\mathrm{Sel}(A)\{l\}$ is of cofinite type of corank $\tau + \mathrm{rank}\, A(K)$, hence $T_l(\mathrm{Sel}(A)) \cong \mathbb{Z}_l^{\tau+\mathrm{rank}\, A(K)}$ and $T_l(\mathrm{Sel}(A))^* \cong (\mathbb{Q}_l/\mathbb{Z}_l)^{\tau+\mathrm{rank}\, A(K)}$. Moreover, $T_l H^1(K_v, A) \cong \mathbb{Z}_l^{[K_v:\mathbb{Q}_l]\dim A}$ if char $K_v = 0$ and $v|l$, and zero otherwise. Since $Ш(A)^{\wedge l}$ is finite, we obtain up to finite groups an exact sequence of Tate-modules

$$0 \to \mathbb{Z}_l^{\tau} \to T_l H^1(K, A) \to \mathbb{Z}_l^{[K:\mathbb{Q}]\dim A} \to \mathbb{Z}_l^{\tau+\mathrm{rank}\, A(K)}, \tag{4}$$

where we set $[K:\mathbb{Q}] = 0$ if K is a function field.

Proposition 6.4. *The vanishing of* $T_l H^1(K, A)$ *implies that* $Ш(A)\{l\}$ *is finite and* $\mathrm{rank}\, A(K) \geqslant [K:\mathbb{Q}]\dim A$. *The converse holds if* $K = \mathbb{Q}$ *and* A *is an elliptic curve.*

Proof. From the sequence (4) we see that the vanishing of $T_l H^1(K, A)$ implies finiteness of $Ш(A)\{l\}$, hence $\tau = 0$ and $A(K)$ has rank at least $[K:\mathbb{Q}]\dim A$. Conversely, if $Ш(A)\{l\}$ is finite, then the first terms of (4) become

$$0 \to T_l H^1(K, A) \to \mathbb{Z}_l^{[K:\mathbb{Q}]\dim A} \to \mathbb{Z}_l^{\mathrm{rank}\, A(K)}.$$

The last map is the map on Tate-modules induced by

$$\prod_{v|l} H^1(K_v, A) \cong \prod_{v|l} (A(K_v)^{\wedge l})^* \to (A(K)^{\wedge l})^* \cong T_l(\mathrm{Sel}(A))^*,$$

and if $\dim A = [K:\mathbb{Q}] = 1$, then this is a nontrivial map from a $\mathbb{Z}_l$-module of rank one, hence injective. ∎

The Proposition generalizes a result of Kriz [17], who considers the question when $T_l\,\mathrm{Br}^0(E) = \mathrm{coker}\, T_l\,\mathrm{Br}(K) \to T_l\,\mathrm{Br}(E)$ vanishes for elliptic curves over $\mathbb{Q}$. Since E has a rational point and $\mathrm{Br}(\bar{E}) = 0$, the Hochschild–Serre spectral sequence (1) gives a split short exact sequence

$$0 \to \mathrm{Br}(K) \to \mathrm{Br}(E) \to H^1(\mathbb{Q}, E) \to 0,$$

hence this is equivalent to the vanishing of $T_l H^1(\mathbb{Q}, E)$.

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§7. Connections to Galois cohomology

We relate the work of Jannsen [15] to the groups $S^{i,n}(X)$. Let m be prime to the characteristic of K.

PROPOSITION 7.1. *There is a map of spectral sequences*

$$
(5)\qquad \begin{array}{ccc}
H^s(K, H^t_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))) & \Rightarrow & H^{s+t}_{\text{et}}(X, \mathbb{Z}/m(n)) \\
\downarrow & & \downarrow \\
\prod'_v H^s(K_v, H^t_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))) & \Rightarrow & \prod'_v H^{s+t}_{\text{et}}(X_v, \mathbb{Z}/m(n))
\end{array}
$$

where $\prod'_v$ on the left denotes the product, restricted product with respect to unramified cohomology, and sum for $s = 0, 1, 2$, respectively, and on the right the restricted product with respect to the cohomology $H^{s+t}_{\text{et}}(\mathcal{X}_v, \mathbb{Z}/m(n))$ of a model (which are subgroups for almost all v). The same statement holds with $\mathbb{Q}_l/\mathbb{Z}_l(n)$-coefficients.

Note that the right vertical map is defined by the argument of Lemma 4.2.

Proof. Assume that v does not divide m and that X has good reduction at v, that is, there is a smooth $\mathcal{X}_v$ over $\mathcal{O}_v$ with generic fiber X_v and closed fiber Y_v. If $\mathfrak{g}$ is the Galois group of the reside field, then we have short exact sequences

$$H^1(\mathfrak{g}, H^{i-1}_{\text{et}}(\bar{Y}_v, \mathbb{Z}/m(n))) \to H^i_{\text{et}}(Y_v, \mathbb{Z}/m(n)) \to H^0(\mathfrak{g}, H^i_{\text{et}}(\bar{Y}_v, \mathbb{Z}/m(n))).$$

By the smooth and proper base-change theorem,

$$H^t_{\text{et}}(\bar{Y}_v, \mathbb{Z}/m(n)) \cong H^t_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))^{I_v} \cong H^t_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))$$

for all t, hence using the proper base-change theorem in the middle, we can identify this with the short exact sequence in the upper row of the diagram

$$
(6)\qquad \begin{array}{ccccc}
H^1(\mathfrak{g}, H^{i-1}_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))) & \longrightarrow & H^i_{\text{et}}(\mathcal{X}_v, \mathbb{Z}/m(n)) & \longrightarrow & H^0(\mathfrak{g}, H^i_{\text{et}}(\bar{X}, \mathbb{Z}/m(n))) \\
 & & \downarrow & & \downarrow \\
 & & H^i_{\text{et}}(X_v, \mathbb{Z}/m(n)) & \longrightarrow & H^0(K_v, H^i_{\text{et}}(\bar{X}, \mathbb{Z}/m(n)))
\end{array}
$$

The outer groups are unramified cohomology groups of K_v by definition. The diagram implies that the lower (edge) map is surjective, hence the differential

$$d_2^v : H^0(K_v, H^i_{\mathrm{et}}(\bar{X}, \mathbb{Z}/m(n))) \to H^2(K_v, H^{i-1}_{\mathrm{et}}(\bar{X}, \mathbb{Z}/m(n)))$$

vanishes for all v as above. It follows that the differential (d_2^v) maps the product to the sum.

Finally, the upper row in the diagram (6) shows the spectral sequence converges to the abutment. ∎

Jannsen [15, Theorem 3(c,d)] shows that if $i \neq 2n-2$ and K is a number field, then

$$H^2(K, H^i_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) \to \bigoplus_v H^2(K_v, H^i_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n)))$$

has finite kernel and cokernel.

PROPOSITION 7.2. *If $i \neq 2n$, then there is a homomorphism from $S^{i,n}(X)\{l\}$ to*

$$\ker H^1(K, H^{i-1}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) \to \prod_v H^1(K_v, H^{i-1}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n)))$$

with finite kernel and cokernel.

Proof. Since the map

$$E_2^{0,i}(X) = H^i_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))^{G_K} \to E_2^{0,i}(X_v) = H^i_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))^{G_v}$$

is injective for any $v \nmid l$, $S^{i,n}(X)\{l\}$ agrees with the kernel of

$$F^1 H^i_{\mathrm{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \to F^1 {\prod_v}' H^i_{\mathrm{et}}(X_v, \mathbb{Q}_l/\mathbb{Z}_l(n)),$$

with respect to the filtration induced by the spectral sequence (5). Now the diagram with exact sequences

$$\begin{array}{ccccc}
H^2(K, H^{i-2}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) & \longrightarrow & F^1 & \longrightarrow & H^1(K, H^{i-1}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_v H^2(K_v, H^{i-2}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) & \longrightarrow & F^1 {\prod_v}' & \longrightarrow & {\prod_v}' H^1(K_v, H^{i-1}_{\mathrm{et}}(\bar{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) \to 0
\end{array}$$

shows that the kernels of the two right vertical map differ by a finite group in view of Jannsen result. ∎

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The kernel of the map

$$H^{s+t}_{\text{et}}(X, \mathbb{Z}/m(n)) \to \prod_v{}' H^{s+t}_{\text{et}}(X_v, \mathbb{Z}/m(n))$$

was examined in [25] and [11], but it is not clear what happens in the colimit. We hope that the groups with integral coefficients are better behaved.

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