ON HOMOGENEOUS IDEALS

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Introduction. The modern algebraic treatment of geometry in projective spaces focuses attention on the properties of homogeneous ideals in polynomial and power-series rings. This inevitably raises questions concerning how far ordinary ideal theory needs to be modified if only homogeneous ideals are to be regarded as significant. In practice, one can usually answer any particular question of this type without undue difficulty when it arises but, it seems to the author, the topic has enough intrinsic interest to merit a connected discussion by itself.

In the present paper the concept of a *graduated ring* makes it possible to treat, in a more abstract way than usual, this notion of homogeneous elements and homogeneous ideals. This brings out the way in which the concept of homogeneity combines with the concepts of general ideal theory without disturbing the familiar pattern of results. Thus, to give one example, it is shown (Theorem 14) that if a homogeneous ideal can be represented as an intersection of primary ideals, then it can also be represented as an intersection of *homogeneous* primary ideals. It is hoped that this account includes proofs of most of the results of this kind which are needed in applications.

1. *Graduated rings.* Let $\mathfrak{R}$ be a commutative ring with a unit element. In addition, suppose that with each non-negative integer n there is associated a subset $\mathfrak{M}^{(n)}$ of $\mathfrak{R}$ and that these subsets $\mathfrak{M}^{(n)}$ have the following properties.

(*a*) $\mathfrak{M}^{(n)}$ *is a subgroup of the additive group of* $\mathfrak{R}$.

(*b*) *If* $X \in \mathfrak{M}^{(r)}$ *and* $Y \in \mathfrak{M}^{(s)}$ *then* $XY \in \mathfrak{M}^{(r+s)}$.

(*c*) *Each element* $r \in \mathfrak{R}$ *can be written as a finite sum* $r = X^{(0)} + X^{(1)} + \ldots + X^{(l)}$, *where* $X^{(i)} \in \mathfrak{M}^{(i)}$.

(*d*) *If* $X^{(0)} + X^{(1)} + \ldots + X^{(l)} = Y^{(0)} + Y^{(1)} + \ldots + Y^{(m)}$, *where* $X^{(i)} \in \mathfrak{M}^{(i)}$, $Y^{(j)} \in \mathfrak{M}^{(j)}$ *and* $l \leqslant m$, *then* $X^{(i)} = Y^{(i)}$ *for* $1 \leqslant i \leqslant l$, *and* $Y^{(j)} = 0$ *for* $j > l$.

We shall describe this situation by saying that $\mathfrak{R}$ is a *graduated* ring. The elements of $\mathfrak{M}^{(n)}$ will be referred to as *homogeneous elements of degree* n and we shall normally use $X^{(n)}$ or $Y^{(n)}$ to denote such elements. The assumptions (*c*) and (*d*) can now be summed up briefly by saying that every element of $\mathfrak{R}$ is uniquely expressible as a finite sum of homogeneous elements of different degrees. The expression *homogeneous element* will be used to mean an element belonging to at least one of the sets $\mathfrak{M}^{(n)}$. It follows from (*d*) that a non-zero homogeneous element has a definite degree. We shall normally use X and Y to denote homogeneous elements. If $r = \sum_0^n X^{(i)}$, where $X^{(i)} \in \mathfrak{M}^{(i)}$, then the terms in this sum will be referred to as *the homogeneous constituents of* r.

An example of a graduated ring can be obtained as follows. Let R be a commutative ring with a unit element and let $\mathfrak{R}$ be the polynomial ring $R[X_1, X_2, \ldots, X_n]$. Then $\mathfrak{R}$ is a graduated ring if we take for $\mathfrak{M}^{(n)}$ the set of all forms of degree n.

In what follows, we suppose that $\mathfrak{R}$ is a fixed (but arbitrary) graduated ring.

Theorem 1. *The unit element of* $\mathfrak{R}$ *is homogeneous and of degree zero.*

Proof. Let $1 = \sum_0^m X^{(\nu)}$, and let $r = \sum_0^n Y^{(\nu)}$ be an arbitrary element of $\mathfrak{R}$. We have

$$Y^{(\mu)} = 1\,Y^{(\mu)} = X^{(0)}Y^{(\mu)} + X^{(1)}Y^{(\mu)} + \ldots + X^{(m)}Y^{(\mu)},$$

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whence, comparing terms of degree μ, $Y^{(\mu)} = X^{(0)} Y^{(\mu)}$. It follows that $X^{(0)} r = r$; consequently, since r was arbitrary, $X^{(0)} = 1$.

2. *Homogeneous ideals.* As before, $\mathfrak{R}$ is a given graduated ring. An ideal $\mathfrak{a}$ of $\mathfrak{R}$ will be said to be *homogeneous* if it has a base (possibly infinite) which is composed entirely of homogeneous elements.

Theorem 2. *Let $\mathfrak{a}$ be a homogeneous ideal and let $r \in \mathfrak{R}$. Then r belongs to $\mathfrak{a}$ if and only if all the homogeneous constituents of r are in $\mathfrak{a}$.*

Proof. Suppose, first, that $r \in \mathfrak{a}$. We then have

$$r = r_1 X_1^{(m_1)} + r_2 X_2^{(m_2)} + \ldots + r_s X_s^{(m_s)},$$

where $r_i \in \mathfrak{R}$, $X_i^{(m_i)} \in \mathfrak{a}$ and $X_i^{(m_i)}$ is homogeneous of degree m_i. Let $r = \sum_{\nu=0}^{n} Y^{(\nu)}$, $r_i = \sum_{\mu=0}^{n_i} Y_i^{(\mu)}$: then

$$\sum_0^n Y^{(\nu)} = \sum_{\mu=0}^{n_1} Y_1^{(\mu)} X_1^{(m_1)} + \ldots + \sum_{\mu=0}^{n_s} Y_s^{(\mu)} X_s^{(m_s)}.$$

Comparing terms of degree ν we see that

$$Y^{(\nu)} = Y_1^{(\nu-m_1)} X_1^{(m_1)} + Y_2^{(\nu-m_2)} X_2^{(m_2)} + \ldots + Y_s^{(\nu-m_s)} X_s^{(m_s)},$$

where, if $\nu < m_i$, $Y_i^{(\nu-m_i)} X_i^{(m_i)}$ must be omitted from the sum on the right-hand side. This shows that $Y^{(\nu)} \in \mathfrak{a}$. Thus half the theorem is proved and the other half is, in this case, trivial.

Theorem 3. *Let $\mathfrak{a}$ be an ideal with the property that " if r belongs to $\mathfrak{a}$ then all the homogeneous constituents of r are in $\mathfrak{a}$ "; then $\mathfrak{a}$ is homogeneous.*

Proof. Let $\mathfrak{a}_0$ be the ideal generated by all the homogeneous elements that belong to $\mathfrak{a}$; then $\mathfrak{a}_0$ is homogeneous and $\mathfrak{a}_0 \subseteq \mathfrak{a}$. If now $r = \sum_0^n X^{(\nu)}$ belongs to $\mathfrak{a}$, then, by hypothesis, $X^{(\nu)} \in \mathfrak{a}$ and so $X^{(\nu)} \in \mathfrak{a}_0$. It follows that $r \in \mathfrak{a}_0$. This proves that $\mathfrak{a} \subseteq \mathfrak{a}_0$ and establishes the theorem.

Theorem 4. *If $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \frown \mathfrak{b}$, $\mathfrak{a} : \mathfrak{b}$ and* Rad $\mathfrak{a}$.

Proof. That $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}$ are homogeneous follows immediately from the definition. Assume now that $r = \sum_0^n X^{(\nu)}$ belongs to $\mathfrak{a} \frown \mathfrak{b}$. If we show that, for each ν, $X^{(\nu)} \in \mathfrak{a} \frown \mathfrak{b}$, then it will follow, by Theorem 3, that $\mathfrak{a} \frown \mathfrak{b}$ is homogeneous. But $\sum_0^n X^{(\nu)} \in \mathfrak{a}$ and so (Theorem 2) $X^{(\nu)} \in \mathfrak{a}$, while a precisely similar argument yields $X^{(\nu)} \in \mathfrak{b}$.

To prove that $\mathfrak{a} : \mathfrak{b}$ is homogeneous, assume that $\sum_0^n X^{(\nu)}$ belongs to $\mathfrak{a} : \mathfrak{b}$ and let $(\ldots, Y_j, \ldots)$ be a (possibly infinite) homogeneous base for $\mathfrak{b}$. Then $\sum_{\nu=0}^n X^{(\nu)} Y_j \in \mathfrak{a}$, whence (Theorem 2) $X^{(\nu)} Y_j \in \mathfrak{a}$ for all ν and j. We therefore have $X^{(\nu)} \mathfrak{b} \subseteq \mathfrak{a}$ and so $X^{(\nu)} \in \mathfrak{a} : \mathfrak{b}$. The homogeneity of $\mathfrak{a} : \mathfrak{b}$ follows by Theorem 3.

Finally, suppose that $r = X^{(s)} + X^{(s+1)} + \ldots + X^{(m)}$ belongs to Rad $\mathfrak{a}$. Then $r^t = [X^{(s)}]^t + \ldots$ belongs to $\mathfrak{a}$ for a suitable value of t. Accordingly, by Theorem 2, $[X^{(s)}]^t \in \mathfrak{a}$ and so $X^{(s)} \in$ Rad $\mathfrak{a}$; consequently $r - X^{(s)} = X^{(s+1)} + X^{(s+2)} + \ldots + X^{(m)}$ is in Rad $\mathfrak{a}$. We can now repeat the argument and show that $X^{(s+1)} \in$ Rad $\mathfrak{a}$ and, proceeding in this way, we finally see that all the homogeneous constituents of r are in the radical of $\mathfrak{a}$. This proves the theorem.

Proposition 1. *Let $\mathfrak{a}$ be a homogeneous ideal with the property that " if X, Y are homogeneous and $XY \in \mathfrak{a}$, then either $X \in \mathfrak{a}$ or $Y \in \mathfrak{a}$ ". In these circumstances, $\mathfrak{a}$ is a prime ideal.*

Proof. Assume that $r\rho \in \mathfrak{a}$, where $r = \sum_0^m X^{(\mu)}$ and $\rho = \sum_0^n Y^{(\nu)}$. We wish to show that either

$r \in \mathfrak{a}$ or $\rho \in \mathfrak{a}$. Assume that neither of these is true; then there will be a first $X^{(\mu)}$, say $X^{(r)}$, and a first $Y^{(\nu)}$, say $Y^{(s)}$, which are not in $\mathfrak{a}$. Since

$$X^{(r+s)}Y^{(0)} + \ldots + X^{(r+1)}Y^{(s-1)} + X^{(r)}Y^{(s)} + X^{(r-1}Y^{(s+1)} + \ldots + X^{(0)}Y^{(r+s)}$$

is the homogeneous constituent of $r\rho$ of degree $r+s$ and since $\mathfrak{a}$ is homogeneous, this sum is in $\mathfrak{a}$. The construction now shows that $X^{(r)}Y^{(s)} \in \mathfrak{a}$; consequently, by our hypothesis, either $X^{(r)} \in \mathfrak{a}$ or $Y^{(s)} \in \mathfrak{a}$. This, however, is a contradiction.

Proposition 2. *Let* $\mathfrak{a}$ *be a homogeneous ideal with the property that "if* X, Y *are homogeneous,* $XY \in \mathfrak{a}$ and $X \notin \mathfrak{a}$, *then some power of* Y *is in* $\mathfrak{a}$*". In these circumstances,* $\mathfrak{a}$ *is a primary ideal.*

Proof. Let $r = \sum_0^m X^{(\mu)}$, $\rho = Y^{(s)} + Y^{(s+1)} + \ldots + Y^{(n)}$ and assume that $r\rho \in \mathfrak{a}$, $r \notin \mathfrak{a}$. We wish to show that $\rho \in \text{Rad } \mathfrak{a}$. Let $X^{(r)}$ be the first of the $X^{(\mu)}$ which is not in $\mathfrak{a}$; then

$$[X^{(r)} + X^{(r+1)} + \ldots + X^{(m)}]\,[Y^{(s)} + Y^{(s+1)} + \ldots + Y^{(n)}] = X^{(r)}Y^{(s)} + \ldots$$

belongs to $\mathfrak{a}$ and therefore, since $\mathfrak{a}$ is homogeneous, $X^{(r)}Y^{(s)} \in \mathfrak{a}$. Again, since $X^{(r)}$, $Y^{(s)}$ are homogeneous and $X^{(r)} \notin \mathfrak{a}$, it follows by hypothesis that $Y^{(s)} \in \text{Rad } \mathfrak{a}$. There exists, therefore, an integer t, which may be zero, such that

$$r[Y^{(s)}]^t \notin \mathfrak{a}, \quad r[Y^{(s)}]^{t+1} \in \mathfrak{a}.$$

Put $r_1 = r[Y^{(s)}]^t$; then $r_1\rho \in \mathfrak{a}$ because $r\rho \in \mathfrak{a}$. But

$$r_1\rho = r[Y^{(s)}]^t\,[Y^{(s)} + Y^{(s+1)} + \ldots + Y^{(n)}] \equiv r_1[Y^{(s+1)} + \ldots + Y^{(n)}] \pmod{\mathfrak{a}},$$

which shows that

$$r_1[Y^{(s+1)} + \ldots + Y^{(n)}] \in \mathfrak{a} \quad \text{and} \quad r_1 \notin \mathfrak{a}.$$

We now have a situation similar to that with which we started and so we can repeat the argument and show that $Y^{(s+1)} \in \text{Rad } \mathfrak{a}$. In this way, it is seen that all the homogeneous constituents of ρ belong to Rad $\mathfrak{a}$ and so ρ itself belongs to Rad $\mathfrak{a}$.

3. *The primary decomposition.* Still supposing that $\mathfrak{R}$ is a given graduated ring we shall now establish the important result that, when $\mathfrak{R}$ is Noetherian, *every homogeneous ideal can be expressed as a finite intersection of homogeneous primary ideals.* It will be convenient, however, not to assume the Noetherian condition in its ordinary form but to postulate it only for homogeneous ideals. More precisely, it is a simple matter to verify that the following assertions are equivalent:

(*a*) Every strictly increasing sequence of homogeneous ideals is finite.

(*b*) Every non-empty set of homogeneous ideals contains one that is maximal for the set.

(*c*) Every homogeneous ideal can be generated by a finite number of homogeneous elements.

We shall therefore say that a graduated ring $\mathfrak{R}$, which has these properties, is *H-Noetherian.*

Theorem 5. *Let* $\mathfrak{R}$ *be H-Noetherian and let* $\mathfrak{a}$ *be a homogeneous ideal; then* $\mathfrak{a}$ *can be expressed as a finite intersection of homogeneous primary ideals.*

The proof proceeds on familiar lines but, since there are some new considerations, we shall go over the steps briefly. Let us call a homogeneous ideal *H-reducible* if it is the intersection of two homogeneous ideals both of which strictly contain it. A homogeneous ideal which is not H-reducible will then be said to be *H-irreducible.* It is a straightforward matter to show that, when $\mathfrak{R}$ is H-Noetherian, every homogeneous ideal is a finite intersection of homogeneous H-irreducible ideals. Now suppose that $\mathfrak{a}$ is homogeneous and H-irreducible; then, to complete the proof, it will be enough to show that $\mathfrak{a}$ is primary. *Assume that* X, Y

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are homogeneous, that $XY \in \mathfrak{a}$ *and* $X \notin \mathfrak{a}$. If we can deduce from this that some power of Y is in $\mathfrak{a}$, the required result will follow by Proposition 2. By Theorem 4,

$$\mathfrak{a} \subset \mathfrak{a}:(Y) \subseteq \mathfrak{a}:(Y^2) \subseteq \mathfrak{a}:(Y^3) \subseteq \dots$$

is a chain of *homogeneous* ideals. Consequently we can choose m so that $\mathfrak{a} : (Y^n) = \mathfrak{a} : (Y^m)$ for $n \geqslant m$. If this is done then

$$\mathfrak{a} = [\mathfrak{a} : (Y^m)] \frown [\mathfrak{a} + (Y^m)],$$

and both $\mathfrak{a} : (Y^m)$ and $\mathfrak{a} + (Y^m)$ are homogeneous. Accordingly, since $\mathfrak{a}$ is H-irreducible and $\mathfrak{a} \subset [\mathfrak{a}:(Y^m)]$, we have $\mathfrak{a} = \mathfrak{a} + (Y^m)$ and therefore $Y^m \in \mathfrak{a}$.

An immediate and important consequence of Theorem 5 is

Theorem 6. *Let* $\mathfrak{R}$ *be H-Noetherian and let* $\mathfrak{a}$ *be a homogeneous ideal. Then all the prime ideals which belong to* $\mathfrak{a}$ *are homogeneous and all the isolated components of* $\mathfrak{a}$ *are homogeneous.*

Theorem 7. *Let* $\mathfrak{R}$ *be H-Noetherian, let* $\mathfrak{q}$ *be a homogeneous primary ideal and let* $\mathfrak{p}$ *be the prime ideal (necessarily homogeneous) to which* $\mathfrak{q}$ *belongs. Then there exist homogeneous* $\mathfrak{p}$*-primary ideals* $\mathfrak{q}_0, \mathfrak{q}_1, \dots, \mathfrak{q}_s$ *which satisfy*

$$\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_s = \mathfrak{p}$$

and are such that there is no $\mathfrak{p}$*-primary ideal, homogeneous or non-homogeneous, between* $\mathfrak{q}_i$ *and* $\mathfrak{q}_{i+1}$.

This theorem shows that, when $\mathfrak{R}$ is H-Noetherian, every homogeneous primary ideal has a definite length and, moreover, that composition series exist in which every term is homogeneous. The proof, after our previous results, presents no difficulty, but we shall give details for the reader's convenience.

Proof. Let us suppose that $\mathfrak{q}$ is strictly contained in $\mathfrak{p}$; then $\mathfrak{q} \subset \mathfrak{q}:\mathfrak{p} \subseteq \mathfrak{p}$ and $\mathfrak{q}:\mathfrak{p}$ is homogeneous. We can therefore choose a homogeneous element $X \in \mathfrak{q} : \mathfrak{p}$ so that $X \notin \mathfrak{q}$. The ideal $\mathfrak{q} + (X)$ is homogeneous and has $\mathfrak{p}$ as a minimal prime ideal; consequently the $\mathfrak{p}$-primary component $\mathfrak{q}_1$ of $\mathfrak{q} + (X)$ is homogeneous (Theorem 6). By construction, $\mathfrak{q} \subset \mathfrak{q}_1$. Now suppose that $\mathfrak{q}'$ is a $\mathfrak{p}$-primary (but not necessarily homogeneous) ideal which satisfies $\mathfrak{q} \subset \mathfrak{q}' \subseteq \mathfrak{q}_1$. Choose $r \in \mathfrak{q}'$ so that $r \notin \mathfrak{q}$. Since $r \in \mathfrak{q}_1$, we can find $c \notin \mathfrak{p}$ so that $cr \in \mathfrak{q} + (X)$, say $cr = q + \rho X$, where $q \in \mathfrak{q}$. Now $\rho \notin \mathfrak{p}$, for otherwise we should have $\rho X \in \mathfrak{p}X \subseteq \mathfrak{q}$ and therefore $cr \in \mathfrak{q}$ which would imply $r \in \mathfrak{q}$, contrary to hypothesis. Accordingly, $\rho X = cr - q \in \mathfrak{q}'$ and $\rho \notin \mathfrak{p}$; consequently $X \in \mathfrak{q}'$. This shows that $\mathfrak{q} + (X) \subseteq \mathfrak{q}'$ which, when combined with $\mathfrak{q}' \subseteq \mathfrak{q}_1$, yields $\mathfrak{q}' = \mathfrak{q}_1$. It has now been proved that there is no $\mathfrak{p}$-primary ideal between $\mathfrak{q}$ and $\mathfrak{q}_1$. We next construct $\mathfrak{q}_2$ from $\mathfrak{q}_1$ in the same way that $\mathfrak{q}_1$ was constructed from $\mathfrak{q}$. Proceeding in this way the whole chain is constructed because, since $\mathfrak{R}$ is H-Noetherian, the process must eventually stop.

Another result which is sometimes needed in applications is

Theorem 8. *Let* $\mathfrak{R}$ *be H-Noetherian, let* $\mathfrak{p}$ *be a homogeneous prime ideal, and let* $\mathfrak{q}, \mathfrak{q}_1$ *be homogeneous* $\mathfrak{p}$*-primary ideals. If now* $\mathfrak{q} \subset \mathfrak{q}_1$ *and there are no homogeneous* $\mathfrak{p}$*-primary ideals between* $\mathfrak{q}$ *and* $\mathfrak{q}_1$, *then there are no non-homogeneous* $\mathfrak{p}$*-primary ideals between* $\mathfrak{q}$ *and* $\mathfrak{q}_1$.

Proof. $(\mathfrak{q} : \mathfrak{p}) \frown \mathfrak{q}_1$ is homogeneous and $\mathfrak{p}$-primary and it satisfies $\mathfrak{q} \subseteq (\mathfrak{q}:\mathfrak{p}) \frown \mathfrak{q}_1 \subseteq \mathfrak{q}_1$; consequently either $(\mathfrak{q} : \mathfrak{p}) \frown \mathfrak{q}_1 = \mathfrak{q}_1$, that is, $\mathfrak{q}_1 \subseteq (\mathfrak{q} : \mathfrak{p})$ or $(\mathfrak{q} : \mathfrak{p}) \frown \mathfrak{q}_1 = \mathfrak{q}$. *We assert that* $\mathfrak{q}_1 \subseteq (\mathfrak{q} : \mathfrak{p})$. For suppose the contrary; then $(\mathfrak{q} : \mathfrak{p}) \frown \mathfrak{q}_1 = \mathfrak{q}$, whence, dividing by $\mathfrak{p}$ and then intersecting with $\mathfrak{q}_1$, we find

$$(\mathfrak{q} : \mathfrak{p}^2) \frown \mathfrak{q}_1 = (\mathfrak{q} : \mathfrak{p}) \frown \mathfrak{q}_1 = \mathfrak{q}.$$

Repeating this device we obtain $(\mathfrak{q} : \mathfrak{p}^n) \frown \mathfrak{q}_1 = \mathfrak{q}$ for all n, which, on taking n very large, becomes $\mathfrak{q}_1 = \mathfrak{q}$. This is a contradiction. We have now established that $\mathfrak{q}_1 \subseteq (\mathfrak{q} : \mathfrak{p})$. Choose a homogeneous element $X \in \mathfrak{q}_1$ so that $X \notin \mathfrak{q}$; then $\mathfrak{q}_1$ will be the $\mathfrak{p}$-primary component of $\mathfrak{q} + (X)$

and X will belong to $\mathfrak{q} : \mathfrak{p}$. The proof that there are no $\mathfrak{p}$-primary ideals between $\mathfrak{q}$ and $\mathfrak{q}_1$ now proceeds exactly as in the proof of Theorem 7.

4. *Graduated series-rings.* Let $\mathfrak{R}$ be a graduated ring. We shall denote by $\mathfrak{R}^*$ the set of all formal series $\sum_0^\infty X^{(\nu)}$, where $X^{(\nu)} \in \mathfrak{M}^{(\nu)}$ for each ν. Let

$$r^* = X^{(0)} + X^{(1)} + X^{(2)} + \dots,$$
$$\rho^* = Y^{(0)} + Y^{(1)} + Y^{(2)} + \dots$$

be two elements of $\mathfrak{R}^*$; then we can define $r^* + \rho^*$ and $r^*\rho^*$ by

$$r^* + \rho^* = [X^{(0)} + Y^{(0)}] + [X^{(1)} + Y^{(1)}] + [X^{(2)} + Y^{(2)}] + \dots$$
$$r^*\rho^* = [X^{(0)}Y^{(0)}] + [X^{(0)}Y^{(1)} + X^{(1)}Y^{(0)}] + [X^{(0)}Y^{(2)} + X^{(1)}Y^{(1)} + X^{(2)}Y^{(0)}] + \dots,$$

and this will turn $\mathfrak{R}^*$ into a commutative ring. Clearly $\mathfrak{R}$ is a subring of $\mathfrak{R}^*$ and it is easily seen, by using Theorem 1, that the unit element of $\mathfrak{R}$ is also the unit element of $\mathfrak{R}^*$.

The ring $\mathfrak{R}^*$ will be called the *graduated series-ring derived from* $\mathfrak{R}$. As an example, let us note that if $\mathfrak{R}$ is the polynomial ring $R[X_1, X_2, \dots, X_n]$ considered in § 1, then $\mathfrak{R}^*$ is the power-series ring $R[[X_1, X_2, \dots, X_n]]$.

The theory of homogeneous ideals in a general graduated series-ring has some rather disagreeable complications. *We shall, however, assume throughout* § 4 *that the original graduated ring* $\mathfrak{R}$ *is H-Noetherian.* This, as we shall see, causes the complications to disappear and our results will still be sufficiently general for the more frequent applications. As before we make the

Definition. An ideal $\mathfrak{a}^*$ *of* $\mathfrak{R}^*$ *will be said to be " homogeneous " if it has a base composed entirely of homogeneous elements.*

According to this definition, an ideal $\mathfrak{a}^*$ in $\mathfrak{R}^*$ is homogeneous if and only if it is the extension of a homogeneous ideal $\mathfrak{a}$ of $\mathfrak{R}$. But $\mathfrak{R}$ is H-Noetherian and so $\mathfrak{a}$, and therefore $\mathfrak{a}^*$, can be generated by a *finite* number of homogeneous elements. It follows that every ascending chain of homogeneous ideals in $\mathfrak{R}^*$ terminates and, also, that the maximal condition holds for homogeneous ideals. In other words, our assumption that $\mathfrak{R}$ is H-Noetherian implies that $\mathfrak{R}^*$ is H-Noetherian as well.

The ring $\mathfrak{R}^*$ is, of course, not a graduated ring in the sense of § 1, because an element of $\mathfrak{R}^*$ need not be a *finite* sum of homogeneous elements. There is, however, no great danger of confusion because all our results, namely Theorems 2–8 and Propositions 1–2, do in fact hold in $\mathfrak{R}^*$. The necessary demonstrations of this will be given very briefly at suitable places in the subsequent discussion, but our main concern will be to examine the relations which hold between the homogeneous ideals of $\mathfrak{R}$ and those of $\mathfrak{R}^*$. For convenience, we shall denote by Theorem 2*, to give one example, the original Theorem 2 with such minor modifications as are necessary to make it applicable to $\mathfrak{R}^*$.

It has been stated that the assumption that $\mathfrak{R}$ is H-Noetherian is used to avoid certain new complications. The nature of these complications is revealed in the proof of

Theorem 2*. *Let* $\mathfrak{a}^*$ *be a homogeneous ideal and let* $r^* \in \mathfrak{R}^*$. *Then* r^* *belongs to* $\mathfrak{a}^*$ *if and only if all the homogeneous constituents of* r^* *are in* $\mathfrak{a}^*$.

Proof. If $r^* \in \mathfrak{a}^*$ then the argument used in Theorem 2 can be employed, with trivial alterations, to show that the homogeneous constituents of r^* are in $\mathfrak{a}^*$. It is the converse that presents a new problem. Assume then that $r^* = \sum_0^\infty Y^{(\nu)}$ and that $Y^{(\nu)} \in \mathfrak{a}^*$ for each ν. Since $\mathfrak{R}^*$ is H-Noetherian, $\mathfrak{a}^*$ can be generated by a finite number of homogeneous elements, say $\mathfrak{a}^* = (X_1^{(m_1)}, X_2^{(m_2)}, \dots, X_s^{(m_s)})$ where $X_i^{(m_i)}$ is homogeneous and of degree m_i. Put

$$m = \max(m_1, m_2, \ldots, m_s)$$

and suppose that $\nu \geqslant m$; then, since $Y^{(\nu)} \epsilon\, \mathfrak{a}^*$, we have

$$Y^{(\nu)} = Y_1^{(\nu-m_1)} X_1^{(m_1)} + Y_2^{(\nu-m_2)} X_2^{(m_2)} + \ldots + Y_s^{(\nu-m_s)} X_s^{(m_s)},$$

where $Y_i^{(\nu-m_i)}$ is homogeneous and of degree $\nu - m_i$. Thus if $r_i^* = \sum_{\nu=m}^{\infty} Y_i^{(\nu-m_i)}$, then

$$Y^{(m)} + Y^{(m+1)} + Y^{(m+2)} + \ldots = r_1^* X_1^{(m_1)} + r_2^* X_2^{(m_2)} + \ldots + r_s^* X_s^{(m_s)},$$

which belongs to $\mathfrak{a}^*$. But, by hypothesis, $Y^{(0)}, Y^{(1)}, \ldots, Y^{(m-1)}$ are in $\mathfrak{a}^*$; consequently $r^* \epsilon\, \mathfrak{a}^*$ as required.

It will now be found that, with the aid of Theorem 2*, the arguments used to prove Theorems 3, 4 and Propositions 1, 2 work equally well for Theorems 3*, 4* and Propositions 1*, 2*.

Theorem 9. *There is a* 1–1 *correspondence between the homogeneous ideals* $\mathfrak{a}$ *of* $\mathfrak{R}$ *and the homogeneous ideals* $\mathfrak{a}^*$ *of* $\mathfrak{R}^*$ *such that, if* $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *correspond, then* $\mathfrak{a}^* = \mathfrak{R}^*\mathfrak{a}$ *and* $\mathfrak{a} = \mathfrak{R} \frown \mathfrak{a}^*$.

Proof. After what has already been said, it will be enough to show that if $\mathfrak{a}$ is homogeneous and $\mathfrak{a}^* = \mathfrak{R}^*\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{a}^* \frown \mathfrak{R}$. Let $\mathfrak{a} = (X_1^{(m_1)}, X_2^{(m_2)}, \ldots, X_s^{(m_s)})$, where $X_i^{(m_i)} \epsilon\, \mathfrak{M}^{(m_i)}$, and assume that $r^* = \sum_0^{\infty} Y^{(\nu)}$ belongs to $\mathfrak{a}^*$. Since $\mathfrak{a}^* = \mathfrak{R}^* X_1^{(m_1)} + \mathfrak{R}^* X_2^{(m_2)} + \ldots + \mathfrak{R}^* X_s^{(m_s)}$, and since $Y^{(\nu)} \epsilon\, \mathfrak{a}^*$, we have $Y^{(\nu)} = Y_1^{(\nu-m_1)} X_1^{(m_1)} + Y_2^{(\nu-m_2)} X_2^{(m_2)} + \ldots + Y_s^{(\nu-m_s)} X_s^{(m_s)}$, where $Y_i^{(\nu-m_i)}$ is homogeneous, and this shows that $Y^{(\nu)} \epsilon\, \mathfrak{a}$. In particular, if r^* is in $\mathfrak{R}$ then, since r^* is a *finite* sum of the $Y^{(\nu)}$, we must have $r^* \epsilon\, \mathfrak{a}$. In other words, we have shown that $\mathfrak{a}^* \frown \mathfrak{R} \subseteq \mathfrak{a}$, and from this the theorem follows.

Note that the argument yields rather more, for it establishes the

Corollary. Suppose that $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *are corresponding homogeneous ideals; then* $\sum_0^{\infty} X^{(\nu)}$ *belongs to* $\mathfrak{a}^*$ *if and only if* $X^{(\nu)} \epsilon\, \mathfrak{a}$ *for every* ν.

Theorem 10. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be homogeneous ideals in* $\mathfrak{R}$ *and let* $\mathfrak{a}^*$ *and* $\mathfrak{b}^*$ *be the corresponding homogeneous ideals in* $\mathfrak{R}^*$. *Then to* $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a}\mathfrak{b}$, $\mathfrak{a} \frown \mathfrak{b}$ *and* $\mathfrak{a} : \mathfrak{b}$ *correspond* $\mathfrak{a}^* + \mathfrak{b}^*$, $\mathfrak{a}^*\mathfrak{b}^*$, $\mathfrak{a}^* \frown \mathfrak{b}^*$ *and* $\mathfrak{a}^* : \mathfrak{b}^*$.

Proof. The relations

$$\mathfrak{R}^*(\mathfrak{a} + \mathfrak{b}) = \mathfrak{R}^*\mathfrak{a} + \mathfrak{R}^*\mathfrak{b} = \mathfrak{a}^* + \mathfrak{b}^*,$$
$$\mathfrak{R}^*(\mathfrak{a}\mathfrak{b}) = \mathfrak{R}^*\mathfrak{a} \,.\, \mathfrak{R}^*\mathfrak{b} = \mathfrak{a}^*\mathfrak{b}^*,$$
$$\mathfrak{a}^* \frown \mathfrak{b}^* \frown \mathfrak{R} = \mathfrak{a}^* \frown \mathfrak{R} \frown \mathfrak{b}^* \frown \mathfrak{R} = \mathfrak{a} \frown \mathfrak{b},$$

establish the truth of the first three assertions. Let $\mathfrak{c}_0 = \mathfrak{a} : \mathfrak{b}$, $\mathfrak{c}_0^* = \mathfrak{R}^*\mathfrak{c}_0$ and $\mathfrak{c}^* = \mathfrak{a}^* : \mathfrak{b}^*$; then from $\mathfrak{c}_0\mathfrak{b} \subseteq \mathfrak{a}$ we obtain $\mathfrak{c}_0^*\mathfrak{b}^* \subseteq \mathfrak{a}^*$, which shows that $\mathfrak{c}_0^* \subseteq \mathfrak{c}^*$. Let $\mathfrak{b} = \mathfrak{R}X_1 + \mathfrak{R}X_2 + \ldots + \mathfrak{R}X_s$, where the X_i are homogeneous; then $\mathfrak{b}^* = \mathfrak{R}^*X_1 + \mathfrak{R}^*X_2 + \ldots + \mathfrak{R}^*X_s$. If now $r^* = \sum_0^{\infty} Y^{(\nu)}$ belongs to $\mathfrak{c}^*$, then $\sum_0^{\infty} Y^{(\nu)} X_i \epsilon\, \mathfrak{a}^*$, whence, by Theorem 9, Corollary, $Y^{(\nu)} X_i \epsilon\, \mathfrak{a}$. This shows that $Y^{(\nu)}\mathfrak{b} \subseteq \mathfrak{a}$, so that $Y^{(\nu)} \epsilon\, \mathfrak{c}_0 \subseteq \mathfrak{c}_0^*$ and therefore $r^* \epsilon\, \mathfrak{c}_0^*$. We now have $\mathfrak{c}^* \subseteq \mathfrak{c}_0^*$ and, as the opposite inclusion has already been established, this implies that $\mathfrak{c}^* = \mathfrak{c}_0^* = \mathfrak{R}^*\mathfrak{c}_0$. The proof is now complete.

Theorem 11. *Let* $\mathfrak{a}$ *and* $\mathfrak{a}^*$ *be corresponding homogeneous ideals. Then if one is prime so is the other or if one is primary so is the other.*

Proof. We shall establish the second assertion. The first can be proved in almost the same way but the details are slightly simpler.

If $\mathfrak{a}^*$ is primary then so is $\mathfrak{a}$ because $\mathfrak{a}$ is the projection of $\mathfrak{a}^*$. Suppose then that $\mathfrak{a}$ is primary. Assume that X, Y are homogeneous, that $XY \in \mathfrak{a}^*$ and that $X \notin \mathfrak{a}^*$. Then $XY \in \mathfrak{a}$ and $X \notin \mathfrak{a}$ and so, for a suitable integer s, we have $Y^s \in \mathfrak{a} \subseteq \mathfrak{a}^*$. That $\mathfrak{a}^*$ is primary now follows from Proposition 2*.

Corollary. Let $\mathfrak{q}$ *and* $\mathfrak{q}^*$ *be corresponding homogeneous primary ideals; then the prime ideals to which they belong are corresponding ideals.*

Proof. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and let $\mathfrak{q}^*$ be $\mathfrak{p}^*$-primary; then $\mathfrak{p}$ and $\mathfrak{p}^*$ are homogeneous by Theorems 4 and 4*. Since $\mathfrak{q}^*$ is $\mathfrak{p}^*$-primary, $\mathfrak{q}^* \cap \mathfrak{R} = \mathfrak{q}$ is $(\mathfrak{p}^* \cap \mathfrak{R})$-primary; consequently $\mathfrak{p}^* \cap \mathfrak{R} = \mathfrak{p}$.

By combining Theorems 10 and 11 we obtain

Theorem 12. *Let* $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_s$, *where* $\mathfrak{q}_i$ *is a homogeneous primary ideal in* $\mathfrak{R}$, *and let* $\mathfrak{a}^* = \mathfrak{R}^*\mathfrak{a}$ *and* $\mathfrak{q}_i^* = \mathfrak{R}^*\mathfrak{q}_i$. *Then* $\mathfrak{a}^* = \mathfrak{q}_1^* \cap \mathfrak{q}_2^* \cap \ldots \cap \mathfrak{q}_s^*$, *and this is a primary decomposition. Further, if one decomposition is irredundant (normal), so is the other.*

Theorem 12 shows, in passing, that Theorem 5* is true and this implies immediately the validity of Theorem 6*. Theorems 7* and 8* can now be proved by using the arguments that were employed in the discussion of Theorems 7 and 8. It should be noted that, in the formal statements of Theorems 5*–8*, it is not necessary to assume explicitly that $\mathfrak{R}^*$ is H-Noetherian. This is because the original assumptions, which apply to the whole of § 4, ensure this anyway.

The next result is included because it is useful in the theory of the Hilbert function.

Theorem 13. *Let* $\mathfrak{q}$ *and* $\mathfrak{q}^*$ *be corresponding homogeneous primary ideals; then*

$$\text{length } \mathfrak{q} = \text{length } \mathfrak{q}^*.$$

Proof. Let $\mathfrak{q}$ be $\mathfrak{p}$-primary, let $\mathfrak{q}^*$ be $\mathfrak{p}^*$-primary and let $\mathfrak{q}^* = \mathfrak{q}_0^* \subset \mathfrak{q}_1^* \subset \ldots \subset \mathfrak{q}_s^* = \mathfrak{p}^*$ be a composition series of $\mathfrak{p}^*$-primary ideals in which all the $\mathfrak{q}_i^*$ are homogeneous (see Theorem 7*). Put $\mathfrak{q}_i = \mathfrak{R} \cap \mathfrak{q}_i^*$; then $\mathfrak{q}_i$ is $\mathfrak{p}$-primary and $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_s = \mathfrak{p}$. Now $\mathfrak{q}_i$ and $\mathfrak{q}_{i+1}$ are homogeneous and there is no homogeneous $\mathfrak{p}$-primary ideal between them (otherwise there would be a $\mathfrak{p}^*$-primary ideal between $\mathfrak{q}_i^*$ and $\mathfrak{q}^*_{i+1}$); consequently, by Theorem 8, $\mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_s = \mathfrak{p}$ is a composition series. It follows that length $\mathfrak{q}$ = length $\mathfrak{q}^*$.

5. *Further remarks.* If we abandon the condition that $\mathfrak{R}$ and $\mathfrak{R}^*$ shall be H-Noetherian we can pose a number of basic questions about homogeneous ideals to which the foregoing discussion provides no answer. The following theorem, however, answers one of these questions and is sufficiently simple and interesting to seem worth including.

Theorem 14. *Let* $\mathfrak{R}$ *be a graduated ring (not necessarily H-Noetherian) and let* $\mathfrak{a}$ *be a homogeneous ideal which can be represented as a finite intersection of primary (but not necessarily homogeneous) ideals. Then* $\mathfrak{a}$ *can be expressed as a finite intersection of homogeneous primary ideals.*

Proof. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_s$, where the $\mathfrak{q}_i$ are primary, but not necessarily homogeneous ideals. Denote by $\overline{\mathfrak{q}_i}$ the homogeneous ideal generated by all the homogeneous elements in $\mathfrak{q}_i$; then $\mathfrak{a} \subseteq \overline{\mathfrak{q}_i} \subseteq \mathfrak{q}_i$ and so $\mathfrak{a} = \overline{\mathfrak{q}_1} \cap \overline{\mathfrak{q}_2} \cap \ldots \cap \overline{\mathfrak{q}_s}$. The proof will therefore be complete if we show that $\overline{\mathfrak{q}_i}$ is primary. Assume that X, Y are homogeneous, that $XY \in \overline{\mathfrak{q}_i}$ and that $X \notin \overline{\mathfrak{q}_i}$. Then $XY \in \mathfrak{q}_i$ and $X \notin \mathfrak{q}_i$; consequently $Y^s \in \mathfrak{q}_i$ for a suitable integer s. But, since $Y^s \in \mathfrak{a}_i$, it follows that $Y^s \in \overline{\mathfrak{q}_i}$ and now the primary character of $\overline{\mathfrak{q}_i}$ is seen by applying Proposition 2.

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