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SMOOTH BOUNDARY VALUES ALONG TOTALLY REAL SUBMANIFOLDS

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The main result of this paper is the following regularity result:

THEOREM. *Let $D \subset \mathbf{C}^N$ be a bounded, strongly pseudoconvex domain with bD of class C^k, $k \geqq 3$. Let $\Sigma \subset bD$ be an N-dimensional totally real submanifold, and let $f \in A(D)$ satisfy $|f| = 1$ on Σ, $|f| < 1$ on $\bar{D} \setminus \Sigma$. If Σ is of class C^r, $3 \leqq r < k$, then the restriction $f_\Sigma = f|\Sigma$ of f to Σ is of class C^{r-0}, and if Σ is of class C^k, then f_Σ is of class C^{k-1}.*

Here, of course, $A(D)$ denotes the usual space of functions continuous on $\bar{D}$, holomorphic on D, and we shall denote by $A^k(D)$, $k = 1, 2, \dots$, the space of functions holomorphic on D whose derivatives or order k lie in $A(D)$.

Saying that f_Σ belongs to C^{r-0} is understood to mean that the derivatives of f of order $r - 1$ satisfy Hölder conditions of all orders less than one.

In the terminology of [**5**], the manifold Σ is the maximum modulus set of the function f. It is shown in [**5**] that if $f \in A^2(D)$ and if the maximum modulus set of f is a smooth manifold, then this manifold is necessarily totally real.

The cases $k = r = \infty$ and $k = r = \omega$ (real analyticity) are admitted. The latter case follows from the edge-of-the-wedge theorem and so involves an entirely different circle of ideas than the differentiable case. The conclusion in case $r = \infty$ is that $f_\Sigma = f|\Sigma$ belongs to $C^\infty(\Sigma)$.

No corresponding conclusion can be drawn for manifolds Σ of dimension less than N. To see this, let $\Sigma \subset bD$ be a smooth submanifold that is a peak interpolation set for $A(D)$, and let $\phi:\Sigma \to \mathbf{R}$ be a continuous nowhere differentiable function. If $f \in A(D)$ satisfies $f = e^{i\phi}$ on Σ, $|f| < 1$ on $\bar{D} \setminus \Sigma$, then f has Σ as its maximum modulus set, but f_Σ is differentiable at no point of Σ. Such a Σ is necessarily totally real [**1**], [**13**]. There exist $(N - 1)$-dimensional Σ's of the desired kind in the boundary of an arbitrary domain. Spheres and tori can be found explicitly in the boundary of the ball in $\mathbf{C}^N$ [**13**]. In general, there are many submanifolds with the property [**4**].

The stated result for $3 \leqq r < k$ is the best one should expect as the classical one-dimensional theory shows: The Riemann map from the unit

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disc to a domain in the plane bounded by a $\mathscr{C}^1$ curve is generally not of class $\mathscr{C}^1$ but only of class $\mathscr{C}^{1-0}$ [**15**, p. 377]. It seems probable that in case $r = k$, the conclusion should be $f_\Sigma \in \mathscr{C}^{k-0}$ rather than $f_\Sigma \in \mathscr{C}^{k-1}$. However, our methods do not yield this.

We shall denote by $\mathscr{C}^{m,\alpha}$, m a natural number, $\alpha \in (0, 1)$, the usual space of functions whose derivatives of order m satisfy a Hölder condition of order α.

Finally, let us remark that, as we shall see, the analysis involved in the proof below is entirely local, so a local version of the theorem could be established.

Proof. The proof of the theorem consists of two steps. In the first, we show that the theorem is true if the function f_Σ is assumed to lie in $\mathscr{C}^1(\Sigma)$; in the second, we show that with f as in the theorem, f_Σ necessarily lies in $\mathscr{C}^1(\Sigma)$.

Step 1. *Reduction to a* $\mathscr{C}^1$ *problem.* Assume to begin with that $3 \leqq r < k$. To show that if $f_\Sigma \in \mathscr{C}^1(\Sigma)$, then necessarily $f_\Sigma \in \mathscr{C}^{r-o}(\Sigma)$, we need a simple fact about functions on products.

LEMMA A. *If* $F:\mathbf{R} \times \mathbf{R}^n \to \mathbf{C}$ *satisfies* (1) $F(t,\cdot)$ *is constant for each* $t \in \mathbf{R}$ *and* (2) $F \circ \gamma \in \mathscr{C}^{s,\alpha}(\mathbf{R})$ *where* $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n): \mathbf{R} \to \mathbf{R} \times \to \mathbf{R}^n$ *is a curve of class* $\mathscr{C}^{s,\alpha}$ *that is transverse to* $\{t\} \times \mathbf{R}^n$ *for each* t, *then locally* F *is of class* $\mathscr{C}^{s,\alpha}$ *on the open set* $\gamma_0(\mathbf{R}) \times \mathbf{R}^n$ *in* $\mathbf{R} \times \mathbf{R}^n$.

The qualification "locally" is required, for the constants involved in the Hölder estimates may well increase off compact sets.

Proof. The transversality hypothesis implies that γ_0' is zero-free so that γ_0 is strictly monotonic. Thus $\gamma_0(\mathbf{R})$ is open in $\mathbf{R}$, and $\gamma_0^{-1}:\gamma_0(\mathbf{R}) \to \mathbf{R}$ is of class $\mathscr{C}^{s,\alpha}$. Define $g:\gamma_0(\mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}$ by $g(t, x) = \gamma_0^{-1}(t)$; g is also of class $\mathscr{C}^{s,\alpha}$. As $F \circ \gamma \circ g = F$, we see that F is of class $\mathscr{C}^{s,\alpha}$ locally and the lemma is proved.

We have supposed that D is strongly pseudoconvex with $\mathscr{C}^k$ boundary, so there is a $\mathscr{C}^k$ strongly plurisubharmonic function Q defined on a neighborhood of $\bar{D}$ so that $D = \{Q < 0\}$ and $dQ \neq 0$ on bD. Denote by η the contact form $d^cQ = i(\bar{\partial} - \partial)Q$. (See [**5**], [**14**].) It is of class $\mathscr{C}^{k-1}$.

LEMMA B. *The differential* df_Σ *vanishes at no point of* Σ.

This is given in [**5**] as Corollary 1.3. For our purposes, it is convenient to have the mechanism of the following proof in hand.

Proof. Fix $p \in \Sigma$ and invoke Lemma 4.5 of [**5**] to find a holomorphic map $\phi:\Delta \to D$, Δ the open unit disc in $\mathbf{C}$, such that ϕ extends to be of class

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$\mathscr{C}^{k-2}$ on $\bar{\Delta}$, $\phi(1) = p$, and $\phi(e^{i\theta}) \in \Sigma$ when $|\theta| < \pi/2$. The function $f \circ \phi$ is holomorphic in Δ, is of modulus less than one there, and

$$|f \circ \phi(e^{i\theta})| = 1 \quad \text{if } |\theta| < \pi/2.$$

Thus,

$$\frac{d}{d\theta} f \circ \phi(e^{i\theta}) \neq 0 \quad \text{for } |\theta| < \pi/2$$

whence $df_\Sigma(p) \neq 0$.

It follows that the map $f_\Sigma: \Sigma \to \mathbf{T}$, $\mathbf{T} = b\Delta$, is a regular map, and thus Σ is foliated by the fibers $f_\Sigma^{-1}(\zeta)$, $\zeta \in \mathbf{T}$. Each of these fibers is a $\mathscr{C}^1$ submanifold of Σ since $f_\Sigma \in \mathscr{C}^1(\Sigma)$, and, moreover, each of them is an integral manifold of the form η-each $f_\Sigma^{-1}(\zeta)$ is a peak set for $A(D)$ and so a peak interpolation set. The $\mathscr{C}^1$ submanifolds of bD that are peak interpolation sets are the integral manifolds of the form η. See [**8**], [**12**].

For $p \in \Sigma$, let $T_p(\Sigma)$ and $T_p(bD)$ denote, respectively, the tangent space to Σ and the tangent space to bD, and let $T_p^{\mathbf{C}}(bD)$ be the maximal complex subspace of $T_p(bD)$ so that $T_p^{\mathbf{C}}(bD)$ is the kernel of the map $\eta: T_p(bD) \to \mathbf{R}$. Let

$$\mathscr{D}(p) = T_p(\Sigma) \cap T_p^{\mathbf{C}}(bD)$$

so that $\mathscr{D}$ is a distribution (or differential system) of rank $N - 1$ on Σ. It is of class $\mathscr{C}^r$, for $\mathscr{D}(p) = \ker i^*\eta$ if $i: \Sigma \to bD$ denotes the inclusion (which is of class $\mathscr{C}^r$) and η is a form of class $\mathscr{C}^{k-1}$. Moreover, it is involutive since the function f defines a foliation of Σ. It follows that if we fix $p \in \Sigma$, there is a $\mathscr{C}^r$ diffeomorphism Φ from the open unit cube I^N in $\mathbf{R}^N$ onto a neighborhood U_p of p in Σ such that for each choice of x_1, $\Phi(x_1, \cdot)$ takes $\{x_1\} \times I^{N-1}$ onto the leaf of the foliation induced in U_p through $\Phi(x_1, 0, \ldots, 0)$. (As a reference for the Frobenius theorem with the differentiability conditions we have, see [**9**].)

The function $f \circ \Phi$ is of class $\mathscr{C}^1$ on I^N and satisfies the condition that for fixed x_1, the partial function

$$f \circ \Phi(x_1, \cdot): I^{N-1} \to \mathbf{C}$$

is constant. Moreover, we show below that there is a $\mathscr{C}^{r-0}$ curve $\gamma: (-1, 1) \to I^N$ transverse to the sections $\{x_1\} \times I^{N-1}$ such that $f \circ \Phi \circ \gamma$ is of class $\mathscr{C}^{r-0}$. Once we have γ, we can invoke Lemma A to conclude that $F \circ \Phi \in \mathscr{C}^{r-0}(I^N)$ whence $F = F \circ \Phi \circ \Phi^{-1}$ is of class $\mathscr{C}^{r-0}$ near p.

To construct γ, revert to the map $\phi: \bar{\Delta} \to \bar{D}$ used above. It is of class $\mathscr{C}^{k-2}$ on $\bar{\Delta}$, and, as we noted,

$$\frac{d}{d\theta} f \circ \phi\,(e^{i\theta}) = 0 \quad \text{for no } \theta \in (-\pi/2, \pi/2).$$

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It follows that the curve χ: $(-\pi/2, \pi/2) \rightarrow \Sigma$ given by $\chi(t) = \phi(e^{it})$ is never tangent to the level sets of the function f, i.e., χ is transverse to the leaves of the foliation of Σ by f.

The curve $\gamma = \Phi^{-1} \circ \chi$ has the geometric property we seek, and from what has been said so far, it is seen to be of class $\mathscr{C}^{r-2}$. In fact, it is of class $\mathscr{C}^{r-0}$ as follows from [**2**, Theorem 33]. (The simpler Theorem 31 of [**2**] together with its proof suffice for our purposes).

Note added in proof. An alternative reference for what we need is Theorem 4.5 of the paper of Bedford and Gaveau, *Envelopes of holomorphy of certain 2-spheres in* $\mathbf{C}^2$, American Journal of Mathematics *105* (1983), 975-1009.

This completes the proof in case that $r < k$. For $r = k$, the conclusion is only that $f_\Sigma \in \mathscr{C}^{k-1}$. The argument proceeds exactly as above except that in this case, the form $i^*\eta$ is of class $\mathscr{C}^{k-1}$ and so the map Φ will be of class $\mathscr{C}^{k-1}$ whence $f_\Sigma \in \mathscr{C}^{k-1}$ rather than $f_\Sigma \in \mathscr{C}^{k-0}$.

Thus to complete the proof of the theorem, it is enough to show that $f \in \mathscr{C}^1(\Sigma)$.

Step 2. The function f_Σ is of class $\mathscr{C}^1$. The proof we give for this fact involves the construction of certain analytic discs abutting Σ from within D. This argument is similar to the proof of Lemma 4.5 of [**5**]. See also [**7**] and [**10**].

The problem is local, so by fixing $p \in \Sigma$ and replacing f by $\log f$ defined on a simply connected, strongly pseudoconvex neighborhood of p in $\bar{D}$, we see that we can replace the hypothesis that $|f| = 1$ on Σ by the hypothesis that $\operatorname{Re} f = 0$ on Σ.

Having fixed $p \in \Sigma$, we choose holomorphic coordinates $\zeta_1, \dots, \zeta_N$, $\zeta_j = \xi_j + i\eta_j$, on $\mathbf{C}^N$ centered at p so that

$$T_p(bD) = \{\eta_N = 0\}$$

and

$$T_p^{\mathbf{C}}(bD) = \{\zeta_N = 0\}.$$

Thus, near 0, D is given by

$$D = \{\zeta \in \mathbf{C}^N : \eta_N > G(\xi_1, \dots, \xi_N, \eta_1, \dots, \eta_{N-1})\}$$

for a $\mathscr{C}^k$ function G with $G(0) = 0$, $dG(0) = 0$. Also, there is a $\mathscr{C}^r$ function h: $\mathbf{R}^N \rightarrow \mathbf{R}^N$ with $h(0) = 0$, $dh(0) = 0$ so that near 0, Σ coincides with the manifold Σ_0 given by

$$\Sigma_0 = \{\xi + ih(\xi) : \xi \in \mathbf{R}^N\}.$$

Denote by $\mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N)$ the space of all $\mathbf{R}^N$-valued functions on $\mathbf{T}$ each component of which has s derivatives, the s^{th} of which satisfies a Hölder condition of order α, $\alpha \in (0, 1)$. For $\xi \in \mathbf{R}^N$, let $\mathscr{C}_\xi^{s,\alpha}(\mathbf{T}, \mathbf{R}^N)$ be the affine subspace of $\mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N)$ consisting of those f with $f(1) = \xi$.

Let T be the conjugation operator on $\mathbf{T}$ so that if u is a suitably restricted function on $\mathbf{T}$, then $u + iTu$ is the boundary value of a function holomorphic on Δ and real at 0. According to a theorem of Privaloff [7], [11], T is a continuous linear operator from $\mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R})$ to itself. We let T act on $\mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N)$ coordinate by coordinate.

The map $u \mapsto u(1)$ is a continuous linear map from $\mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N)$ onto $\mathbf{R}^N$. Denote it by E. For $\xi \in \mathbf{R}^N$ we define

$$E_\xi: \mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N) \to \mathbf{R}^N$$

by

$$E_\xi(u) = E(u) - \xi$$

so that $E = E_0$. Also, for $\xi \in \mathbf{R}^N$, define

$$S_\xi: \mathscr{C}^{s,\alpha}(\mathbf{T}, \mathbf{R}^N) \to \mathscr{C}^{s,\alpha}_\xi(\mathbf{T}, \mathbf{R}^N)$$

by

$$S_\xi(u) = Tu - E_\xi T(u).$$

The operators E_ξ and S_ξ are affine but linear only where $\xi = 0$. Notice that

$$\begin{aligned} S_\xi(u)(1) &= T(u)(1) - (T(u)(1) - \xi) \\ &= \xi. \end{aligned}$$

The function h is of class $\mathscr{C}^r$, so we can define a map

$$H: \mathscr{C}^{r-1,\alpha}(\mathbf{T}, \mathbf{R}^N) \to \mathscr{C}^{r-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$$

by

$$Hf = h \circ f.$$

The following fact about H is proved in [7]:

LEMMA C. *The map H is of class $\mathscr{C}^1$.*

Let $u_1 \in \mathscr{C}^\infty(\mathbf{T})$ satisfy $u_1(e^{i\theta}) = 0$ for $|\theta| < \pi/2$ and $u_1(e^{i\theta}) < 0$ for $\theta \in (\pi/2, 3\pi/2)$. Assume also that u_1 is normalized so that if $\widetilde{u}_1$ is the harmonic function on Δ with boundary values u_1, then

$$\frac{\partial \widetilde{u}_1}{\partial \rho}(1) = 1.$$

Define $u \in \mathscr{C}^\infty(\mathbf{T}, \mathbf{R}^N)$ by $u = (u_1, \dots u_1)$, and if $\tau = (\tau_1, \dots, \tau_N) \in \mathbf{R}^N$, let

$$\tau u = (\tau_1 u_1, \dots, \tau_N u_1) \in \mathscr{C}^\infty(\mathbf{T}, \mathbf{R}^N).$$

Let

$$F: \mathscr{C}^{r-2,\alpha}(\mathbf{T}, \mathbf{R}^N) \times \mathbf{R}^N \times \mathbf{R}^N \to \mathscr{C}^{r-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$$

be given by

$$F(x, \xi, \tau) = x - S_\xi(h \circ x + \tau u).$$

If (x, ξ, τ) is a zero of F, then the function $g_{\xi,\tau}$ defined on $\mathbf{T}$ by

$$g_{\xi,\tau}(e^{i\theta}) = x(e^{i\theta}) + i(h \circ x(e^{i\theta}) + \tau u(e^{i\theta}))$$

is the boundary value function of a function, denoted by $\widetilde{g}_{\xi,\tau}$, holomorphic on Δ, and a theorem of Hardy and Littlewood **[6]** implies that

$$\widetilde{g}_{\xi,\rho} \in A^{r-2,\alpha}(\Delta),$$

i.e., the $(r-2)^{\text{nd}}$ derivative of $\widetilde{g}_{\xi,\tau}$ satisfies a Hölder condition of order α on $\bar{\Delta}$. We have

$$\begin{aligned} g_{\xi,\tau}(1) &= x(1) + ih(x(1)) \\ &= S_\xi(h \circ x + \tau u)(1) + ih(x(1)) \\ &= \xi + ih(\xi), \end{aligned}$$

and, for as $|\theta| < \tau/2$,

$$g_{\xi,\tau}(e^{i\theta}) = x(e^{u\theta}) + ih \circ x(e^{i\theta})$$

so that $g_{\xi,\tau}(e^{i\theta}) \in \Sigma$, $|\theta| < \pi/2$.

For a given τ, the map $\xi \mapsto g_{\xi,\tau}(1)$ is simply $\xi \mapsto \xi + ih(\xi)$ and so is a $\mathscr{C}^r$ diffeomorphism of $\mathbf{R}^N$ onto Σ_0.

A computation given in **[5]**, **[7]** shows that if D_1F denotes the partial derivative of F with respect to the first variable, then $D_1F(0, 0, 0) = I$, the identity map from $\mathscr{C}^{r-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$ to itself. Thus, if for small $\delta > 0$, J_δ denotes the open ball of radius δ about $0 \in \mathbf{R}^N$, the implicit function theorem **[3]** provides a $\mathscr{C}^1$ map

$$x^*: J_\delta \times J_\delta \to \mathscr{C}^{r-2,\alpha}(\mathbf{T}, \mathbf{R}^N)$$

such that

$$x^*(0, 0) = 0 \quad \text{and} \quad F(x^*(\xi, \tau), \xi, \tau) = 0.$$

The function $x{:}\mathbf{T} \times J_\delta \times J_\delta \to \mathrm{R}^N$ given by

$$x(e^{i\theta}, \xi, \tau) = x(\xi, \tau)(e^{i\theta})$$

belongs to $\mathscr{C}^1(\mathbf{T} \times J_\delta \times J_\delta)$. We also have the function

$$g: \bar{\Delta} \times J_\delta \times J_\delta \to \mathbf{C}^N$$

given by

$$g(\cdot, \xi, \tau) = \widetilde{g}_{\xi,\tau},$$

$\widetilde{g}_{\xi,\tau}$ as above.

By definition we have

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$$\frac{\partial g}{\partial \theta}(e^{i\theta}, 0, \tau)|_{\theta=0} = \frac{\partial x}{\partial \theta}(e^{i\theta}, 0, \tau)|_{\theta=0} + i\frac{\partial}{\partial \theta}(h \circ x(e^{i\theta}, 0, \tau) + \tau u(e^{i\theta}))|_{\theta=0},$$

and since $u(e^{i\theta}) = 0$ for $|\theta| < \pi/2$ and $h(0) = 0$, $dh(0) = 0$, this is just

$$\frac{\partial x}{\partial \theta}(e^{i\theta}, 0, \tau)|_{\theta=0}$$

which, by the Cauchy-Riemann equations, is

$$-\frac{\partial}{\partial \rho}(\widetilde{h \circ x}(\rho, 0, \tau) + \tau u(\rho))|_{\rho=1}$$

where $\widetilde{h \circ x}$ denotes the harmonic extension of $h \circ x$ through Δ and similarly for $\widetilde{\tau u}$. By the choice of u,

$$\frac{\partial}{\partial \rho}\widetilde{\tau u}(\rho)|_{\rho=1} = \tau,$$

so for fixed $\tau \in \mathbf{R}^N$, the derivative of the map $e^{i\theta} \mapsto g(e^{i\theta}, 0, \tau)$ at $\theta = 0$, thought of as a real linear map from $\mathbf{R}$ to $\mathbf{R}^N$, is

$$t \mapsto t\tau - t\frac{\partial h \circ x}{\partial \rho}(\rho, 0, \tau)|_{\rho=1}.$$

We have that $x(0, 0) = 0$ and $dh(0) = 0$, so for small values of τ, this is a small perturbation of the map $t \mapsto t\tau$. Thus, if we fix τ, then we can find N arbitrarily small perturbations of τ, say $\tau^{(1)}, \ldots, \tau^{(N)}$, such that the vectors

$$\frac{\partial g}{\partial \theta}(e^{1\theta}, 0, \tau^{(j)}) = \tau^{(j)} - \frac{\partial \widetilde{h \circ x}}{\partial \rho}(\rho, 0, \tau^{(j)})|_{\rho=1}$$

span $\mathbf{R}^N$.

If $\tau_1 = \cdots = \tau_{N-1} = 0$ and $\tau_N > 0$, τ_N sufficiently small, then $g(w, 0, \tau) \in D$ provided $w \in \Delta$ is near the right half, P, of $\mathbf{T}$. The same is true, therefore, for all τ, provided $\tau_1, \ldots, \tau_{N-1}$ are small and $\tau_N > 0$. This will persist also for $g(w, \xi, \tau)$, for all ξ in a small neighborhood of $0 \in \mathbf{R}^N$.

Given a vector $\tau \in \mathbf{R}^N$ near 0 such that for small $\xi \in \mathbf{R}^N$, $g(\cdot, \xi, \tau)$ takes points in Δ near P into D, denote by $\tau^\perp$ the subspace of $\mathbf{R}^N$ orthogonal to τ and consider the map (of class $\mathscr{C}^1$) $\bar{\Delta} \times \tau^\perp \to \mathbf{C}^N$ defined by

(1) $\quad (w, \xi) \mapsto g(w, \xi, \tau).$

The partial map $\xi \mapsto g(e^{i\theta}, \xi, \tau)$ is a diffeomorphism of $\tau^\perp$ into Σ_0, and its image is transverse to the vector τ at 0 and so to the tangent space of the curve $e^{i\theta} \mapsto g(e^{i\theta}, 0, \tau)$. Thus, the map given by (1) is regular, i.e., of real rank N, on a neighborhood of $(1, 0) \in \mathbf{T} \times \tau^{\perp}$. As $\tau^{\perp}$ is isomorphic to $\mathbf{R}^{N-1}$, we have constructed a $\mathscr{C}^1$ map

$$G_\tau: \bar{\Delta} \times \mathbf{R}^{N-1} \to \mathbf{C}^N$$

such that $G_\tau(\cdot, \xi) \in A^{r-2,\alpha}(\Delta, \mathbf{C}^N)$, $G_\tau(w, \xi) \in D$ when $w \in \Delta$, and so that G_τ takes a neighborhood of $(1, 0) \in \mathbf{T} \times \mathbf{R}^{N-1}$ diffeomorphically onto a neighborhood of $0 \in \Sigma$. Moreover, the tangent space, at 1, to the curve $G_\tau(e^{i\theta}, 0)$ is very nearly τ.

If we choose $\tau^{(1)}, \dots, \tau^{(N)}$ appropriately and denote by Ξ_j the vector field on Σ_0 given by

$$\Xi_j = G_{\tau^{(j)}*}\frac{\partial}{\partial\theta}$$

where $\dfrac{\partial}{\partial\theta}$ is the vector field on $\mathbf{T} \times \mathbf{R}^N$ given by differentiating with respect to the coordinate θ on $\mathbf{T}$, then near $0 \in \Sigma$, the vector fields $\Xi_1, \dots, \Xi_N$ span the tangent bundle of Σ_j; they are of class $\mathscr{C}^1$. Accordingly, in order to prove the function f_Σ to be of class $\mathscr{C}^1$ near 0, it suffices to show that each of the functions $\Xi_j f_\Sigma$ is continuous.

To this end, write G_j for $G_{\tau(j)}$ for notational convenience, and consider $F \circ G_j$. Recall that we are assuming $\operatorname{Re} f = 0$ on Σ, so if we write $f \circ G_j = U + iV$, then for $w \in \Delta$,

$$f \circ G_j(w, \xi) = \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} U(e^{i\theta}, \xi)\frac{e^{i\theta} + w}{e^{i\theta} - w}\, d\theta + iV(0),$$

so

$$\frac{f \circ G_j}{\partial w}(w, \xi) = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} U(e^{i\theta}, \xi)\frac{e^{i\theta}}{(e^{i\theta} - w)^2}\, d\theta.$$

The last integral is continuous in (w, ξ), $w \in \Delta \cup \{e^{i\theta}: |\theta| < \pi/2\}$ and the desired continuity of $\dfrac{\partial f \circ G}{\partial\theta}j(e^{i\theta}, \xi)$ in θ and ξ follows. Thus, $\Xi_j f_\Sigma$ is continuous, and the theorem is proved.

It is worth remarking that the second step of the proof does not invoke the full hypothesis that f have Σ as its maximum modulus set. All that is required is that $f \in A(D)$ be of constant modulus along Σ. It seems probable that the full conclusion of the theorem can be drawn under this weaker hypothesis, but we have no proof.

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