

ON RADICAL FORMULA IN MODULES

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Abstract. We will state some conditions under which if a quotient of a module M satisfies the radical formula of degree k (s.t.r.f of degree k), so does M . Especially, we will introduce some particular modules M' such that $M' \oplus M''$ s.t.r.f of degree k , when M'' s.t.r.f of degree k . Furthermore, we will show that, under certain conditions, if the completion of a module M s.t.r.f of degree k , then there is a non-negative integer k' such that M s.t.r.f of degree k' . Moreover, we state a corrected version of Leung and Man's theorem (K. H. Leung and S. H. Man, On commutative Noetherian rings which satisfy the radical formula, *Glasgow Math. J.* 39 (1997), 285–293) on Noetherian rings that satisfies the radical formula.

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1. Introduction. In this paper all rings are commutative and with identity, all modules are unitary, R denotes a ring and M denotes an R -module. Also, by \mathbb{N} we mean the set of positive integers and $\mathbb{N}^* = \mathbb{N} \cup \{0, \infty\}$.

A proper submodule P of M is called *prime* when from $rm \in P$ for some $r \in R$ and $m \in M$, we can conclude either $m \in P$ or $rM \subseteq P$ (see for example [1, 3, 4, 6, 9, 10, 11, 13, 15, 16]). Let $(P : M)$ be the set of all $r \in R$ such that $rM \subseteq P$. If P is a prime submodule, then $\mathfrak{P} = (P : M)$ is a prime ideal of R and we say that P is \mathfrak{P} -prime. Recall that if R is an integral domain, then M is called *torsion-free*, if for every $0 \neq r \in R$ and $0 \neq m \in M$, we have $rm \neq 0$. One can easily verify that P is a \mathfrak{P} -prime submodule of M if and only if $\frac{M}{P}$ is a torsion-free $\frac{R}{\mathfrak{P}}$ -module.

Another generalisation of prime ideals to modules was proposed in [7]. There a proper submodule W of M is said to be *weakly prime*, if from $rs m \in W$ for $r, s \in R$ and $m \in M$, we can conclude either $rm \in W$ or $sm \in W$. One can easily see that it is equivalent to saying that $(W : m)$ is a prime ideal for every $m \in M \setminus W$. So every prime submodule of M is weakly prime.

Recall that for an ideal \mathfrak{J} of R , the intersection of all prime ideals of R containing \mathfrak{J} is called the radical of \mathfrak{J} and is denoted by $\sqrt{\mathfrak{J}}$. Similarly, if N is a submodule of M , the intersection of prime (weakly prime) submodules of M containing N is called the *radical* (weak radical) of N and we denote it by $\text{rad}_M(N)$ or $\text{rad}(N)$ ($\text{wrad}_M(N)$ or $\text{wrad}(N)$).

It is well known that $\sqrt{\mathfrak{J}} = \{r \in R \mid r^k \in \mathfrak{J} \text{ for some } k \in \mathbb{N}\}$. In order to find a similar characterisation for $\text{rad}(N)$, the notion of envelope of a submodule was introduced by McCasland and Moore [13]. The *envelope* of a submodule N of M , $E_M(N)$ (or $E(N)$ if no subtlety), is the set of all $x \in M$ for which there exist $r \in R$,

$m \in M$ and $k \in \mathbb{N}$ such that $x = rm$ and $r^k m \in N$. In general, $E(B)$ is not a submodule, so we consider $RE(B)$ (or $RE_M(B)$) to be the submodule of M generated by $E(B)$.

Also in [3], the k th envelope of N was defined recursively by $E_0(N) = N$ and $E_k(N) = E(RE_{k-1}(N))$, where $RE_k(N)$ (or $RE_{k,M}(N)$) is a submodule of M generated by $E_k(N)$. Moreover, here by $RE_\infty(N)$ we mean $\bigcup_{i=0}^\infty RE_k(N)$.

One can easily verify that for every submodule N of M

$$N = RE_0(N) \subseteq RE_1(N) \subseteq RE_2(N) \cdots \subseteq RE_\infty(N) \subseteq \text{wrad}(N) \subseteq \text{rad}(N).$$

Now if $\text{rad}(N) = RE_k(N)$ ($\text{wrad}(N) = RE_k(N)$) for a $k \in \mathbb{N}^*$, then we say that N satisfies (weakly satisfies) the radical formula of degree k , or N s.t.r.f. (weakly s.t.r.f.) of degree k in M . A module M s.t.r.f. (weakly s.t.r.f.) of degree k , when every submodule of M s.t.r.f. (weakly s.t.r.f.) of degree k in M . Also, we say that R s.t.r.f. (weakly s.t.r.f.) of degree k , if every R -module s.t.r.f. (weakly s.t.r.f.) of degree k . For $k = 1$ we simply drop k and write R s.t.r.f. (weakly s.t.r.f.) and write $RE(N)$ instead of $RE_1(N)$.

In [3], it was proved that every arithmetical ring with the finite Krull dimension k , s.t.r.f. of degree k and that an arithmetical ring with DCC on prime ideals s.t.r.f. of degree ∞ . Also, [4, Theorem 2.1] states that every arithmetical ring weakly s.t.r.f. of degree ∞ . In [16] the rings that s.t.r.f. of degree 0 were named absolutely radical, and it was proved that a ring is absolutely radical if and only if it is the von Neumann regular ring. Moreover, in [4] it is proved that R s.t.r.f. of degree 0 if and only if it weakly s.t.r.f. of degree 0. Also in [9], the characterisation of Noetherian rings that s.t.r.f. is given, which, in Section 2, we will show that it needs a small correction.

A proper submodule N of M is called *semi-prime* if $E(N) = N$. It is said that R has the *s.p.a.r. property* when every semi-prime submodule N of an R -module is a radical submodule (that is, N equals an intersection of prime submodules). One can easily see that a proper submodule B of M is semi-prime if and only if B is of the form $RE_\infty(N)$ for some submodule N of M . Thus, having the s.p.a.r. property is equivalent to s.t.r.f. of degree ∞ . In [10], Noetherian rings with the s.p.a.r. property were characterized.

It is known that if M s.t.r.f. (weakly s.t.r.f.) of degree k and N is a submodule of M , then $\frac{M}{N}$ s.t.r.f. (weakly s.t.r.f.) of degree k . In Section 3 we state conditions on M and N under which the converse of this is true. Especially in Sections (3.4), (3.7), (3.11), (3.14) and (3.16), we will introduce some particular modules M' such that $M' \oplus M''$ s.t.r.f. of degree k , when M'' s.t.r.f. of degree k .

Also, we will prove that if $s \in \mathbb{N}$ and \mathcal{J} is a finitely generated ideal of R such that $\mathcal{J} \subseteq \sqrt{(RE_s(0) : M)}$ and the \mathcal{J} -adic completion of M s.t.r.f. of degree t , then M s.t.r.f. of degree $s + t$ and moreover as an application of this, in (3.21) we will show that for Noetherian rings, having the s.p.a.r. property is equivalent to s.t.r.f. of degree 2. This result generalises Theorem 3.5 and Corollary 2.6 of [4], which state that a Noetherian domain R s.t.r.f. of degree k for some $k \in \mathbb{N}^*$, if and only if R is a Dedekind domain.

2. A corrected version of Leung and Man’s theorem. In [9] the following theorem is claimed to be proved.

THEOREM 2.1. *Let R be a Noetherian ring and $\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_n$ be all the minimal prime ideals of R . R s.t.r.f. if and only if R is Artinian or all of the following conditions are satisfied:*

- (a) R is one dimensional and for each $i = 1, \dots, n$, $\frac{R}{\mathfrak{P}_i}$ is a Dedekind domain and \mathfrak{P}_i is the only \mathfrak{P}_i -primary ideal.

- (b) For $k = 1, \dots, n - 1$, $(\bigcap_{i=1}^k \mathfrak{P}_i) + \mathfrak{P}_{k+1} = \bigcap_{i=1}^k (\mathfrak{P}_i + \mathfrak{P}_{k+1})$ if $n \geq 2$.
- (c) For all $1 \leq i < j \leq n$, $R = \mathfrak{P}_i + \mathfrak{P}_j$ or $\frac{R}{\mathfrak{P}_i + \mathfrak{P}_j}$ is semi-simple Artinian, if $n \geq 2$.

Now we give a counter example of this theorem. Recall that a ring is called a *special principal ideal ring (SPIR)* if it has a unique prime ideal which is principal. These rings are sometimes called special primary rings (see [8]).

EXAMPLE 2.2. Let R_1 be an SPIR, which is not a field, and R_2 be discrete valuation domain, which is not filed. Set $R = R_1 \times R_2$. By [8, Theorem 9.10] any finite direct sum of SPIR and Dedekind’s domains is a ZPI-ring; therefore, R is a ZPI-ring, hence s.t.r.f. by [16, Theorem 2.10]. Also, R is clearly Noetherian and one dimensional.

Now if \mathfrak{P}_1 is the maximal ideal of R_1 , it is easy to see that $\mathfrak{P} = \mathfrak{P}_1 \times R_2$ is both a maximal and a minimal ideal of R . Also, $0 \times R_2$ is a \mathfrak{P} -primary ideal of R not equal to $\mathfrak{P} \times R_2$, which contradicts condition (a) of (2.1).

THEOREM 2.3. For correcting Theorem 2.1, we need to replace condition (a) of this theorem with the following condition:

(a’) R is one dimensional and for each $i = 1, \dots, n$; if \mathfrak{P}_i is not maximal, then $\frac{R}{\mathfrak{P}_i}$ is a Dedekind domain and \mathfrak{P}_i is the only \mathfrak{P}_i -primary ideal.

Proof. First note that if R satisfies the conditions of (2.1), then by the argument stated in [9], R s.t.r.f. Now let R be a ring that satisfies the conditions of the corrected version of the theorem. If \mathfrak{M} is a maximal ideal of R , then $R_{\mathfrak{M}}$ is either Artinian or a one-dimensional local ring that satisfies the conditions of (2.1). Thus, in both cases $R_{\mathfrak{M}}$ s.t.r.f., whence by [9, Theorem 3.4], R s.t.r.f.

Now, suppose that R is a one-dimensional Noetherian ring that s.t.r.f. By a careful reading of proof of [9, Corollary 2.3], we see that for every minimal prime ideal \mathfrak{P} of R , which is not maximal, $\frac{R}{\mathfrak{P}}$ is the Dedekind domain and \mathfrak{P} is the only \mathfrak{P} -primary ideal of R (in fact, in that proof its authors have assumed that if \mathfrak{P} is a minimal prime ideal of a one-dimensional Noetherian ring, then it is not maximal, which is not generally true). So condition (a’) holds.

Now note that if (b) (or (c)) holds locally, then it holds globally. Now if \mathfrak{P} is a height zero prime ideal of R , then (b) and (c) hold trivially for $R_{\mathfrak{P}}$. If \mathfrak{P} is a height one prime ideal of R , then $R_{\mathfrak{P}}$ is a Noetherian local ring of dimension one and hence by [9, Theorem 4.2], $R_{\mathfrak{P}}$ satisfies (b) and (c). Thus, R satisfies these conditions. □

COROLLARY 2.4. If R is a reduced Noetherian ring, then R s.t.r.f. if and only if R has the s.p.a.r property.

Proof. In fact this was noted in [10, p. 33], but there [9, Theorem 1.1] was used, which we showed is incorrect. Using (2.3) instead, one can prove the claim with a similar argument. □

3. Quotient of modules and the radical formula. The proof of the following lemma is easy and it is left to the reader.

LEMMA 3.1. Let $k \in \mathbb{N}^*$ and N be a submodule of M . For every submodule L of M containing N , we have

$$\frac{RE_{k,M}(L)}{N} = RE_{k,\frac{M}{N}}\left(\frac{L}{N}\right), \quad \frac{rad_M(L)}{N} = rad_{\frac{M}{N}}\left(\frac{L}{N}\right) \quad \text{and} \quad \frac{wrad_M(L)}{N} = wrad_{\frac{M}{N}}\left(\frac{L}{N}\right).$$

By the above lemma if M' is a submodule of M and M s.t.r.f. (weakly s.t.r.f.) of degree k , then $\frac{M}{M'}$ s.t.r.f. (weakly s.t.r.f.) of degree k too. Here we study some conditions on M and M' , under which the converse of this statement is true. Throughout this section, we assume that M' is a submodule of M . Consider the following properties of M' , where $k \in \mathbb{N}^*$ (note that each of (I_k) and (II_k) have two versions: one for radicals and another for weak radicals):

- (I_k) For every submodule N of M , $RE_{k,M}(N + M') = RE_{k,M}(N) + M'$ and $M' \cap \text{rad}_M(N) \subseteq RE_{k,M}(N)$ [$M' \cap \text{wrad}_M(N) \subseteq RE_{k,M}(N)$].
- (II_k) For every submodule N of M , $RE_{k,M}(N + M') \cap \text{rad}_M(N) = RE_{k,M}(N)$ [$RE_{k,M}(N + M') \cap \text{wrad}_M(N) = RE_{k,M}(N)$].

THEOREM 3.2. *The following statements are equivalent for all $k \in \mathbb{N}^*$:*

- (a) M s.t.r.f. (weakly s.t.r.f.) of degree k .
- (b) There is a submodule M' of M such that (II_k) for radicals (weak radicals) holds for M' and $\frac{M}{M'}$ s.t.r.f. (weakly s.t.r.f.) of degree k .
- (c) Every proper submodule M' of M satisfies (II_k) for radicals (weak radicals).

Proof. We prove the radical version of the statement, the proof of the statement on weak radicals is similar.

(a) \Rightarrow (b), (c): Suppose that M s.t.r.f. of degree k and M' and N are arbitrary submodules of M . Now $RE_k(N + M') \cap \text{rad}_M(N) = \text{rad}_M(N + M') \cap \text{rad}_M(N) = \text{rad}_M(N) = RE_k(N)$. Thus, every submodule of M has the property (II_k) for radicals. Also by (3.1), every quotient module of M s.t.r.f. of degree k .

(b) \Rightarrow (a): Since $\frac{M}{M'}$ s.t.r.f. of degree k by (3.1), $RE_k(N + M') = \text{rad}_M(N + M')$. Hence, $\text{rad}_M(N) = \text{rad}_M(N + M') \cap \text{rad}_M(N) = RE_k(N + M') \cap \text{rad}_M(N) = RE_k(N)$.

(c) \Rightarrow (a): Let N be an arbitrary submodule of M . First, suppose that $\text{rad}_M(N) \neq M$. This means that there exists a prime submodule P of M containing N . P being proper, P satisfies (II_k) for radicals. Thus, $\text{rad}_M(N) = P \cap \text{rad}_M(N) = RE_k(P) \cap \text{rad}_M(N) = RE_k(P + N) \cap \text{rad}_M(N) = RE_k(N)$. Now suppose that $\text{rad}_M(N) = M$. If $RE_k(N) \neq M$, then let $m \in M \setminus RE_k(N)$ and so $Rm \neq M$ as $\text{rad}_M(N) = M$. Now $m \in RE_k(N + Rm) = RE_k(N + Rm) \cap M = RE_k(N + Rm) \cap \text{rad}_M(N) = RE_k(N)$, which is a contradiction. □

LEMMA 3.3. *Let $k \in \mathbb{N}^*$. Then*

- (a) *The condition (I_k) ((II_k)) for radicals implies (I_k) ((II_k)) for weak radicals.*
- (b) *The condition (I_k) for radicals (weak radicals) implies (II_k) for radicals (weak radicals).*
- (c) *If (I_1) holds for radicals (weak radicals), then for all $0 \neq k \in \mathbb{N}^*$, (I_k) holds for radicals (weak radicals).*

Proof. (a) Obvious.

(b) If (I_k) holds for radicals and N is an arbitrary submodule of M , then using the modularity law, we get

$$\begin{aligned} RE_k(N + M') \cap \text{rad}(N) &= (RE_k(N) + M') \cap \text{rad}(N) \\ &= RE_k(N) + (M' \cap \text{rad}(N)) = RE_k(N). \end{aligned}$$

(c) This follows from a simple induction, definition of $RE_\infty(N)$ and the fact that $RE_i(N) \subseteq RE_j(N)$ whenever $i \leq j$ and N is a submodule of M . □

Let R be an integral domain. An R -module M is called *divisible* if $rM = M$ for every $0 \neq r \in R$. In [15] a module M was called *generalized torsion divisible* when $M = \sum_{i \in I} M_i$ for submodules M_i such that for each $i \in I$ there exists a prime ideal \mathfrak{P}_i of R such that $\mathfrak{P}_i M_i = 0$ and M_i is a torsion divisible $\frac{R}{\mathfrak{P}_i}$ -module. The following generalises [15, Theorem 2.5].

COROLLARY 3.4. *Let $0 \neq k \in \mathbb{N}^*$. Suppose that M' is a generalised torsion divisible submodule of M . If $\frac{M}{M'}$ s.t.r.f. (weakly s.t.r.f.) of degree k , then M s.t.r.f. (weakly s.t.r.f.) of degree k . Particularly if M'' is a module that s.t.r.f. (weakly s.t.r.f.) of degree k , then $M' \oplus M''$ s.t.r.f. (weakly s.t.r.f.) of degree k .*

Proof. Let N be an arbitrary submodule of M . By [15, Lemma 2.4], $E(N + M') \subseteq RE(N)$. Hence, $RE(N) + M' \subseteq RE(N + M') \subseteq RE(N) \subseteq RE(N) + M'$, that is $RE(N + M') = RE(N)$. Also, $M' \cap \text{rad}_M(N) \subseteq M' \subseteq RE(N + M') = RE(N)$. Thus, M' satisfies (I_1) for radicals and the result follows from (3.3) and (3.2). \square

For every prime ideal \mathfrak{P} of R and every submodule N of M , let

$$\mathcal{F}_{\mathfrak{P}}(N) = \{m \in M \mid sm \in N \text{ for some } s \in R \setminus \mathfrak{P}\}.$$

Evidently $\mathcal{F}_{\mathfrak{P}}(N) = N_{\mathfrak{P}} \cap M$, where $N_{\mathfrak{P}}$ is the localisation of N at \mathfrak{P} .

The proof of the following lemma follows from [11, Proposition 2.5].

LEMMA 3.5. *For every submodule N of M , we have $\text{rad}_M(N) = \bigcap \mathcal{F}_{\mathfrak{P}}(N + \mathfrak{P}M)$, where \mathfrak{P} runs through all the prime ideals of R .*

A *secondary module* M is a non-zero module in which for every $r \in R$, either $rM = M$ or $r^n M = 0$ for some $n \in \mathbb{N}$. If M is secondary, then $\mathfrak{P} = \sqrt{(0 : M)}$ is a prime ideal of R and we say that M is \mathfrak{P} -secondary.

A *minimal secondary representation* of M is an equation of the form $M = \sum_{i=1}^n M_i$, where each M_i is \mathfrak{P}_i -secondary for distinct \mathfrak{P}_i s and none of the M_i s in the sum is redundant. If M has such a representation, then M is called *representable*. By [12, Theorem 6.9], the set $\{\mathfrak{P}_i \mid 1 \leq i \leq n\}$ is independent of the specific representation and it is denoted by $\text{Att}(M)$.

LEMMA 3.6. *Let M' be a \mathfrak{P} -secondary submodule of M . Then $\mathfrak{P}M' + (M' : \mathfrak{P}M) \subseteq RE(0)$.*

Proof. We prove that the generators of the left-hand side are in $RE(0)$. Let $p \in \mathfrak{P}$, $m \in M$, $m' \in M'$ and $r \in (M' : \mathfrak{P}M)$. Then $p^k M' = 0$ for some $k \in \mathbb{N}$. Thus, $pm' \in RE(0)$. Also, since $rpm \in M'$, $p^{k+1}(rm) \in p^k M' = 0$ and hence $p(rm) \in RE(0)$. \square

THEOREM 3.7. *Suppose that M' is a \mathfrak{P} -secondary submodule of M , $0 \neq k \in \mathbb{N}^*$ and $\frac{M}{M'}$ s.t.r.f. (weakly s.t.r.f.) of degree k . Then M s.t.r.f. (weakly s.t.r.f.) of degree k if one of the following holds:*

- (a) $k \geq 2$ and $(M' : \mathfrak{P}M) \not\subseteq \mathfrak{P}$.
- (b) $k = 1$ and $(\mathfrak{P}M' : \mathfrak{P}M) \not\subseteq \mathfrak{P}$.

Proof. By (3.3) and (3.2), in each case it suffices to show that M' satisfies (I_k) for radicals. So let N be an arbitrary submodule of M . First we show that in both cases $RE_k(N + M') = RE_k(N) + M'$. We just need to prove this for $k = 1$, and for $k > 1$ it is proved inductively.

Let $x \in E(N + M')$. Then $x = rx'$, where $r \in R$ and $x' \in M$ and there exist $t \in \mathbb{N}$, $n \in N$ and $m \in M'$ such that $r^t x' = n + m$. We will find an $m' \in M'$ such that $r(x' - m') \in E(N)$. If $r \notin \mathfrak{P}$, by M' being \mathfrak{P} -secondary, there is an $m' \in M'$ such that $r^t m' = m$. Thus, $r^t(x' - m') = n \in N$ and this m' has the required property. If $r \in \mathfrak{P}$, then there is a $t' \in \mathbb{N}$ such that $r^{t'} M = 0$. In this case we choose $m' = 0$, therefore in this case $r^{t+t'}(x' - m') = r^{t'} n \in N$ and $r(x' - m') \in E(N)$. Whence, in both cases $x = r(x' - m') + rm' \in RE(N) + M'$. Therefore, every generator of $RE(N + M')$ is in $RE(N) + M'$ and thus $RE(N + M') \subseteq RE(N) + M'$. The converse inclusion is obvious.

Now in each of the above cases we must prove that $M' \cap \text{rad}(N) \subseteq RE_k(N)$.

Let $x_1 \in M' \cap \text{rad}_M(N)$. By (3.5), $x_1 \in \mathcal{F}_{\mathfrak{P}}(N + \mathfrak{P}M)$ and hence there exist $r_1 \in R \setminus \mathfrak{P}$, $n \in N$ and $m \in \mathfrak{P}M$ such that $r_1 x_1 = n + m$.

(a) By the assumption of this case of the theorem, there exists an $r' \in (M' : \mathfrak{M}M) \setminus \mathfrak{P}$. If we set $r'' = r_1 r'$, then $r'' \in R \setminus \mathfrak{P}$ and $r'' x_1 = r' n + r' m \in N + (M' : \mathfrak{P}M)\mathfrak{P}M$. Thus, by (3.6) $r'' x_1 \in RE(N)$. Now, because $r'' \notin \mathfrak{P}$ and $x_1 \in M'$, there is a $y \in M'$ such that $r'' y = x_1$. Therefore, $r''^2 y = r'' x_1 \in RE(N)$ and so $x_1 = r'' y \in RE_2(N) \subseteq RE_k(N)$. This means $M' \cap \text{rad}(N) \subseteq RE_k(N)$ and hence (I_k) for radicals holds for M' , which completes the proof of this case.

(b) By our assumption, there exists an $r' \in (\mathfrak{P}M' : \mathfrak{P}M) \setminus \mathfrak{P}$. If we set $r'' = r' r_1$, then $r'' x_1 = r' n + r' m$ and $r' m \in \mathfrak{P}M'$. Put $m' = r' m$. Then $m' = \sum_{i=1}^t p_i m'_i$, where $p_i \in \mathfrak{P}$ and $m'_i \in M'$. As $r'' \notin \mathfrak{P}$, for each $i \leq t$, there exists $m''_i \in M'$ such that $m'_i = r'' m''_i$. Let $m'' = \sum_{i=1}^t p_i m''_i \in \mathfrak{P}M'$, which is in $RE(N)$ by (3.6). Now $r'' m'' = m'$ and whence $r''(x_1 - m'') = r'' x_1 - m' = r' n \in N$. Moreover, $x_1 - m'' \in M'$, thus $x_1 - m'' = r'' y$ for some $y \in M'$, as $r'' \notin \mathfrak{P}$. Therefore, $r''^2 y = r''(x_1 - m'') \in N$ and so $x_1 - m'' = r'' y \in RE(N)$, and as $m'' \in RE(N)$, we have $x_1 \in RE(N)$. This means $M' \cap \text{rad}(N) \subseteq RE(N)$, which completes the proof. □

Note that if R is an integral domain, then every divisible R -module is 0-secondary. Thus, (3.7) generalises [15, Theorem 2.2], which states if M is a direct sum of a divisible module and a module which s.t.r.f., then M s.t.r.f. Now we will state another application of (3.7).

PROPOSITION 3.8. Let $n \in \mathbb{N}$, and suppose for each $1 \leq i \leq n$, M_i is a \mathfrak{P}_i secondary R -module such that $M_i = M_j$ if $\mathfrak{P}_i = \mathfrak{P}_j$. If \mathfrak{J} is an ideal of R with $\mathfrak{J} \not\subseteq \cup_{i=1}^n \mathfrak{P}_i$, then $M = M_1 \oplus M_2 \oplus \dots \oplus M_n \oplus \frac{R}{\mathfrak{J}}$ s.t.r.f.

Proof. We use induction on n . As an application of the previous theorem, we have the result for $n = 1$. Now suppose the result is correct for $n - 1$.

By rearranging the indices, we may suppose that \mathfrak{P}_1 is a minimal element of the set $\{\mathfrak{P}_1, \mathfrak{P}_2, \dots, \mathfrak{P}_n\}$. Then for each $1 \leq i \leq n$, either $\mathfrak{P}_i = \mathfrak{P}_1$ or $\mathfrak{P}_i \not\subseteq \mathfrak{P}_1$. So by our assumption either $M_i = M_1$ or $\mathfrak{P}_i \not\subseteq \mathfrak{P}_1$. Put $M'' = M_2 \oplus \dots \oplus M_n \oplus \frac{R}{\mathfrak{J}}$. We show that $(\mathfrak{P}_1 M_1 : \mathfrak{P}_1 M) \not\subseteq \mathfrak{P}_1$, and by induction hypothesis $\frac{M}{M_1} \cong M''$ s.t.r.f., therefore by the previous theorem, M s.t.r.f.

For each $1 \leq i \leq n$, if $M_i = M_1$, then put $t_i = 1$. Otherwise consider $r_i \in \mathfrak{P}_i \setminus \mathfrak{P}_1$. Then $r_i \in \mathfrak{P}_i = \sqrt{(0 : M_i)}$, and so there exists $n_i \in \mathbb{N}$ such that $r_i^{n_i} M_i = 0$. Thus, $t_i = r_i^{n_i} \in (\mathfrak{P}_1 M_1 : \mathfrak{P}_1 M_i) \setminus \mathfrak{P}_1$. Let $i \in \mathfrak{J} \setminus \mathfrak{P}_1$, and put $r = t_1 t_2 t_3 \dots t_n$. Then $r \in (\mathfrak{P}_1 M_1 : \mathfrak{P}_1 M_i)$, for each $1 \leq i \leq n$, and whence $r \in (\mathfrak{P}_1 M_1 : \mathfrak{P}_1 M) \setminus \mathfrak{P}_1$, which completes the proof. □

LEMMA 3.9. *Suppose that $M' = \sum_{i=1}^t M_i$ is a minimal secondary representation of M' , where M_i is \mathfrak{P}_i -secondary. Also assume that \mathfrak{P}_1 is minimal in $\text{Att}(M)$. Let \mathfrak{J} be an ideal of R and N a submodule of M . Then $(\mathfrak{J}M' : N) \not\subseteq \mathfrak{P}_1$, if and only if $(\mathfrak{J}M_1 : N) \not\subseteq \mathfrak{P}_1$.*

Proof. Assume that there exists an $r \in (\mathfrak{J}M' : N) \setminus \mathfrak{P}_1$. If $t = 1$, then the proof of the claim is trivial. So assume that $t > 1$. By minimality of \mathfrak{P}_1 and minimality of the representation, no \mathfrak{P}_i is a subset of \mathfrak{P}_1 , for $i \neq 1$. Thus, $\mathfrak{P}_2\mathfrak{P}_3 \cdots \mathfrak{P}_t \not\subseteq \mathfrak{P}_1$. Let $r' \in \mathfrak{P}_2 \cdots \mathfrak{P}_t \setminus \mathfrak{P}_1$. Then for each $2 \leq i \leq t$, there is a $k_i \in \mathbb{N}$ such that $r'^{k_i} M_i = 0$. Set $k = \max_{i=2}^t \{k_i\}$. Since $r' \notin \mathfrak{P}_1$, we have $r'^k M' = M_1$. Now $r'^k r N \subseteq \mathfrak{J}r'^k M = \mathfrak{J}M_1$ and $r'^k r \notin \mathfrak{P}_1$. The other side of the claim is obvious. □

LEMMA 3.10. *If M_1 and M_2 are submodules of M such that $M_1 \not\subseteq M_2$ and M_1 is \mathfrak{P} -secondary, then $\frac{M_1+M_2}{M_2}$ is \mathfrak{P} -secondary.*

Proof. The proof is easy and it is left to the reader. □

THEOREM 3.11. *Assume that $M' = \sum_{i=1}^t M_i$ is a minimal secondary representation of M' , where M_i is \mathfrak{P}_i -secondary. Also suppose that $\frac{M'}{M}$ s.t.r.f. (weakly s.t.r.f.) of degree k . Then M s.t.r.f. (weakly s.t.r.f.) of degree k , if one of the following holds:*

- (a) $k \geq 2$ and $(M' : \mathfrak{P}M) \not\subseteq \mathfrak{P}$, for each maximal element \mathfrak{P} of $\text{Att}(M)$.
- (b) $k = 1$ and $(\mathfrak{P}M' : \mathfrak{P}M) \not\subseteq \mathfrak{P}$, for each maximal element \mathfrak{P} of $\text{Att}(M)$.

Proof. We state the proof for s.t.r.f., the claim for weakly s.t.r.f. is proved similarly.

(a) We use induction on t . The base step is true by (3.7)(a). One can easily check that our assumption implies $(M' : \mathfrak{P}M) \not\subseteq \mathfrak{P}$, for every element \mathfrak{P} of $\text{Att}(M)$.

Suppose \mathfrak{P}_1 is a minimal element of $\text{Att}(M)$. Thus, by applying (3.9) with $\mathfrak{J} = R$ and $N = \mathfrak{P}_1 M$, we see that $(M_1 : \mathfrak{P}_1 M) \not\subseteq \mathfrak{P}_1$. So by (3.7)(a), it suffices to prove $\frac{M'}{M_1}$ s.t.r.f. of degree k .

By (3.10), $\frac{M'}{M_1} = \sum_{i=2}^t \frac{M_i+M_1}{M_1}$ is a secondary representation of $\frac{M'}{M_1}$. Also, since

$$\frac{\frac{M}{M_1}}{\frac{M'}{M_1}} \cong \frac{M}{M'} \quad \text{and} \quad \left(\frac{M'}{M_1} : \mathfrak{P} \frac{M}{M_1} \right) \supseteq (M' : \mathfrak{P}M),$$

for every $\mathfrak{P} \in \text{Att}(M)$, we can apply the induction hypothesis and get $\frac{M}{M_1}$ s.t.r.f. of degree k as required.

(b) Similar to (a), just use (3.7)(b) instead of (3.7)(a) and apply (3.9) with $\mathfrak{J} = \mathfrak{P}_1$ instead of R . □

In [1], it is proved that if R is an integral domain with the Krull dimension one, then every Noetherian representable module s.t.r.f. ([1, Theorem 1.7]). The following corollary generalises this result.

COROLLARY 3.12. *Every representable module s.t.r.f.*

Proof. Just apply (3.11), with $M' = M$, where M is a representable module. □

According to [15] an R -module M is called *special* when for each maximal ideal \mathfrak{M} of R , each $a \in \mathfrak{M}$ and each $m \in M$, there exist $c \in R \setminus \mathfrak{M}$ and $k \in \mathbb{N}$ such that $ca^k m = 0$. *Semi-simple modules* (direct sum of simple modules), *locally Artinian modules* (modules in which every cyclic submodule is Artinian) and *semi-Artinian modules* (modules every homomorphic image of which has a non-zero simple submodule) are special (see [15, Section 3]).

It was proved there that if $M = M' \oplus M''$, where M' is special and M'' s.t.r.f., then M s.t.r.f. ([15, Theorem 4.8]). Here we will generalise this to s.t.r.f. (weakly s.t.r.f.) of degree k . Let \mathfrak{M} be a maximal ideal of R and $k \in \mathbb{N}^*$, consider the following property on a submodule M' of M :

(III_k) For every submodule N of M , $RE_k(N + \mathfrak{M}M') \subseteq \mathcal{F}_{\mathfrak{M}}(RE_k(N))$.

LEMMA 3.13. *Let \mathfrak{M} be a maximal ideal of R and M' a submodule of M .*

- (a) *For each submodule N of M , $RE(\mathcal{F}_{\mathfrak{M}}(N)) \subseteq \mathcal{F}_{\mathfrak{M}}(RE(N))$.*
- (b) *If M' satisfies (III₁), then it satisfies (III_k) for each $0 \neq k \in \mathbb{N}^*$.*

Proof. (a) It suffices to show that $E(\mathcal{F}_{\mathfrak{M}}(N)) \subseteq \mathcal{F}_{\mathfrak{M}}(RE(N))$, which one can easily prove.

(b) We prove (III_k) by induction on k . Let N be an arbitrary submodule of M . For $k > 1$, if $N' = RE_{k-1}(N + \mathfrak{M}M')$, then by induction hypothesis we have $N' \subseteq \mathcal{F}_{\mathfrak{M}}(RE_{k-1}(N))$. Now $RE_k(N + \mathfrak{M}M') = RE(N') \subseteq RE(N' + \mathfrak{M}M') \subseteq \mathcal{F}_{\mathfrak{M}}(RE(N')) \subseteq \mathcal{F}_{\mathfrak{M}}(RE(\mathcal{F}_{\mathfrak{M}}(RE_{k-1}(N))))$. Thus, by part (a) of this lemma, $RE_k(N + \mathfrak{M}M') \subseteq \mathcal{F}_{\mathfrak{M}}(\mathcal{F}_{\mathfrak{M}}(RE_k(N))) = \mathcal{F}_{\mathfrak{M}}(RE_k(N))$. The case of $k = \infty$ follows the definition of $RE_{\infty}(N)$ and the case of $k \in \mathbb{N}$. □

LEMMA 3.14. *Let $k \in \mathbb{N}^*$. If $M = M' \oplus M''$, where M' is semi-simple and M'' s.t.r.f. (weakly s.t.r.f.) of degree k , then M s.t.r.f. (weakly s.t.r.f.) of degree k .*

Proof. In [15], through a series of lemmas, the s.t.r.f. case was proved for $k = 1$. In the same way, one can prove the corresponding lemmas for each $k \in \mathbb{N}^*$ and weak radicals, and hence prove the claim. □

LEMMA 3.15. *If M' is special, then for every maximal ideal \mathfrak{M} of R , (III₁) holds for M' .*

Proof. This is in fact [15, Lemma 4.7]. □

THEOREM 3.16. *Let $k \in \mathbb{N}^*$. If for every maximal ideal \mathfrak{M} of R , a submodule M' of M satisfies (III_k) and $\frac{M}{\mathfrak{M}M'}$ s.t.r.f. of degree k , then M s.t.r.f. of degree k . In particular, if $k \neq 0$ and $M = M' \oplus M''$, where M' is special and M'' s.t.r.f. of degree k , then M s.t.r.f. of degree k . The statement also holds if we replace s.t.r.f. with weakly s.t.r.f.*

Proof. Let N be an arbitrary submodule of M , $x \in \text{rad}(N)$ and $\mathcal{J} = (RE_k(N) : Rx)$. We must show that $R = \mathcal{J}$. If $\mathcal{J} \neq R$, then it is contained in a maximal ideal of R , say \mathfrak{M} . By our assumption, $\frac{M}{\mathfrak{M}M'}$ s.t.r.f. of degree k , hence by (3.1), for every maximal ideal \mathfrak{M} of R , $\text{rad}(N + \mathfrak{M}M')/\mathfrak{M}M' = RE_k(N + \mathfrak{M}M')/\mathfrak{M}M'$, whence

$$x \in \text{rad}(N) \subseteq \text{rad}(N + \mathfrak{M}M') = RE_k(N + \mathfrak{M}M') \subseteq \mathcal{F}_{\mathfrak{M}}(RE_k(N)).$$

This means that there exists an $s \in R \setminus \mathfrak{M}$ such that $sx \in RE_k(N)$. But then $s \in \mathcal{J} \subseteq \mathfrak{M}$, a contradiction. Thus, M s.t.r.f. of degree k .

Now suppose that $k \neq 0$ and $M = M' \oplus M''$, where M' is an special module and M'' s.t.r.f. of degree k . Let \mathfrak{M} be an arbitrary maximal ideal of R . According to (3.15) and (3.13), M' satisfies (III_k), for \mathfrak{M} . Thus, to complete the proof, we just need to show that $\frac{M}{\mathfrak{M}M'}$ s.t.r.f. of degree k . But $\frac{M}{\mathfrak{M}M'} \cong \frac{M'}{\mathfrak{M}M'} \oplus M''$ and $\frac{M'}{\mathfrak{M}M'}$ is semi-simple. Thus, the result follows from (3.14). The assertion on satisfying the weak radical formula can be proved similarly. □

Next we state a condition on M' under which M s.t.r.f. of a higher degree than $\frac{M}{M'}$ does. Recall that if $\mathcal{C} : M_1 \supseteq M_2 \supseteq \dots$ is a chain of submodules of M , then the

completion \widehat{M} of M with respect to \mathcal{C} is the set $\{[(x_i)] \mid (x_i) \text{ is a Cauchy sequence in } M\}$, where $[(x_i)]$ denotes the equivalency class of the sequence (x_i) with respect to the relation $(x_i) \equiv (y_i)$ if and only if for each $n \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that for all $n_0 \leq i$ we have $x_i - y_i \in M_n$ (see [2, Section 10]). In the particular case that $M_i = \mathfrak{I}^i M$, for an ideal \mathfrak{I} of R , \widehat{M} is called the \mathfrak{I} -adic completion of M .

Clearly \widehat{M} forms a module with element-wise operations. In the following, for each submodule N of M , we set $\widehat{N} = \{[(x_i)] \in \widehat{M} \mid [(x_i)] = [(y_i)] \text{ for some } y_i\text{'s in } N\}$, which is a submodule of \widehat{M} . Moreover, by \widehat{m} for an $m \in M$ we mean $[(m)] \in \widehat{M}$, where (m) denotes a constant sequence, all terms of which equals m . Also, we denote the natural homomorphism from M to \widehat{M} , which maps m to \widehat{m} , by \widehat{f} .

THEOREM 3.17. *Suppose that $M_1 \supseteq M_2 \supseteq \dots$ is a chain of submodules of M and \widehat{M} is the completion of M with respect to this chain. Also let N be a submodule of M .*

- (a) $\widehat{f}(\text{rad}_M(N)) \subseteq \text{rad}_{\widehat{M}}(\widehat{N})$ and if $M_l \subseteq \text{rad}_M(N)$, for some $l \in \mathbb{N}$, then $\text{rad}_{\widehat{M}}(\widehat{N}) \subseteq \widehat{\text{rad}_M(N)}$.
- (b) $\widehat{f}(E_M(N)) \subseteq E_{\widehat{M}}(\widehat{N})$ and if $M_l \subseteq RE_k(N)$, for some $l \in \mathbb{N}$ and $k \in \mathbb{N}^*$, then $E_{\widehat{M}}(\widehat{N}) \subseteq RE_{k+1}(N)$.
- (c) If $M_l \subseteq RE_k(N)$, for some $l \in \mathbb{N}$ and $k, k' \in \mathbb{N}^*$ and \widehat{M} s.t.r.f. of degree k' , then $\frac{M}{N}$ s.t.r.f. of degree $k + k'$.

Proof. (a) Let $x \in \text{rad}(N)$. Then by (3.5), for every prime ideal \mathfrak{P} of R there exists an $s \in R \setminus \mathfrak{P}$ such that $sx \in N + \mathfrak{P}M$, say $sx = m + \sum_{i=1}^t p_i m_i$, for some m_i 's in M , p_i 's in \mathfrak{P} and $m \in N$. Thus, $s\widehat{x} = \widehat{m} + \sum_{i=1}^t p_i \widehat{m}_i \in \widehat{N} + \mathfrak{P}\widehat{M}$. Therefore, according to (3.5) $\widehat{x} \in \text{rad}_{\widehat{M}}(\widehat{N})$.

Now suppose that $M_l \subseteq \text{rad}(N)$ and $[(x_i)] \in \text{rad}_{\widehat{M}}(\widehat{N})$. Then for each prime ideal \mathfrak{P} of R there is an $s \in R \setminus \mathfrak{P}$ such that $s[(x_i)] \in \widehat{N} + \mathfrak{P}\widehat{M} \subseteq \widehat{N} + \mathfrak{P}M$. So there are n_i 's in N and m_i 's in $\mathfrak{P}M$ such that $s[(x_i)] = [(n_i)] + [(m_i)]$. Hence, if i is large enough we have $sx_i = n_i + m_i + \delta_i$, where $\delta_i \in M_l \subseteq \text{rad}(N)$. Consequently, for some $s_i \in R \setminus \mathfrak{P}$ we have $s_i \delta_i \in N + \mathfrak{P}M$. Thus, $s s_i x_i \in N + \mathfrak{P}M$ and since \mathfrak{P} was arbitrary, we deduce that $x_i \in \text{rad}_M(N)$ for all large enough i s and whence $[(x_i)] \in \widehat{\text{rad}_M(N)}$.

(b) It is obvious that $\widehat{f}(E_M(N)) \subseteq E_{\widehat{M}}(\widehat{N})$. So suppose that $M_l \subseteq RE_k(N)$ and let $[(x_i)] \in E_{\widehat{M}}(\widehat{N})$. Then for some $r \in R$, $[(m_i)] \in \widehat{M}$, $[(n_i)] \in \widehat{N}$ and $t \in \mathbb{N}$, we have $[(x_i)] = r[(m_i)]$ and $r^t[(m_i)] = [(n_i)]$. Therefore, for large enough i s we have $r^t m_i - n_i \in M_l \subseteq RE_k(N)$. Thus, $r^t m_i \in RE_k(N)$ and hence $r m_i \in RE_{k+1}(N)$. But for large enough i s $x_i = r m_i + \delta_i$, where $\delta_i \in M_l \subseteq RE_k(N)$. From this we conclude that $x_i \in RE_{k+1}(N)$ for all large enough i s, and the result follows.

(c) Let A be a submodule of M containing N and $x \in \text{rad}_M(A)$. If $k' > 0$, then $M_l \subseteq RE_k(N) \subseteq RE_k(A) \subseteq \text{rad}(A)$. Therefore, by (a), $\widehat{x} \in \text{rad}_{\widehat{M}}(\widehat{A}) = RE_{k'}(\widehat{A}) = RE_{k'-1}(RE(\widehat{A}))$, which according to (b) is contained in $RE_{k'-1}(RE_{k+1}(A))$. Now if we put $B = RE_{k+1}(A)$, then $M_l \subseteq B = RE_0(B)$. So by repeated use of (b) (with 0 as the k in (b)), we get

$$RE_{k'-1}(\widehat{B}) = RE_{k'-2}(RE(\widehat{B})) \subseteq RE_{k'-2}(\widehat{RE(B)}) \subseteq \dots \subseteq \widehat{RE_{k'-1}(B)} = \widehat{RE_{k'+k}(A)}.$$

Therefore, $\widehat{x} = [(y_i)]$ for some y_i 's in $RE_{k'+k}(A)$, and hence for a large enough i , we have $x - y_i \in RE_k(N)$. Consequently, $x \in RE_{k'+k}(A)$, as required. If $k' = 0$, then $\widehat{x} \in \widehat{A}$ and by an argument similar to that of (b), we get $x \in RE_k(A)$. □

COROLLARY 3.18. *Let \mathfrak{J} be a finitely generated ideal of R and $k, k' \in \mathbb{N}^*$. If $\mathfrak{J} \subseteq \sqrt{(RE_k(0) : M)}$ and the \mathfrak{J} -adic completion of M (as an R -module) s.t.r.f. of degree k' , then M s.t.r.f. of degree $k + k'$.*

Proof. Note that since \mathfrak{J} is finitely generated, for some $n \in \mathbb{N}$ we have $\mathfrak{J}^n \subseteq (RE_k(0) : M)$. Hence, the result follows from (3.17). □

PROPOSITION 3.19. *If $M' \subseteq RE_k(0)$ and $\frac{M}{M'}$ s.t.r.f. (weakly s.t.r.f.) of degree k' , where $k, k' \in \mathbb{N}^*$, then M s.t.r.f. (weakly s.t.r.f.) of degree $k + k'$.*

Proof. Noting that for every submodule N of M , we have $M' \subseteq RE_k(0) \subseteq RE_k(N) \subseteq \text{wrad}(N) = \text{wrad}(RE_k(N))$, the result for weak radicals follows (3.1). The case on radicals can be proved similarly or can be deduced from (3.17), setting $M_i = M'$ for all $i \in \mathbb{N}$. □

COROLLARY 3.20. *Let R be a one-dimensional ring.*

- (a) *If R is a domain and $(RE_k(0) : m) \neq 0$ for each $m \in M$, then M s.t.r.f. of degree $k + 1$. Moreover, if $(RE_k(0) : M)$ is a non-zero radical ideal, then M s.t.r.f. of degree k .*
- (b) *If $(RE_k(0) : M)$ is a prime ideal, then M weakly s.t.r.f. of degree $k + 1$.*

Proof. (a) By our hypothesis, $\frac{M}{RE_k(0)}$ is a torsion module. By [15, Corollary 3.6], every torsion module over a one-dimensional domain is special and hence s.t.r.f. Now use (3.19). For the second part, note that by [16, Theorem 2.4] if $(RE_k(0) : M)$ is a non-zero radical ideal, then $\frac{M}{RE_k(0)}$ s.t.r.f. of degree 0.

(b) This follows from (3.19) and [5, Corollary 3.3], which states that every one-dimensional domain weakly s.t.r.f. □

The following corollary is proved in [14] but we bring it here for completeness.

COROLLARY 3.21. *Assume that R is a Noetherian ring. The following are equivalent.*

- (a) *R has the s.p.a.r. property.*
- (b) *R s.t.r.f. of degree k , for some $2 \leq k \in \mathbb{N}^*$.*
- (c) *R s.t.r.f. of degree 2.*
- (d) *R is an Artinian ring, or all of the following conditions are satisfied, where $\mathfrak{P}_1, \dots, \mathfrak{P}_n$ are all of the minimal ideals of R :*
 - (1) *$\dim R = 1$ and for each i , $\frac{R}{\mathfrak{P}_i}$ is a Dedekind domain.*
 - (2) *If $n \geq 2$, then for each $k = 1, 2, \dots, n - 1$, $(\bigcap_{i=1}^k \mathfrak{P}_i) + \mathfrak{P}_{k+1} = \bigcap_{i=1}^k (\mathfrak{P}_i + \mathfrak{P}_{k+1})$.*
 - (3) *If $n \geq 2$, then for each i, j , $1 \leq i < j \leq n$, $R = \mathfrak{P}_i + \mathfrak{P}_j$, or $\frac{R}{\mathfrak{P}_i + \mathfrak{P}_j}$ is a semi-simple Artinian ring.*

Proof. As it was noted in the Introduction, R has the s.p.a.r. property if and only if R s.t.r.f. of degree ∞ . So (b) \Rightarrow (a) and also (c) \Rightarrow (b), are clear. Thus, we just need to show that if M has the s.p.a.r. property, then M s.t.r.f. of degree 2. Suppose that R has the s.p.a.r. property and M is an arbitrary R -module. Now $N(R)M \subseteq RE(0)$, where $N(R)$ stands for the nil radical of R , and hence $\frac{M}{RE(0)}$ is an $\frac{R}{N(R)}$ -module. Since every $\frac{R}{N(R)}$ -module is an R -module, $\frac{R}{N(R)}$ has the s.p.a.r. property and hence by (2.4), s.t.r.f. Thus, $\frac{M}{RE(0)}$ s.t.r.f. and by (3.19) M s.t.r.f. of degree 2.

(a) \Leftrightarrow (d) The proof is given by [10, Theorem 1.1]. □

EXAMPLE 3.22. In [10, p. 33] it was noted that $R = \frac{F[x,y]}{\langle x^2, y^2 \rangle}$ has the s.p.a.r. property but does not s.t.r.f. So by the previous corollary R s.t.r.f. of degree 2 but does not s.t.r.f. of degree 1.

At the end of this paper we turn our attention to a stronger form of radical formula. In [6] we stated that M satisfies the simplified radical formula (s.t.s.r.f.) if $\text{rad}_M(N) = E_M(N) + N$, for each submodule N of M . If every R -module s.t.s.r.f., then we say that R s.t.s.r.f. In [6, Theorem 2.15], it is proved that a Noetherian ring s.t.s.r.f. if and only if it is a ZPI-ring. Here we consider the generalisation of the above concept.

DEFINITION 3.23. Let $k \in \mathbb{N} \cup \{0\}$. If $\text{rad}_M(N) = kE_M(N) + N$ ($\text{wrad}_M(N) = kE_M(N) + N$) for every submodule N of M then we say that M satisfies (weakly satisfies) the simplified radical formula of degree k (s.t.s.r.f. (weakly s.t.s.r.f.) of degree k). Also we say that R satisfies (weakly satisfies) the simplified radical formula of degree k , when every R module does so.

Recall that $E_M(N)$ is not necessarily a submodule of M , and $kE_M(N)$ in the above definition is

$$kE_M(N) = \{x \in M \mid \exists x_1, x_2, \dots, x_k \in E_M(N) \ni x = x_1 + x_2 + \dots + x_k\}.$$

In [5] we have proved that a valuation domain of the Krull dimension k , s.t.s.r.f. of degree k but not of $k - 1$ and also we have shown that if R s.t.s.r.f. of degree k , then $\dim R \leq k$.

LEMMA 3.24.

- (a) Let $K \subseteq N$ be submodules of M . Then $\text{rad}_{\frac{M}{K}}\left(\frac{N}{K}\right) = kE_{\frac{M}{K}}\left(\frac{N}{K}\right) + \frac{N}{K}$ if and only if $\text{rad}_M(N) = kE_M(N) + N$.
- (b) Let $k \in \mathbb{N}^*$. If $M = M' \oplus M''$, where M' is semi-simple and M'' s.t.s.r.f. (weakly s.t.s.r.f.) of degree k , then M s.t.s.r.f. (weakly s.t.s.r.f.) of degree k .

Proof. (a) is easy and (b) can be proved similar to [15, Lemma 4.5], using (a) instead of [15, lemma 4.1]. □

A ring R is called weakly Bezout when every finitely generated proper ideal of R is contained in a proper principal ideal (see [6]).

THEOREM 3.25. Let (R, \mathfrak{M}) be a local weakly Bezout ring, $k \in \mathbb{N} \cup \{0\}$ and $M = M' \oplus M''$, where M' and M'' are submodules of M . If M' is a special module and M'' s.t.s.r.f. (weakly s.t.s.r.f.) of degree k , then M s.t.s.r.f. (weakly s.t.s.r.f.) of degree $k + 1$.

Proof. First suppose that $x \in \mathfrak{M}M'$, say $x = \sum_{i=1}^t r_i m_i$ for some r_i s in \mathfrak{M} and some m_i s in M' . Then $\mathfrak{J} = \langle r_1, r_2, \dots, r_n \rangle \subseteq \mathfrak{M}$, and since R is weakly Bezout, $\mathfrak{J} \subseteq Rr$ for some $r \in \mathfrak{M}$, say $r_i = a_i r$ for some a_i s in R . Then $x = r \sum_{i=1}^t a_i m_i$ and because M' is special and $r \in \mathfrak{M}$, there is a $t' \in \mathbb{N}$ such that $r^{t'} \sum_{i=1}^t a_i m_i = 0$. Therefore, $\mathfrak{M}M' \subseteq E_M(0)$.

Since $M = M' \oplus M''$, $\frac{M}{\mathfrak{M}M'} \cong \frac{M'}{\mathfrak{M}M'} \oplus M''$. As $\frac{M'}{\mathfrak{M}M'}$ is semi-simple, by (3.24)(b) $\frac{M}{\mathfrak{M}M'}$ s.t.s.r.f. of degree k . Consequently if N is a submodule of M , then $\text{rad}(N) \subseteq \text{rad}(N + \mathfrak{M}M') = kE(N + \mathfrak{M}M') + N + \mathfrak{M}M'$. We will prove that $E(N + \mathfrak{M}M') \subseteq E(N) + \mathfrak{M}M'$. Then $\text{rad}(N) \subseteq kE(N) + N + \mathfrak{M}M' \subseteq (k + 1)E(N) + N$ and the result follows.

Let $x \in E(N + \mathfrak{M}M')$. Then $x = rm$ and $r'm = n + r'm'$ for some $r \in R$, $r' \in \mathfrak{M}$, $n \in N$, $m \in M$, $m' \in M'$ and $t \in \mathbb{N}$. If r is a unit, then $x \in N + \mathfrak{M}M'$ as required. Suppose that $r \in \mathfrak{M}$. Note that M' is a special module, then there exists $t' \in \mathbb{N}$ with $r^{t'}m' = 0$. Now $r^{t+t'}m = r^t n + r^t r^{t'}m' \in N$, whence $x = rm \in E(N)$, which completes the assertion. The case for weak radicals is quite similar. \square

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