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A NOTE ON A CONTINUED FRACTION OF RAMANUJAN

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Ramanujan recorded many beautiful continued fractions in his notebooks. In this paper, we derive several identities involving the Ramanujan continued fraction $c(q)$, including relations between $c(q)$ and $c(q^n)$. We also obtain explicit evaluations of $c(e^{-\pi\sqrt{n}})$ for various positive integers n.

1. Introduction

The celebrated Rogers–Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1+}\ \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+\cdots}, \quad |\, q \,|< 1\,. \tag{1.1}$$

In his first two letters to Hardy [12, pp. xxvii, xxviii], Ramanujan communicated several theorems about $R(q)$ and $S(q) := -R(-q)$. In these two letters, Ramanujan claimed that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\,,$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}\,.$$

Ramanujan recorded other values for $R(q)$ and $S(q)$ in his first notebook [11] and in his 'lost' notebook [13]. Most of these results were proved by Ramanathan [8, 9, 10], Berndt and Chan [5]. On page 365 of his 'lost' notebook, Ramanujan wrote five modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$ and $R(q^5)$. Ramanujan eventually found several generalisations and ramifications of (1.1) which are recorded in his 'lost' notebook.

Recently, Berndt and Chan [5], Chan[6], Chan and Huang [7], Adiga, Vasuki and Naika[2] and Adiga, Vasuki and Shivashankara [3] have established several new interesting evaluations of the Rogers–Ramanujan continued fraction, Ramanujan's cubic continued fraction, and the Ramanujan–Göllnitz–Gordon continued fraction.

Motivated by these works, in this paper we study the Ramanujan continued fraction

$$c(q) := \frac{1}{1+}\ \frac{2q}{1-q^2+}\ \frac{q^2(1+q^2)^2}{1-q^6+}\ \frac{q^4(1+q^4)^2}{1-q^{10}+\cdots}, \qquad |\, q \,|< 1\,. \tag{1.2}$$

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In Section 2, we obtain a product representation for $c(q)$. In Section 3, we deduce some basic identities satisfied by $c(q)$ and also establish relations between $c(q)$ and $c(-q)$ and $c(q)$ and $c(q^2)$.

In Section 4, we derive relations between $c(q)$ and $c(q^n)$ using modular equations of degree n. In the final section we shall establish two formulas for evaluating $c(e^{-\pi\sqrt{n}})$ in terms of Ramanujan–Weber class invariants. We use these formulas to evaluate explicitly $c(e^{-\pi\sqrt{n}})$ for various positive integers n.

2. Product representation for $c(q)$

Theorem 2.1. *Let $c(q)$ be as defined in (1.2). Then,*

$$c(q) = \frac{\phi(q^4)}{\phi(q)} ,$$

where

$$\phi(q) = \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}$$

and

$$(a)_\infty = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n) .$$

Proof: From Entry 11 of [**1**, p. 14], we have

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{(a-b)}{1-q+}\ \frac{(a-bq)(aq-b)}{1-q^3+}\ \frac{q(a-bq^2)(aq^2-b)}{1-q^5+\cdots} ,$$

which can be rewritten as

$$1+\frac{(-a)_\infty(b)_\infty}{(a)_\infty(-b)_\infty} = \frac{2}{1-}\ \frac{(a-b)}{1-q+}\ \frac{(a-bq)(aq-b)}{1-q^3+}\ \frac{q(a-bq^2)(aq^2-b)}{1-q^5+\cdots} . \tag{2.1}$$

Changing q to q^2, a to zq and b to $-q/z$ in (2.1) we obtain

$$1+\frac{(-zq;q^2)_\infty(-q/z;q^2)_\infty}{(zq;q^2)_\infty(q/z;q^2)_\infty} = \frac{2}{1-}\ \frac{zq+q/z}{1-q^2+}\ \frac{(zq+q^3/z)(zq^3+q/z)}{1-q^6+\cdots} . \tag{2.2}$$

Employing Entry 30(ii) [**1**, p. 43] in (2.2) we deduce that

$$\frac{f(z^2q^4,q^4/z^2)}{f(-zq,-q/z)} = \frac{1}{1-}\ \frac{zq+q/z}{1-q^2+}\ \frac{(zq+q^3/z)(zq^3+q/z)}{1-q^6+\cdots} , \tag{2.3}$$

where

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2} = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty .$$

Putting $z=-1$ in (2.3) and noting that $f(q,q)=\phi(q)$, we complete the proof. ∎

3. Some Identities involving $c(q)$

In Chapter 16 of his second notebook [**11**], Ramanujan develops the theory of Theta-function using his function $f(a,b)$ and its restrictions

$$\phi(q) := f(q,q) = \frac{(-q;-q)_\infty}{(q;-q)_\infty}$$

and

$$\psi(q) := f(q;q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\,.$$

Now we derive some identities involving $c(q)$.

Theorem 3.1. *We have,*

(i) $c^2(q) = \dfrac{\psi^4(q^4)}{\phi^2(q)\psi^2(q^8)}$,

(ii) $c^2(q) - c^2(-q) = \dfrac{-8q\psi^6(q^4)}{\psi^2(q^8)\phi^4(-q^4)}$,

(iii) $c^2(q).c^2(-q) = \frac{\phi^4(q^4)}{\phi^4(-q^2)}$, *and*

(iv) $\dfrac{c^{-1}(q) - c(q)}{c^{-1}(q) + c(q)} = \dfrac{\phi^2(q) - \phi^2(q^4)}{\phi^2(q) + \phi^2(q^4)}$.

Proof of (i): We have

$$c(q) = \frac{\phi(q^4)}{\phi(q)}\,. \tag{3.1}$$

Also from [**1**, p. 36, Entry 25(iv)], we have

$$\phi(q) = \frac{\psi^2(q)}{\psi(q^2)}\,,$$

and

$$\phi(q^4) = \frac{\psi^2(q^4)}{\psi(q^8)}\,. \tag{3.2}$$

Substituting (3.2) in (3.1) and squaring, we deduce that

$$c^2(q) = \frac{\psi^4(q^4)}{\phi^2(q)\psi^2(q^8)}\,.$$

This completes the proof of (i). □

Proof of (ii): From (i) we have,

$$c^2(q) = \frac{\psi^4(q^4)}{\phi^2(q)\psi^2(q^8)}\,.$$

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Replacing q by $-q$, we obtain

$$c^2(-q) = \frac{\psi^4(q^4)}{\phi^2(-q)\psi^2(q^8)} .$$

Taking the difference of the above two equations and using [**1**, p. 36, Entry 25(iii), (v)], we complete the proof. ∎

PROOF OF (iii): The result follows from (i) and [**1**, p. 36, Entry 25 (iii) and (iv)]. ∎

PROOF OF (iv): By Theorem 2.1, we deduce that

$$c^{-1}(q) - c(q) = \frac{\phi^2(q) - \phi^2(q^4)}{\phi(q)\phi(q^4)} , \tag{3.3}$$

and

$$c^{-1}(q) + c(q) = \frac{\phi^2(q) + \phi^2(q^4)}{\phi(q)\phi(q^4)} . \tag{3.4}$$

On dividing (3.3) by (3.4) we complete the proof. ∎

REMARK: Identities in Theorem 2.1 are similar to those identities satisfied by $H(q)$ in [**7**, Theorem 2.1].

THEOREM 3.2. *Let $u = c(q)$, $v = c(-q)$ and $w = c(q^2)$. Then,*

(i) $u + v = 2uv$,

(ii) $u = \frac{1}{2}\Big(\frac{1+\sqrt{1-(2w-1)^4}}{(2w-1)^2} + 1\Big)$, *and*

(iii) $(u-v)^2(2w-1)^2 + 8uvw(w-1) = 0$.

PROOF OF (i): Follows easily from Theorem 2.1 and [**1**, Entry 25(i), p. 35]. ∎

PROOF OF (ii): We have,

$$w = \frac{\phi(q^2) + \phi(-q^2)}{2\phi(q^2)} ,$$

and hence

$$(2w-1)^2 = \frac{\phi^2(-q^2)}{\phi^2(q^2)} .$$

From [**1**, p. 36, Entry 25 (iii), (vi)], we have

$$(2w-1)^2 = \frac{2(\phi(-q)/\phi(q))}{1 + (\phi^2(-q)/\phi^2(q))} . \tag{3.5}$$

Thus,

$$(2w-1)^2 = \frac{2(2u-1)}{1+(2u-1)^2} .$$

On simplification, we obtain the desired result. ∎

PROOF OF (iii): We have,

$$u = \frac{\phi(q) + \phi(-q)}{2\phi(q)} ,$$
$$v = \frac{\phi(q) + \phi(-q)}{2\phi(-q)} ,$$

and hence

$$\frac{u}{v} = \frac{\phi(-q)}{\phi(q)} .$$

From (3.5), we have

$$(2w-1)^2 = \frac{2(u/v)}{1+(u^2/v^2)} = \frac{2uv}{u^2+v^2} .$$

On simplification, we obtain,

$$(u-v)^2(2w-1)^2 + 8uvw(w-1) = 0 .$$

This completes the proof of Theorem 3.2(iii). □

4. MODULAR EQUATIONS OF DEGREE n AND RELATIONS BETWEEN $c(q)$AND $c(q^n)$

To briefly define a modular equation, we first write as usual

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} .$$

We say that the modulus β has degree n over the modulus α when

$$n \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha)}{{}_2F_1(1/2, 1/2; 1; \alpha)} = \frac{{}_2F_1(1/2, 1/2; 1; 1-\beta)}{{}_2F_1(1/2, 1/2; 1; \beta)} \tag{4.1}$$

where

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} \quad x^k, \qquad |x| < 1 .$$

A modular equation of degree n is an equation relating α and β which is induced by (4.1).

THEOREM 4.1. *If*

$$q = \exp\Big(-\pi \frac{{}_2F_1(1/2, 1/2; 1; 1-\alpha)}{{}_2F_1(1/2, 1/2; 1; \alpha)}\Big) , \tag{4.2}$$

then

$$\alpha = 1 - (2c(q) - 1)^4 .$$

PROOF: We have

$$c(q) = \frac{1}{2}\Big(1 + \frac{\phi(-q)}{\phi(q)}\Big) . \tag{4.3}$$

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From [**4**, p. 100, Entry 5] we know that the identity (4.2) implies that

$$\alpha = 1 - \frac{\phi^4(-q)}{\phi^4(q)} . \tag{4.4}$$

Using (4.3) and (4.4), we find that

$$(2c(q)-1)^4 = 1-\alpha ,$$

that is,

$$\alpha = 1-(2c(q)-1)^4 . \tag{4.5}$$

Now, let α and q be related by (4.2). If β has degree n over α, then Theorem 4.1 gives us

$$\beta = 1-(2c(q^n)-1)^4. \tag{4.6}$$

Hence by (4.5), (4.6) and any given modular equation of degree n, we can obtain a relation between $c(q)$ and $c(q^n)$. We illustrate this with n=3 and n=4. ∎

Corollary 4.2. *Let* $u = c(q)$, $v = c(q^3)$ *and* $w = c(q^4)$. *Then,*

(i) $(u^2+v^2)^2 - 12uv(u+v)^2 - 4uv\Big[(1+2uv)\big(2uv-3(u+v)\big)+1\Big] = 0$, *and*

(ii) $u = \dfrac{1}{1+\sqrt[4]{1-(2w-1)^4}}$.

Proof of (i): When β has degree 3 over α, we have [**4**, Entry 5(ii), p. 230]

$$(\alpha\beta)^{1/4} = 1-\big((1-\alpha)(1-\beta)\big)^{1/4} . \tag{4.7}$$

From (4.5) and (4.6), we have

$$(2c(q)-1)^4 = 1-\alpha ,$$

and

$$(2c(q^3)-1)^4 = 1-\beta .$$

Substituting these in (4.7),we obtain,

$$\big(1-(2u-1)^4\big)\big(1-(2v-1)^4\big) = \big[1-(2u-1)(2v-1)\big]^4 .$$

On simplification, we deduce that

$$(u^2+v^2)^2 - 12uv(u+v)^2 - 4uv\Big[(1+2uv)\big(2uv-3(u+v)\big)+1\Big] = 0 .$$ ∎

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PROOF OF (ii): When β has degree 4 over α, we have from [**4**, Equation (24.22) p. 215]

$$\beta = \left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^4 . \tag{4.8}$$

Also we have,

$$1-\alpha = (2u-1)^4 , \tag{4.9}$$

and

$$\beta = 1-(2w-1)^4 . \tag{4.10}$$

Using (4.9) in (4.8) we obtain

$$\beta^{1/4} = \frac{1-(2u-1)}{1+(2u-1)} ,$$

which implies

$$\beta^{1/4} = \frac{1-u}{u} . \tag{4.11}$$

Using (4.10) in (4.11) we get the required result. ∎

5. Explicit formulas for the evaluations of $c(q)$

Let $q_n = e^{-\pi\sqrt{n}}$ and let α_n denote the corresponding value of α in (4.2). Then applying Theorem 4.1, we get,

$$c(e^{-\pi\sqrt{n}}) = \frac{\sqrt[4]{(1-\alpha_n)}+1}{2} . \tag{5.1}$$

It is known that from [4, p. 97] that $\alpha_1 = 1/2$, $\alpha_2 = (\sqrt{2}-1)^2$and $\alpha_4 = (\sqrt{2}-1)^4$. Hence by using (5.1) we deduce that

$$c(e^{-\pi}) = \frac{\sqrt[4]{1/2}+1}{2} , \tag{5.2}$$

$$c(e^{-\pi\sqrt{2}}) = \frac{\sqrt[4]{2(\sqrt{2}-1)}+1}{2} , \tag{5.3}$$

and

$$c(e^{-2\pi}) = \frac{\sqrt[4]{12\sqrt{2}-16}+1}{2} . \tag{5.4}$$

THEOREM 5.1. *Let the Ramanujan–Weber Class invariants be defined by*

$$G_n := 2^{-1/4} {q_n}^{-1/24} (-q_n; {q_n}^2)_\infty$$

and

$$g_n := 2^{-1/4} {q_n}^{-1/24} (q_n; {q_n}^2)_\infty \ ,$$

where $q_n = e^{-\pi\sqrt{n}}$ *and let* $p = {G_n}^{12}$ *and* $p_1 = {g_n}^{12}$*. Then,*

$$\text{(i)} \quad c(e^{-\pi\sqrt{n}}) = \frac{\sqrt[4]{(p+\sqrt{p^2-1})/(2p)}+1}{2}, \quad \text{and}$$

$$\text{(ii)} \quad c(e^{-\pi\sqrt{n}}) = \frac{\sqrt[4]{2p_1(\sqrt{p_1^2+1}-p_1)}+1}{2}.$$

PROOF OF (i): Since from [7],

$$G_n = \big(4\alpha_n(1-\alpha_n)\big)^{-1/24} \ ,$$

we deduce that

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)}+\sqrt{p(p-1)})^2} \ . \tag{5.5}$$

Using (5.5) in (5.1)we get the required result. ∎

PROOF OF (ii): Also from [7], we have,

$$\frac{1}{\sqrt{(\alpha_n)}} - \sqrt{(\alpha_n)} = 2{g_n}^{12} \ .$$

Hence,

$$\sqrt{(\alpha_n)} = \sqrt{({p_1}^2+1)} - p_1 \ . \tag{5.6}$$

Using (5.6) in (5.1), we complete the proof. ∎

EXAMPLES. Let n=1. Since $G_1 = 1$, Theorem 5.1 yields

$$c(e^{-\pi}) = \frac{\sqrt[4]{1/2}+1}{2} \ .$$

Let $n = 2$. Since $g_2 = 1$, Theorem 5.1 yields

$$c(e^{-\pi\sqrt{2}}) = \frac{\sqrt[4]{2(\sqrt{2}-1)}+1}{2} \ ,$$

Using $G_3^{12} = 2$ in Theorem 5.1, we obtain

$$c(e^{-\pi\sqrt{3}}) = \frac{\sqrt[4]{(2+\sqrt{3})/4} + 1}{2} .$$

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