12
QCD in loop space

QCD can be entirely reformulated in terms of the colorless composite field $\Phi(C)$ – the trace of the Wilson loop for closed contours. This fact involves two main steps:

(1) all of the observables are expressed via $\Phi(C)$;

(2) the dynamics is entirely reformulated in terms of $\Phi(C)$.

This approach is especially useful in the large-N limit where everything is expressed via the vacuum expectation value of $\Phi(C)$ – the Wilson loop average. Observables are given by summing the Wilson loop average over paths with the same weight as in free theory. The Wilson loop average itself obeys a close functional equation – the loop equation.

We begin this chapter by presenting the formulas which relate observables to Wilson loops. Then we translate the quantum equation of motion of Yang–Mills theory into loop space. We derive the closed equation for the Wilson loop average as $N \to \infty$ and discuss its various properties, including a nonperturbative regularization. Finally, we briefly comment on what is known concerning solutions of the loop equation.

12.1 Observables in terms of Wilson loops

All observables in QCD can be expressed via the Wilson loops $\Phi(C)$ defined by Eq. (11.115). This property was first advocated by Wilson [Wil74] on a lattice. Calculation of QCD observables can be divided into two steps:

(1) calculation of the Wilson loop averages for arbitrary contours;

(2) summation of the Wilson loop averages over the contours with some weight depending on a given observable.

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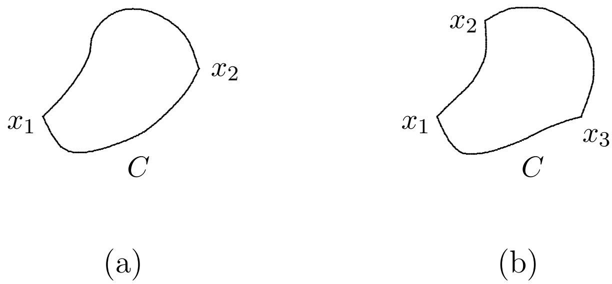

Fig. 12.1. Contours in the sum over paths representing observables: (a) in Eq. (12.3) and (b) in Eq. (12.4). The contour (a) passes x_1 and x_2. The contour (b) passes x_1, x_2, and x_3.

At finite N, observables are expressed via the n-loop averages

$$W_n(C_1, \ldots, C_n) \quad = \quad \langle\, \Phi(C_1) \cdots \Phi(C_n) \,\rangle\,, \tag{12.1}$$

which are analogous to the n-point Green functions (2.45). The appropriate formulas for the continuum theory can be found in [MM81].

Great simplifications occur in these formulas at $N = \infty$, when all observables are expressed only via the one-loop average

$$W(C) \quad = \quad \langle\, \Phi(C) \,\rangle \quad \equiv \quad \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}g \oint_C \mathrm{d}x^\mu A_\mu} \right\rangle. \tag{12.2}$$

This is associated with the quenched approximation discussed in the Remark on p. 158.

For example, the average of the product of two colorless quark vector currents (11.92) is given at large N by

$$\left\langle \bar\psi\gamma_\mu\psi(x_1)\, \bar\psi\gamma_\nu\psi(x_2) \right\rangle \quad = \sum_{C \ni x_1, x_2} J_{\mu\nu}(C)\, \langle\, \Phi(C) \,\rangle\,, \tag{12.3}$$

where the sum runs over contours C passing through the points x_1 and x_2 as is depicted in Fig. 12.1a. An analogous formula for the (connected) correlators of three quark scalar currents can be written as

$$\left\langle \bar\psi\psi(x_1)\, \bar\psi\psi(x_2)\, \bar\psi\psi(x_3) \right\rangle_{\mathrm{conn}} \quad = \sum_{C \ni x_1, x_2, x_3} J(C)\, \langle\, \Phi(C) \,\rangle\,, \tag{12.4}$$

where the sum runs over contours C passing through the three points x_1, x_2, and x_3 as depicted in Fig. 12.1b. A general (connected) correlator of n quark currents is given by a similar formula with C passing through n points x_1, ..., x_n (some of them may coincide).

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The weights $J_{\mu\nu}(C)$ in Eq. (12.3) and $J(C)$ in Eq. (12.4) are completely determined by free theory. If quarks were scalars rather than spinors, then we would have

$$J(C) \;=\; \mathrm{e}^{-\frac{1}{2}m^2\tau-\frac{1}{2}\int_0^\tau \mathrm{d}t\,\dot{z}_\mu^2(t)} \;=\; \mathrm{e}^{-mL(C)} \qquad \boxed{\text{scalar quarks}}\,, \tag{12.5}$$

where $L(C)$ is the length of the (closed) contour C, as was shown in Sect. 1.6. Using the notation (1.156), we can rewrite Eq. (12.4) for scalar quarks as

$$\left\langle \psi^\dagger\psi(x_1)\,\psi^\dagger\psi(x_2)\,\psi^\dagger\psi(x_3)\right\rangle_{\text{conn}} \;=\; \sum_{C\ni x_1,x_2,x_3}{}^{\prime}\,\langle\,\Phi(C)\,\rangle\,. \tag{12.6}$$

Therefore, we obtain the sum over paths of the Wilson loop, likewise in Sect. 1.7 and Problem 5.4 on p. 91.

For spinor quarks, an additional disentangling of the γ-matrices is needed. This can be done in terms of a path integral over the momentum variable, with $k_\mu(t)$ $(0\leq t\leq\tau)$ being an appropriate trajectory. The result is given by [BNZ79]

$$J(C) \;=\; \int \mathcal{D}k_\mu(t)\,\mathrm{sp}\,\boldsymbol{P}\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\,\{\mathrm{i}k_\mu(t)[\dot{x}_\mu(t)-\gamma_\mu(t)]+m\}} \tag{12.7}$$

and

$$J_{\mu\nu}(C) \;=\; \int \mathcal{D}k_\mu(t)\,\mathrm{sp}\,\boldsymbol{P}\left[\gamma_\mu(t_1)\,\gamma_\nu(t_2)\;\mathrm{e}^{-\int_0^\tau \mathrm{d}t\,\{\mathrm{i}k_\mu(t)[\dot{x}_\mu(t)-\gamma_\mu(t)]+m\}}\right], \tag{12.8}$$

where the values t_1 and t_2 of the parameter t are associated with the points x_1 and x_2 in Eq. (12.3), and the symbol of P-ordering puts the matrices γ_μ and γ_ν at a proper order.

Problem 12.1 Derive Eqs. (12.7) and (12.8).

Solution Since the spinor field ψ enters the QCD action quadratically, it can be integrated out in the correlators (12.3) and (12.4), so that they can be represented, in the first quantized language, via the resolvent of the Dirac operator in the external field $\mathcal{A}_\mu$, with subsequent averaging over $\mathcal{A}_\mu$. Proceeding as in Chapter 1, we express the resolvent by

$$\left\langle y\left|\frac{1}{\widehat{\nabla}+m}\right|x\right\rangle \;=\; \int_0^\infty \mathrm{d}\tau\,\left\langle y\left|\,\mathrm{e}^{-\tau\left(\widehat{\nabla}+m\right)}\right|x\right\rangle \tag{12.9}$$

and represent the matrix element of the exponential of the Dirac operator as

$$\left\langle y\left|\,\mathrm{e}^{-\tau\left(\widehat{\nabla}+m\right)}\right|x\right\rangle \;=\; \mathrm{e}^{-\tau m}\boldsymbol{P}\,\mathrm{e}^{-\int_0^\tau \mathrm{d}t\,\widehat{\nabla}(t)}\delta^{(d)}(x-y)\,. \tag{12.10}$$

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In order to disentangle the RHS, we insert unity, represented by

$$1 \;=\; \int\limits_{z_\mu(0)=x_\mu} \mathcal{D}z_\mu(t) \int \mathcal{D}p_\mu(t)\, \mathrm{e}^{-\mathrm{i}\int_0^\tau \mathrm{d}t\, p_\mu(t)\dot{z}_\mu(t)}, \tag{12.11}$$

where the path integration over $p_\mu(t)$ is unrestricted, i.e. the integrals over $p_\mu(0)$ and $p_\mu(\tau)$ are included. Then we obtain

$$\begin{aligned}\left\langle y \left| \mathrm{e}^{-\tau(\hat{\nabla}+m)} \right| x \right\rangle \;&=\; \mathrm{e}^{-\tau m} \int\limits_{z_\mu(0)=x_\mu} \mathcal{D}z_\mu(t) \int \mathcal{D}p_\mu(t) \\ &\times \boldsymbol{P}\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \{\mathrm{i}p_\mu(t)\dot{z}_\mu(t) - [\mathrm{i}p_\mu(t)+\mathrm{i}\mathcal{A}_\mu(t)]\gamma_\mu(t) + \partial_\mu(t)\dot{z}_\mu(t)\}}\, \delta^{(d)}(x-y)\,,\end{aligned} \tag{12.12}$$

the equivalence of which to the original expression is obvious since everything commutes under the sign of the P-ordering (so that we can substitute $p_\mu(t) = -\mathrm{i}\partial_\mu(t)$ in the integrand).

By making the change of the integration variable, $p_\mu(t) \;=\; k_\mu(t) - \mathcal{A}_\mu(t)$, and proceeding as in Problem 1.13 on p. 29, we represent the RHS of Eq. (12.12) by

$$\begin{aligned}\left\langle y \left| \mathrm{e}^{-\tau(\hat{\nabla}+m)} \right| x \right\rangle \;&=\; \mathrm{e}^{-\tau m} \int\limits_{z_\mu(0)=x_\mu} \mathcal{D}z_\mu(t) \int \mathcal{D}k_\mu(t) \\ &\times \boldsymbol{P}\, \mathrm{e}^{-\int_0^\tau \mathrm{d}t\, \{\mathrm{i}k_\mu(t)[\dot{z}_\mu(t)-\gamma_\mu(t)] - \mathrm{i}\dot{z}_\mu(t)\mathcal{A}_\mu(t) + \partial_\mu(t)\dot{z}_\mu(t)\}}\, \delta^{(d)}(x-y) \\ =\; &\mathrm{e}^{-\tau m} \int\limits_{\substack{z_\mu(0)=x_\mu \\ z_\mu(\tau)=y_\mu}} \mathcal{D}z_\mu(t) \int \mathcal{D}k_\mu(t)\, \boldsymbol{P}\, \mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}z_\mu\, \mathcal{A}_\mu(z)}\, \boldsymbol{P}\, \mathrm{e}^{-\mathrm{i}\int_0^\tau \mathrm{d}t\, k_\mu(t)[\dot{z}_\mu(t)-\gamma_\mu(t)]}\,,\end{aligned} \tag{12.13}$$

where the first P-exponential on the RHS depends only on color matrices (it is nothing but the non-Abelian phase factor), and the second one depends only on spinor matrices. In [BNZ79], Eq. (12.13) is derived by discretizing paths.

Equation (12.13) leads to Eqs. (12.7) and (12.8).

Remark on renormalization of Wilson loops

Perturbation theory for $W(C)$ can be obtained by expanding the path-ordered exponential in the definition (12.2) in g (see Eq. (11.46)) and averaging over the gluon field A_μ. Because of ultraviolet divergences, we need a (gauge-invariant) regularization. After such a regularization has been introduced, the Wilson loop average for a smooth contour C of the type in Fig. 12.2a reads as

$$W(C) \;=\; \exp\left[-g^2 \frac{(N^2-1)}{4\pi N} \frac{L(C)}{a}\right] W_{\mathrm{ren}}(C)\,, \tag{12.14}$$

where a is the cutoff, $L(C)$ is the length of C, and $W_{\mathrm{ren}}(C)$ is finite when expressed via the renormalized charge g_{R}. The exponential factor is a

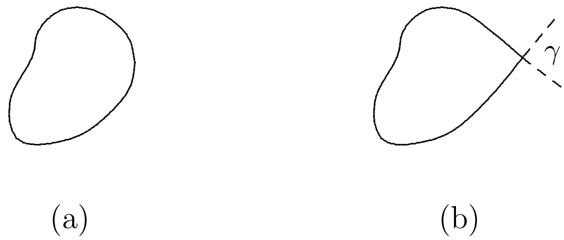

Fig. 12.2. Examples of (a) a smooth contour and (b) a contour with a cusp. The tangent vector to the contour jumps through an angle γ at the cusp.

result of the renormalization of the mass of a heavy test quark, which was already discussed in the Remark on p. 113. This factor does not emerge in the dimensional regularization where $d = 4 - \varepsilon$. The multiplicative renormalization of the smooth Wilson loop was shown in [GN80, Pol80, DV80].

If the contour C has a cusp (or cusps) but no self-intersections as is illustrated by Fig. 12.2b, then $W(C)$ is still multiplicatively renormalizable [BNS81]:

$$W(C) \quad = \quad Z(\gamma)\, W_{\text{ren}}(C)\,, \tag{12.15}$$

while the (divergent) factor of $Z(\gamma)$ depends on the cusp angle (or angles) γ (or γs) and $W_{\text{ren}}(C)$ is finite when expressed via the renormalized charge g_{R}.

Problem 12.2 Calculate the divergent parts of the Wilson loop average (12.2) for contours without self-intersections to order g^2. Consider the cases of a smooth contour C and a contour with a cusp.

Solution Expanding the Wilson loop average (12.2) in g^2 (see Eq. (11.46) and Problem 5.2 on p. 89), we obtain

$$W(C) \quad = \quad 1 + W^{(2)}(C) + \mathcal{O}\big(g^4\big) \tag{12.16}$$

with

$$W^{(2)}(C) \quad = \quad -g^2 \frac{(N^2-1)}{2N} \oint\limits_C \mathrm{d}x_\mu \oint\limits_C \mathrm{d}y_\nu\, D_{\mu\nu}(x-y)\,, \tag{12.17}$$

where $D_{\mu\nu}(x-y)$ is the gluon propagator (11.4).

Since the contour integral in Eq. (12.17) diverges for $x = y$, we introduce the regularization by

$$D_{\mu\nu}(x-y) \quad \stackrel{\text{reg.}}{\Longrightarrow} \quad \frac{1}{4\pi^2} \frac{\delta_{\mu\nu}}{\left[(x-y)^2 + a^2\right]} \tag{12.18}$$

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with a being the ultraviolet cutoff. Parametrizing the contour C using the function $z_\mu(\sigma)$, we rewrite the contour integral in Eq. (12.17) as

$$\oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \frac{1}{(x-y)^2+a^2} = \int \mathrm{d}s \int \mathrm{d}t \frac{\dot{z}_\mu(s)\dot{z}_\mu(s+t)}{\left[z(s+t)-z(s)\right]^2+a^2}. \tag{12.19}$$

Choosing the proper-length parametrization (1.101) when $\dot{z}_\mu(s)\ddot{z}_\mu(s)=0$, expanding in powers of t, and assuming that the contour C is smooth as is depicted in Fig. 12.2a, we obtain for the integral (12.19)

$$\int \mathrm{d}s\, \dot{x}^2(s) \int \mathrm{d}t \frac{1}{\dot{x}^2(s)t^2+a^2} = \frac{\pi}{a}\int \mathrm{d}s \sqrt{\dot{x}^2(s)} = \frac{\pi}{a}L(C). \tag{12.20}$$

Typical values of t in the last integral are $\sim a$, which justifies the expansion in t: the next terms lead to a finite contribution as $a \to 0$.

Thus, we find

$$W^{(2)}(C) = -g^2\frac{(N^2-1)}{4\pi N}\frac{L(C)}{a} + \text{ finite term as } a \to 0 \tag{12.21}$$

for a smooth contour. This is precisely the renormalization of the mass of a heavy test quark owing to the interaction.

If the contour C is not smooth and has a cusp at some value s_0 of the parameter, as depicted in Fig. 12.2b, then an extra divergent contribution in the integral (12.19) emerges when $s \approx s_0$, $t \approx t_0$. Introducing $\Delta s = s - s_0$ and $\Delta t = t - t_0$, we represent this extra divergent term by

$$\begin{aligned}\dot{x}_\mu(s_0+0)\dot{x}_\mu(s_0-0) &\int \mathrm{d}\Delta s \int \mathrm{d}\Delta t \frac{1}{\left[\dot{x}_\mu(s_0+0)\Delta s - \dot{x}_\mu(s_0-0)\Delta t\right]^2+a^2} \\ &= (\gamma\cot\gamma - 1)\ln\frac{L(C)}{a}, \end{aligned} \tag{12.22}$$

where γ is the angle of the cusp ($\cos\gamma \equiv \dot{x}_\mu(s_0+0)\dot{x}_\mu(s_0-0)$) and the upper limit of the integrations is chosen to be $L(C)$ with logarithmic accuracy. Collecting all of this together, we obtain finally for the divergent part of $W^{(2)}(C)$:

$$\begin{aligned} W^{(2)}(C) &= -g^2\frac{(N^2-1)}{4\pi N}\left[\frac{L(C)}{a} + \frac{1}{\pi}(\gamma\cot\gamma-1)\ln\frac{L(C)}{a}\right] \\ &\quad + \text{ finite term as } a \to 0. \end{aligned} \tag{12.23}$$

The second term in square brackets is associated with the bremsstrahlung radiation of a particle changing its velocity when passing the cusp. The answers in the Abelian and non-Abelian cases coincide to this order in g^2.

Problem 12.3 Obtain Coulomb's law of interaction in Maxwell's theory by calculating the average of a rectangular Wilson loop.

Solution Performing the Gaussian averaging over A_μ in Maxwell's theory, we obtain from Eqs. (6.50) and (6.51)

$$\begin{aligned} -\ln W(C) &= \frac{1}{2}\int \mathrm{d}^4x \int \mathrm{d}^4y \, J^\mu(x)\, D_{\mu\nu}(x-y)\, J^\nu(y) \\ &= \frac{e^2}{2}\oint\limits_C \mathrm{d}x^\mu \oint\limits_C \mathrm{d}y^\nu \, D_{\mu\nu}(x-y)\,. \end{aligned} \tag{12.24}$$

The interaction potential is now determined by Eq. (6.43) for a rectangular contour depicted in Fig. 6.6 on p. 111 as $\mathcal{T} \gg R$. The contribution to the interaction potential arises when the photon line is emitted by the upper part of the rectangular contour and absorbed by the lower part. Otherwise, we obtain singular terms associated with the renormalization of the Wilson loop as discussed in the previous Problem.

Choosing the parametrization with $x_\mu = (R, \ldots, s)$ for the upper part and $x_\mu = (0, \ldots, t)$ for the lower part of the rectangular contour with $0 \le s, t \le \mathcal{T}$, we have

$$V(R)\,\mathcal{T} = \frac{e^2}{4\pi^2}\int\limits_0^{\mathcal{T}} \mathrm{d}s \int\limits_0^{\mathcal{T}} \mathrm{d}t \, \frac{1}{(s-t)^2+R^2}\,. \tag{12.25}$$

Introducing $u = (s+t)/2$ and $v = s-t$, we obtain

$$V(R)\,\mathcal{T} = \frac{e^2}{4\pi^2}\int\limits_0^{\mathcal{T}} \mathrm{d}u \int\limits_{-\mathcal{T}}^{\mathcal{T}} \mathrm{d}v \, \frac{1}{v^2+R^2} = \frac{e^2}{4\pi R}\mathcal{T} \tag{12.26}$$

which reproduces Coulomb's law.

12.2 Schwinger–Dyson equations for Wilson loop

The dynamics of (quantum) Yang–Mills theory is described by the quantum equation of motion

$$-\nabla^{ab}_\mu F^b_{\mu\nu}(x) \overset{\text{w.s.}}{=} \hbar \frac{\delta}{\delta A^a_\nu(x)} \tag{12.27}$$

which is analogous to Eq. (2.27) for the scalar field, and is again understood in the weak sense, i.e. for the averages

$$-\left\langle \nabla^{ab}_\mu F^b_{\mu\nu}(x)\, Q\,[A] \right\rangle = \hbar \left\langle \frac{\delta}{\delta A^a_\nu(x)}\, Q\,[A] \right\rangle. \tag{12.28}$$

The standard set of Schwinger–Dyson equations of Yang–Mills theory emerges when the functional $Q[A]$ is chosen in the form of the product of A_{μ_i} as in Eq. (11.45).

Strictly speaking, the last statement is incorrect, since in Eqs. (12.27) and (12.28) we have not added contributions coming from the variation

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of gauge-fixing and ghost terms in the Yang–Mills action. However, these two contributions are mutually canceled for gauge-invariant functionals $Q[A]$. We shall deal only with such gauge-invariant functionals (the Wilson loops). This is why we have not considered the contribution of the gauge-fixing and ghost terms.

It is also convenient to use the matrix notation (5.5), when Eq. (12.27) for the Wilson loop takes the form

$$-\left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\nabla_\mu \mathcal{F}_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \right\rangle \;=\; \left\langle \frac{g^2}{N}\,\mathrm{tr}\,\frac{\delta}{\delta \mathcal{A}_\nu(x)}\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \right\rangle, \tag{12.29}$$

where we have restored the units with $\hbar = 1$.

The variational derivative on the RHS can be calculated by virtue of the formula

$$\frac{\delta \mathcal{A}^{ij}_\mu(y)}{\delta \mathcal{A}^{kl}_\nu(x)} \;=\; \delta_{\mu\nu}\,\delta^{(d)}(x-y)\left(\delta^{il}\delta^{kj} - \frac{1}{N}\delta^{ij}\delta^{kl}\right) \tag{12.30}$$

which is a consequence of

$$\frac{\delta A^a_\mu(y)}{\delta A^b_\nu(x)} \;=\; \delta_{\mu\nu}\,\delta^{(d)}(x-y)\,\delta^{ab}\,. \tag{12.31}$$

The second term in the parentheses in Eq. (12.30) – same as in Eq. (11.6) – is because $\mathcal{A}_\mu$ is a matrix from the adjoint representation of $SU(N)$.

By using Eq. (12.30), we obtain for the variational derivative on RHS of Eq. (12.29):

$$\begin{aligned}\mathrm{tr}\,\frac{\delta}{\delta \mathcal{A}_\nu(x)}\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;&=\; \mathrm{i}\oint_C \mathrm{d}y_\nu\,\delta^{(d)}(x-y)\\ &\times\left[\frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}} \mathrm{d}\xi^\mu \mathcal{A}_\mu}\frac{1}{N}\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{xy}} \mathrm{d}\xi^\mu \mathcal{A}_\mu} - \frac{1}{N^3}\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C} \mathrm{d}\xi^\mu \mathcal{A}_\mu}\right].\end{aligned} \tag{12.32}$$

The contours C_{yx} and C_{xy}, which are depicted in Fig. 12.3, are the parts of the loop C: from x to y and from y to x, respectively. They are always closed owing to the presence of the delta-function. It implies that x and y should be the same points of *space* but not necessarily of the *contour* (i.e. they may be associated with different values of the parameter σ).

Finally, we rewrite Eq. (12.29) as

$$\begin{aligned}&\mathrm{i}\left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\nabla_\mu \mathcal{F}_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \right\rangle\\ &\quad=\; \lambda\oint_C \mathrm{d}y_\nu\,\delta^{(d)}(x-y)\left[\langle\,\Phi(C_{yx})\,\Phi(C_{xy})\,\rangle - \frac{1}{N^2}\langle\,\Phi(C)\,\rangle\right],\end{aligned} \tag{12.33}$$

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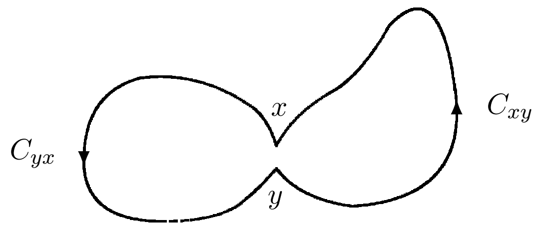


Fig. 12.3. Contours C_{yx} and C_{xy} which enter the RHSs of Eqs. (12.29) and (12.33).

where we have introduced the *'t Hooft coupling*

$$\lambda \;=\; g^2 N\,. \tag{12.34}$$

Note that the RHS of Eq. (12.33) is completely represented via the (closed) Wilson loops.

Problem 12.4 Prove the cancellation of the contributions of the gauge-fixing and ghost terms in the Lorentz gauge.

Solution The Yang–Mills action, associated with the Lorentz gauge, is given by

$$S_{\mathrm{gf}} \;=\; \frac{1}{g^2}\int \mathrm{d}^d x \left[\frac{1}{4}\,\mathrm{tr}\,\mathcal{F}_{\mu\nu}^2 + \frac{1}{2\alpha}\,\mathrm{tr}\,(\partial_\mu \mathcal{A}_\mu)^2\right]. \tag{12.35}$$

Since $\Phi(C)$ is gauge invariant, the (infinite) group-volume factors, in the numerator and denominator in the definition of the average, cancel when fixing the gauge (see the Remark on p. 109), and we obtain

$$W(C) \;\equiv\; \frac{\displaystyle\int \mathcal{D}A_\mu\, \mathrm{e}^{-S}\,\Phi(C)}{\displaystyle\int \mathcal{D}A_\mu\, \mathrm{e}^{-S}} \;=\; \frac{\displaystyle\int \mathcal{D}A_\mu\, \det\left(\partial_\mu\nabla_\mu\right)\, \mathrm{e}^{-S_{\mathrm{gf}}}\,\Phi(C)}{\displaystyle\int \mathcal{D}A_\mu\, \det\left(\partial_\mu\nabla_\mu\right)\, \mathrm{e}^{-S_{\mathrm{gf}}}}\,, \tag{12.36}$$

where $\det\left(\partial_\mu\nabla_\mu\right)$ is associated with ghosts.

The Schwinger–Dyson equation for the Yang–Mills theory in the Lorentz gauge is

$$-\nabla_\mu^{ab} F_{\mu\nu}^b(x) \;\overset{\text{w.s.}}{=}\; \hbar\,\frac{\delta}{\delta A_\nu^a(x)} + \frac{1}{\alpha}\partial_\nu\partial_\mu A_\mu^a(x) + \frac{\partial}{\partial x_\nu} f^{abc} G^{bc}(x' = x, x; A)\,, \tag{12.37}$$

where $G^{bc}\left(x', x; A\right)$ is the Green function of the ghost in an external field A_μ. Applying this equation to the Wilson loop and using the gauge Ward identity (the Slavnov–Taylor identity), we transform the contribution from the second term on the RHS to

$$
\begin{aligned}
&\frac{\mathrm{i}}{\alpha}\left\langle \frac{1}{N}\operatorname{tr}\partial_\nu\partial_\mu\mathcal{A}_\mu U(C_{xx})\right\rangle_{\mathrm{gf}} \\
&\quad = g^2\oint\limits_C \mathrm{d}\xi_\mu \left\langle \frac{1}{N}\operatorname{tr}\left[U(C_{\xi x})t^a U(C_{x\xi})t^b\right]\frac{\partial}{\partial x_\nu}\nabla_\mu^{bc}(\xi)\, G^{ca}(\xi,x;A)\right\rangle_{\mathrm{gf}} \\
&\quad = g^2\oint\limits_C \mathrm{d}\xi_\mu \frac{\partial}{\partial \xi_\mu}\left\langle \frac{1}{N}\operatorname{tr}\left[U(C_{\xi x})t^a U(C_{x\xi})t^b\right]\frac{\partial}{\partial x_\nu}G^{ba}(\xi,x;A)\right\rangle_{\mathrm{gf}} \\
&\quad = g^2\left\langle \frac{1}{N}\operatorname{tr}\left\{U(C_{xx})\left[t^a,t^b\right]\right\}\frac{\partial}{\partial x_\nu}G^{ba}(x'=x,x;A)\right\rangle_{\mathrm{gf}}
\end{aligned}
\tag{12.38}
$$

which exactly cancels the contribution from the ghost term in Eq. (12.37).

We have thus proven that the contribution of gauge-fixing and ghost terms in Eq. (12.37) are mutually canceled, when applied to the Wilson loop (and, in fact, to any gauge-invariant functional).

12.3 Path and area derivatives

As we already mentioned, the RHS of Eq. (12.33) is completely represented via the (closed) Wilson loops. It is crucial for the loop-space formulation of QCD that the LHS of Eq. (12.33) can also be represented in loop space as some operator applied to the Wilson loop. To do this we need to develop a differential calculus in loop space.

Loop space consists of arbitrary continuous closed loops, C. They can be described in a parametric form by the functions $x_\mu(\sigma) \in L_2$,* where $\sigma_0 \le \sigma \le \sigma_1$ and $\mu = 1,\ldots,d$, which take on values in a d-dimensional Euclidean space. The functions $x_\mu(\sigma)$ can be discontinuous, generally speaking, for an arbitrary choice of the parameter σ. The continuity of the loop C implies a continuous dependence on parameters of the type of proper length

$$
s(\sigma) \;=\; \int\limits_{\sigma_0}^{\sigma} \mathrm{d}\sigma' \sqrt{\dot{x}_\mu^2(\sigma')}\,,
\tag{12.39}
$$

where $\dot{x}_\mu(\sigma) = \mathrm{d}x_\mu(\sigma)/\mathrm{d}\sigma$.

The functions $x_\mu(\sigma) \in L_2$ which are associated with the elements of loop space obey the following restrictions.

(1) The points $\sigma = \sigma_0$ and $\sigma = \sigma_1$ are identified: $x_\mu(\sigma_0) = x_\mu(\sigma_1)$ – the loops are closed.

* Let us remind the reader that L_2 denotes the Hilbert space of functions $x_\mu(\sigma)$, the square of which is integrable over the Lebesgue measure: $\int_{\sigma_0}^{\sigma_1}\mathrm{d}\sigma\, x_\mu^2(\sigma) < \infty$. We have already mentioned this in the Remark on p. 19.

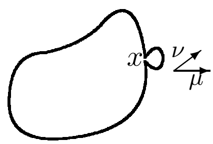
(2) The functions $x_\mu(\sigma)$ and $\Lambda_{\mu\nu}x_\nu(\sigma)+\alpha_\mu$, with $\Lambda_{\mu\nu}$ and α_μ independent of σ, represent the same element of the loop space – rotational and translational invariance.

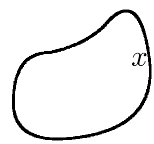
(3) The functions $x_\mu(\sigma)$ and $x_\mu(\sigma')$ with $\sigma'=f(\sigma)$, $f'(\sigma)\geq 0$ describe the same loop – reparametrization invariance.

An example of functionals which are defined on the elements of loop space is the Wilson loop average (12.2) or, more generally, the n-loop average (12.1).

The differential calculus in loop space is built out of the path and area derivatives.

The *area derivative* of a functional $\mathcal{F}(C)$ is defined by the difference

$$\frac{\delta\mathcal{F}(C)}{\delta\sigma_{\mu\nu}(x)} \equiv \frac{1}{\delta\sigma_{\mu\nu}}\Bigg[\mathcal{F}\Bigg($$


$$\Bigg) - \mathcal{F}\Bigg($$


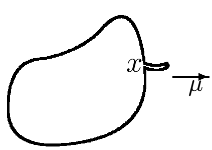
$$\Bigg)\Bigg], \tag{12.40}$$

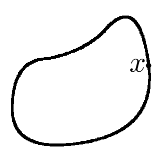
where an infinitesimal loop $\delta C_{\mu\nu}(x)$ is attached to a given loop at the point x in the (μ,ν)-plane and $\delta\sigma_{\mu\nu}$ denotes the area enclosed by $\delta C_{\mu\nu}(x)$. For a rectangular loop $\delta C_{\mu\nu}(x)$, one finds

$$\delta\sigma_{\mu\nu} = dx_\mu \wedge dx_\nu\,, \tag{12.41}$$

where the symbol $\wedge$ implies antisymmetrization. The sign of $\delta\sigma_{\mu\nu}$ is determined by the orientation of $\delta C_{\mu\nu}(x)$.

Analogously, the *path derivative* is defined by

$$\partial^x_\mu\,\mathcal{F}(C_{xx}) \equiv \frac{1}{\delta x_\mu}\Bigg[\mathcal{F}\Bigg($$


$$\Bigg) - \mathcal{F}\Bigg($$


$$\Bigg)\Bigg], \tag{12.42}$$

where the point x is shifted from the loop along an infinitesimal path $\delta\Gamma_\mu$ and δx_μ denotes the length of $\delta\Gamma_\mu$. The sign of δx_μ is determined by the direction of $\delta\Gamma_\mu$.

As is usual in quantum field theory, the typical size of $\delta C_{\mu\nu}$ in the definition of the area derivative as well as the length of $\delta\Gamma_\mu$ in the definition of the path derivative should be smaller than the size of an ultraviolet cutoff.

These two differential operations are well-defined for so-called functionals of the Stokes type which satisfy the backtracking condition – they do

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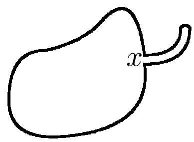
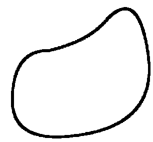
not change when a small path passing back and forth is added to the loop at some point x:

$\mathcal{F}\Big($ 

$\Big) = \mathcal{F}\Big($  $\Big).$ (12.43)

This condition is equivalent to the Bianchi identity of Yang–Mills theory and is obviously satisfied by the Wilson loop (12.2) owing to the properties of the non-Abelian phase factor (see Eq. (5.47)). Such functionals are known in mathematics as Chen integrals.*

A simple example of the Stokes functional is the area of the minimal surface, $A_{\min}(C)$. It obviously satisfies Eq. (12.43). Otherwise, the length $L(C)$ of the loop C is not a Stokes functional, since the lengths of contours on the LHS and RHS of Eq. (12.43) are different.

For the Stokes functionals, the variation on the RHS of Eq. (12.40) is proportional to the area enclosed by the infinitesimally small loop $\delta C_{\mu\nu}(x)$ and does not depend on its shape. Analogously, the variation on the RHS of Eq. (12.42) is proportional to the length of the infinitesimal path $\delta\Gamma_\mu$ and does not depend on its shape.

If x is a regular point (such as any point of the contour for the functional (12.2)), the RHS of Eq. (12.42) vanishes owing to the backtracking condition (12.43). In order for the result to be nonvanishing, the point x should be a *marked* (or irregular) point. A simple example of the functional with a marked point x is

$$\Phi^a[C_{xx}] \quad \equiv \quad \frac{1}{N}\,\mathrm{tr}\left(t^a \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{xx}} \mathrm{d}\xi^\mu \mathcal{A}_\mu(\xi)}\right) \tag{12.44}$$

with the $SU(N)$ generator t^a being inserted in the path-ordered product at the point x.

The area derivative of the Wilson loop is given by the Mandelstam formula

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x)}\frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \quad = \quad \frac{\mathrm{i}}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathcal{F}_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}. \tag{12.45}$$

In order to prove this, it is convenient to choose $\delta C_{\mu\nu}(x)$ to be a rectangle in the (μ,ν)-plane, as was done in Problem 5.8 on p. 94, and use straightforwardly the definition (12.40). The sense of Eq. (12.45) is very simple:

* See, for example, [Tav93] which contains definitions of path and area derivatives in this language.

Table 12.1. Vocabulary for translation of Yang–Mills theory from ordinary space into loop space.

Ordinary space		Loop space	
$\Phi[A]$	Phase factor	$\Phi(C)$	Loop functional
$F_{\mu\nu}(x)$	Field strength	$\dfrac{\delta}{\delta\sigma_{\mu\nu}(x)}$	Area derivative
∇^x_μ	Covariant derivative	∂^x_μ	Path derivative
$\nabla \wedge F = 0$	Bianchi identity		Stokes functionals
$-\nabla_\mu F_{\mu\nu}$ $= \delta/\delta A_\nu$	Schwinger–Dyson equations		Loop equations

$\mathcal{F}_{\mu\nu}$ is a curvature associated with the connection $\mathcal{A}_\mu$, as we discussed in the Remark on p. 95.

The functional on the RHS of Eq. (12.45) has a marked point x, and is of the same type as in Eq. (12.44). When the path derivative acts on such a functional according to the definition (12.42), the result is given by

$$\partial^x_\mu \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,B(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\nabla_\mu B(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}, \tag{12.46}$$

where

$$\nabla_\mu B \;=\; \partial_\mu B - \mathrm{i}\,[\mathcal{A}_\mu, B] \tag{12.47}$$

is the covariant derivative (5.10) in the adjoint representation (see also Problem 5.7 on p. 93).

Combining Eqs. (12.45) and (12.46), we finally represent the expression on the LHS of Eq. (12.29) (or Eq. (12.33)) as

$$\frac{\mathrm{i}}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\nabla_\mu \mathcal{F}_{\mu\nu}(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; \partial^x_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu}, \tag{12.48}$$

i.e. via the action of the path and area derivatives on the Wilson loop. It is therefore rewritten in loop space.

A summary of the results of this section is presented in Table 12.1 as a vocabulary for translation of Yang–Mills theory from the language of ordinary space in the language of loop space.

Remark on Bianchi identity for Stokes functionals

The backtracking condition (12.43) can be represented equivalently as

$$\epsilon_{\mu\nu\lambda\rho}\,\partial^x_\mu \frac{\delta}{\delta\sigma_{\nu\lambda}(x)}\,\Phi(C) \;=\; 0\,, \tag{12.49}$$

by choosing the small path in Eq. (12.43) to be an infinitesimal straight line in the ρ-direction and applying Stokes' theorem geometrically. Using Eqs. (12.45) and (12.46), Eq. (12.49) can in turn be rewritten as

$$\epsilon_{\mu\nu\lambda\rho}\frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\nabla_\mu \mathcal{F}_{\nu\lambda}(x)\,\mathrm{e}^{\mathrm{i}\oint_C \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; 0\,. \tag{12.50}$$

Therefore, Eq. (12.49) represents the Bianchi identity (5.18) in loop space.

Remark on the regularized length

The length $L(C)$ can be approximated by the Stokes functional

$$L_a(C) \;\stackrel{\mathrm{def}}{=}\; \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \frac{1}{\sqrt{2\pi}a}\,\mathrm{e}^{-(x-y)^2/2a^2} \;\stackrel{a\to 0}{\longrightarrow}\; L(C)\,. \tag{12.51}$$

This works for the contours, the size of which is much larger than the ultraviolet cutoff a. The area derivative of the functional $L_a(C)$ is finite at finite a but does not commute with taking the limit $a \to 0$. This illustrates the above statement that the size of the variation should be much smaller than the ultraviolet cutoff.

Problem 12.5 Prove Eq. (12.51).

Solution The calculation is similar to that in Problem 12.2 on p. 253. We have

$$\begin{aligned}
&\int \mathrm{d}s\,\dot{x}_\mu(s) \int \mathrm{d}t\,\dot{x}_\mu(s+t)\frac{1}{\sqrt{2\pi}a}\,\mathrm{e}^{-(x(s+t)-x(s))^2/2a^2} \\
&\quad\stackrel{a\to 0}{\longrightarrow}\; \int \mathrm{d}s\,\dot{x}^2(s) \int_{-\infty}^{+\infty} \mathrm{d}\tau\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\dot{x}^2(s)\,\tau^2/2} \\
&\quad=\; \int \mathrm{d}s\,\sqrt{\dot{x}^2(s)} \;=\; L(C)\,,
\end{aligned} \tag{12.52}$$

where $\tau = t/a$. This proves Eq. (12.51).

Remark on the relation with the variational derivative

The standard variational derivative, $\delta/\delta x_\mu(\sigma)$, can be expressed via the path and area derivatives using the formula

$$\frac{\delta}{\delta x_\mu(\sigma)} \;=\; \dot{x}_\nu(\sigma)\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} + \sum_{i=1}^m \partial^{x_i}_\mu\,\delta(\sigma-\sigma_i)\,, \tag{12.53}$$

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where the sum on the RHS is present for the case of a functional having m marked (irregular) points $x_i \equiv x(\sigma_i)$. The simplest example of the functional with m marked points is just a function of m variables $x_1, \ldots, x_m$.

Using Eq. (12.53), the path derivative can be calculated as the limiting procedure

$$\partial_\mu^{x(\sigma)} = \int\limits_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' \frac{\delta}{\delta x_\mu(\sigma')}\,. \tag{12.54}$$

The result is obviously nonvanishing only when ∂_μ^x is applied to a functional with $x(\sigma)$ being a marked point.

It is nontrivial that the area derivative can also be expressed via the variational derivative [Pol80]:

$$\frac{\delta}{\delta\sigma_{\mu\nu}(x(\sigma))} = \int\limits_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' (\sigma'-\sigma) \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\nu(\sigma)}\,. \tag{12.55}$$

The point is that the six-component quantity, $\delta/\delta\sigma_{\mu\nu}(x(\sigma))$, is expressed via the four-component one, $\delta/\delta x_\mu(\sigma)$, which is possible because the components of $\delta/\delta\sigma_{\mu\nu}(x(\sigma))$ are dependent owing to the loop-space Bianchi identity (12.49).

12.4 Loop equations

By virtue of Eq. (12.48), Eq. (12.33) can be represented completely in loop space:

$$\begin{aligned} &\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \left\langle \Phi(C) \right\rangle \\ &\quad = \lambda \oint\limits_C \mathrm{d}y_\nu\, \delta^{(d)}(x-y) \left\langle \left[\Phi(C_{yx})\, \Phi(C_{xy}) - \frac{1}{N^2}\Phi(C) \right] \right\rangle, \end{aligned} \tag{12.56}$$

or, using the definitions (12.1) and (12.2) of the loop averages, as

$$\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) = \lambda \oint\limits_C \mathrm{d}y_\nu\, \delta^{(d)}(x-y) \left[W_2(C_{yx}, C_{xy}) - \frac{1}{N^2} W(C) \right]. \tag{12.57}$$

This equation is not closed. Having started from $W(C)$, we obtain another quantity, $W_2(C_1, C_2)$, so that Eq. (12.57) connects the one-loop average with a two-loop one. This is similar to the case of the (quantum)

φ^3-theory, whose Schwinger–Dyson equations (2.47) connect the n-point Green functions with different n. We shall derive this complete set of equations for the n-loop averages later in this section.

However, the two-loop average factorizes in the large-N limit:

$$W_2(C_1, C_2) \;=\; W(C_1)\, W(C_2) + \mathcal{O}\big(N^{-2}\big)\,, \tag{12.58}$$

as was discussed in Sect. 11.6. Keeping the constant λ (defined by Eq. (12.34)) fixed in the large-N limit as prescribed by Eq. (11.13), we obtain [MM79]

$$\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \;=\; \lambda \oint\limits_C \mathrm{d}y_\nu\, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy}) \tag{12.59}$$

as $N \to \infty$.

Equation (12.59) is a closed equation for the Wilson loop average in the large-N limit. It is referred to as the *loop equation* or the Makeenko–Migdal equation.

To find $W(C)$, Eq. (12.59) should be solved in the class of Stokes functionals with the initial condition

$$W(0) \;=\; 1 \tag{12.60}$$

for loops which are shrunk to points. This is a consequence of the obvious property of the Wilson loop

$$\mathrm{e}^{\mathrm{i}\oint_0 \mathrm{d}\xi^\mu \mathcal{A}_\mu} \;=\; 1 \tag{12.61}$$

and the normalization $\langle\, 1\,\rangle = 1$ of the averages.

The factorization (12.58) can itself be derived from the chain of loop equations. Proceeding as before, we obtain

$$\begin{aligned}
&\frac{1}{\lambda}\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_n(C_1,\ldots,C_n) \\
&\quad= \oint\limits_{C_1} \mathrm{d}y_\nu\, \delta^{(d)}(x-y) \left[W_{n+1}(C_{xy}, C_{yx}, \ldots, C_n) - \frac{1}{N^2} W_n(C_1,\ldots,C_n)\right] \\
&\qquad\quad + \sum_{j\geq 2} \frac{1}{N^2} \oint\limits_{C_j} \mathrm{d}y_\nu\, \delta^{(d)}(x-y) \left[W_{n-1}\Big(C_1C_j, \ldots, \underline{C_j}, \ldots, C_n\Big) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. - W_n(C_1,\ldots,C_n)\right].
\end{aligned} \tag{12.62}$$

Here x belongs to C_1; C_1C_j denotes the joining of C_1 and C_j; $\underline{C_j}$ denotes that C_j is omitted.

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Equation (12.62) looks like Eq. (2.47) for φ^3-theory. Moreover, the number of colors N enters Eq. (12.62) simply as a scalar factor, N^{-2}, likewise Planck's constant $\hbar$ enters Eq. (2.47). It is the major advantage of the use of loop space. What was mentioned in Sect. 11.8 concerning the "semiclassical" nature of the $1/N$-expansion of QCD is realized explicitly in Eq. (12.62). Its expansion in $1/N$ is straightforward.

At $N = \infty$, Eq. (12.62) is simplified to

$$\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_n(C_1,\ldots) \;=\; \lambda \oint_{C_1} \mathrm{d}y_\nu\, \delta^{(d)}(x-y)\, W_{n+1}(C_{yx}, C_{xy}, \ldots)\,. \tag{12.63}$$

This equation possesses [Mig80] a factorized solution

$$\begin{aligned} W_n(C_1,\ldots,C_n) \;&=\; \langle \Phi(C_1)\rangle \cdots \langle \Phi(C_n)\rangle + \mathcal{O}\big(N^{-2}\big) \\ &\equiv\; W(C_1)\cdots W(C_n) + \mathcal{O}\big(N^{-2}\big) \end{aligned} \tag{12.64}$$

provided $W(C)$ obeys Eq. (12.59) which plays the role of a "classical" equation in the large-N limit. Thus, we have given a nonperturbative proof of the large-N factorization of the Wilson loops.

Problem 12.6 Derive a lattice analog of the loop equation.

Solution The derivation is similar to that in Problem 6.3 on p. 105 for the classical case. We perform the shift (6.22) in the definition (6.42) of the lattice Wilson loop average. Similarly to Eqs. (6.24) and (12.59), we obtain

$$\frac{\beta}{2N^2} \sum_p \left[W(C\,\partial p) - W\big(C\,\partial p^{-1}\big) \right] \;=\; \sum_{l\in C} \delta_{xy} \tau_\nu(l)\, W(C_{yx})\, W(C_{xy})\,. \tag{12.65}$$

Here we use the notations of Problem 5.6 on p. 92 so that the contours $C\,\partial p$ and $C\,\partial p^{-1}$ are obtained from C_{xx} by adding the boundary of the plaquette p (∂p^{-1} denotes that the orientation of the boundary is opposite) and the sum over p goes over the $2(d-1)$ plaquettes involving the link at which the shift of $U_\nu(x)$ is performed. These contours are depicted in Fig. 12.4.

The sum on the RHS goes over the links belonging to the contour C. The unit vector $\tau_\nu(l) = 0, \pm 1$ denotes the projection of the (oriented) link $l \in C$ on the axis ν ($\tau_\nu(l) = 1$, -1 or 0 when the directions are parallel, antiparallel, or perpendicular, respectively). The point y is defined as the beginning of the link l if it has positive direction, or as the end of l if it has negative direction. Such an asymmetry arises from the fact that we have performed the right shift (6.22) of $U_\nu(x)$. The Kronecker symbol δ_{xy} guarantees that C_{yx} and C_{xy} are always closed.

Equation (12.65) is a lattice regularization of the continuum loop equation (12.59). The loop equation on the lattice was first discussed in [Foe79, Egu79] and with quarks in [Wei79].

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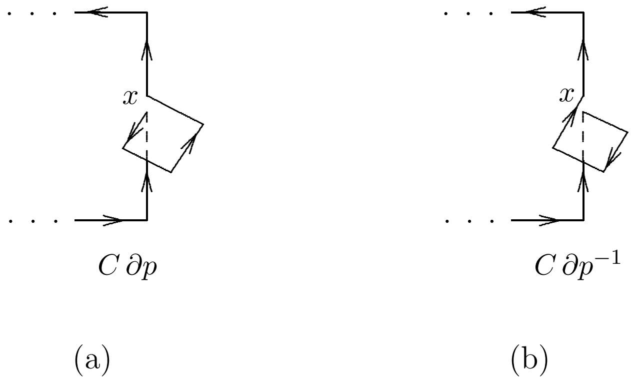


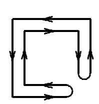
Fig. 12.4. Contours (a) $C\,\partial p$ and (b) $C\,\partial p^{-1}$ on the RHS of the lattice loop equation (12.65).

Problem 12.7 Find a solution to the lattice loop equation (12.65) at small β/N^2.

Solution A strong-coupling solution to Eq. (12.65) can be obtained iteratively in β/N^2. Let us choose the contour C to be the boundary ∂p_0 of a plaquette p_0. Since δ_{xy} on the RHS of Eq. (12.65) is nonvanishing only when y coincides with x, we rewrite Eq. (12.65) as

$$W(\partial p_0) \;=\; \frac{\beta}{2N^2}\sum_p \left[W(\partial p_0\,\partial p) - W(\partial p_0\,\partial p^{-1})\right]. \tag{12.66}$$

One of the terms on the RHS is

$$W\,(\partial p_0\,\partial p) \;=\; W\Big($$

$$\Big) \;=\; W\,(0) \;=\; 1\,, \tag{12.67}$$

when p and p_0 have opposite orientations as depicted in Fig. 6.7 on p. 115, owing to the backtracking condition (12.43) and the initial condition (12.60). We thus obtain

$$W(\partial p) \;=\; \frac{\beta}{2N^2} \tag{12.68}$$

to the leading order in β/N^2, which reproduces Eq. (6.72). The other terms on the RHS of Eq. (12.66) are of the next order in β/N^2.

Analogously, Eq. (6.73) is reproduced for a general contour C to the leading order in β/N^2, since

$$\min\left\{A\,(C\,\partial p)\right\} \;=\; A_{\min}\,(C) - 1 \tag{12.69}$$

in the lattice units.

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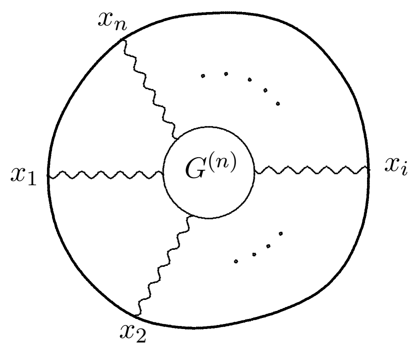

Fig. 12.5. Graphical representation of the terms on the RHS of Eq. (12.70).

12.5 Relation to planar diagrams

The perturbation-theory expansion of the Wilson loop average can be calculated from Eq. (11.46), which we represent in the form

$$\begin{aligned} W(C) &= 1 + \sum_{n=2}^{\infty} \mathrm{i}^n \oint\limits_C \mathrm{d}x_1^{\mu_1} \oint\limits_C \mathrm{d}x_2^{\mu_2} \cdots \oint\limits_C \mathrm{d}x_n^{\mu_n} \\ &\quad \times \theta_{\mathrm{c}}(1,2,\ldots,n)\, G^{(n)}_{\mu_1\mu_2\cdots\mu_n}(x_1,x_2,\ldots,x_n)\,, \end{aligned} \tag{12.70}$$

where $\theta_{\mathrm{c}}(1,2,\ldots,n)$ orders the points $x_1,\ldots,x_n$ along the contour in cyclic order and $G^{(n)}_{\mu_1\cdots\mu_n}$ is given by Eq. (11.71). This θ-function has the meaning of the propagator of a test heavy particle on contour C (see Problem 5.3 on p. 90).

We assume, for definiteness, dimensional regularization throughout this section to make all the integrals well-defined.

Each term on the RHS of Eq. (12.70) can be conveniently represented by the diagram in Fig. 12.5, where the integration over contour C is associated with each point x_i lying on contour C.

These diagrams are analogous to those discussed in Sect. 11.3 with one external boundary – the Wilson loop in the given case. This was already mentioned in the Remark on p. 227. In the large-N limit, only planar diagrams survive. Some of them, which are of the lowest order in λ, are depicted in Fig. 12.6. The diagram in Fig. 12.6a has already been considered in Problem 12.2 (see Eq. (12.17)).

The large-N loop equation (12.59) describes the sum of the planar diagrams. Its iterative solution in λ reproduces the set of planar diagrams for $W(C)$ provided the initial condition (12.60) and some boundary conditions for asymptotically large contours are imposed.

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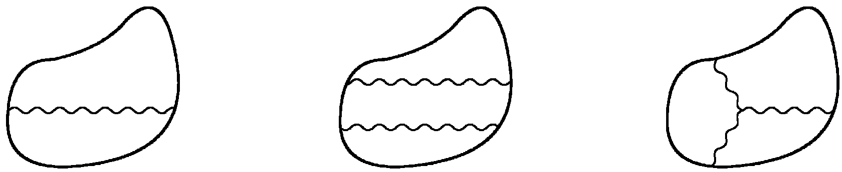

(a) (b) (c)

Fig. 12.6. Planar diagrams for $W(C)$: (a) of order λ with a gluon propagator, and of order λ^2 (b) with two noninteracting gluons and (c) with the three-gluon vertex. Diagrams of order λ^2 with one-loop insertions to the gluon propagator are not shown.

Equation (12.70) can be viewed as an ansatz for $W(C)$ with some unknown functions $G^{(n)}_{\mu_1\cdots\mu_n}(x_1,\ldots,x_n)$ to be determined by substitution into the loop equation. To preserve symmetry properties of $W(C)$, the functions $G^{(n)}$ must be symmetric under a cyclic permutation of the points $1,\ldots,n$ and depend only on $x_i - x_j$ (translational invariance). The main advantage of this ansatz is that it corresponds automatically to a Stokes functional, owing to the properties of vector integrals, and the initial condition (12.60) is satisfied.

The action of the area and path derivatives on the ansatz (12.70) is easily calculable. For instance, the area derivative is given by

$$\begin{aligned}
\frac{\delta W(C)}{\delta \sigma_{\mu\nu}(z)} &= \sum_{n=1}^{\infty} \mathrm{i}^{(n+1)} \oint_C \mathrm{d}x_1^{\mu_1} \ldots \oint_C \mathrm{d}x_n^{\mu_n}\, \theta_{\mathrm{c}}(1,2,\ldots,n) \\
&\quad \times \Big[\left(\partial^z_\mu \delta_{\nu\alpha} - \partial^z_\nu \delta_{\mu\alpha}\right) G^{(n+1)}_{\alpha\mu_1\cdots\mu_n}(z,x_1,\ldots,x_n) \\
&\qquad + \mathrm{i}\left(\delta_{\mu\beta}\delta_{\nu\alpha} - \delta_{\mu\alpha}\delta_{\nu\beta}\right) G^{(n+2)}_{\alpha\beta\mu_1\cdots\mu_n}(z,z,x_1,\ldots,x_n)\Big].
\end{aligned} \tag{12.71}$$

The analogy with the Mandelstam formula (12.45) is obvious.

More concerning solving the loop equation by the ansatz (12.70) can be found in [MM81, BGS82, Mig83].

Problem 12.8 Solve Eq. (12.59) to order λ using the ansatz (12.70).

Solution To order λ, we can restrict ourselves by the $n = 2$ term in the ansatz (12.70). For the θ-function, we have

$$\theta_{\mathrm{c}}(1,2) \;\equiv\; \frac{1}{2}\left[\theta(1,2)+\theta(2,1)\right] \;=\; \frac{1}{2}\,. \tag{12.72}$$

The meaning of this formula is obvious: there is no cyclic ordering for two points. We therefore rewrite the ansatz as

$$W(C) \;=\; 1 - \frac{\lambda}{2} \oint\limits_C \mathrm{d}x_\mu \oint\limits_C \mathrm{d}y_\nu \, D_{\mu\nu}(x-y) + \mathcal{O}\big(\lambda^2\big) \tag{12.73}$$

with some unknown function $D_{\mu\nu}(x-y)$. Its tensor structure reads

$$D_{\mu\nu}(x-y) \;=\; \delta_{\mu\nu} D(x-y) + \partial_\mu \partial_\nu f(x-y)\,. \tag{12.74}$$

The second (longitudinal) term in this formula does not contribute to $W(C)$ since the contour integral of this term vanishes in Eq. (12.73). We can thus write

$$W(C) \;=\; 1 - \frac{\lambda}{2} \oint\limits_C \mathrm{d}x_\mu \oint\limits_C \mathrm{d}y_\mu \, D(x-y) + \mathcal{O}\big(\lambda^2\big)\,. \tag{12.75}$$

The area derivative can be calculated easily using Stokes' theorem, which gives

$$\frac{\delta}{\delta\sigma_{\mu\nu}(z)} \oint\limits_C \mathrm{d}x_\rho \oint\limits_C \mathrm{d}y_\rho \, D(x-y) \;=\; 2\Bigg[\oint\limits_C \mathrm{d}y_\nu \, \partial_\mu D(z-y) - \oint\limits_C \mathrm{d}y_\mu \, \partial_\nu D(z-y) \Bigg] \tag{12.76}$$

and

$$\partial^z_\mu \frac{\delta}{\delta\sigma_{\mu\nu}\,(z)} \oint\limits_C \mathrm{d}x_\rho \oint\limits_C \mathrm{d}y_\rho \, D(x-y) \;=\; 2 \oint\limits_C \mathrm{d}y_\nu \, \partial^2 D(z-y) \tag{12.77}$$

since

$$\partial_\mu \oint dy_\mu \, \partial_\nu D(x-y) \;=\; 0\,. \tag{12.78}$$

Substituting into the loop equation (12.59), we find

$$- \oint\limits_C \mathrm{d}y_\nu \, \partial^2 D(x-y) \;=\; \oint\limits_C \mathrm{d}y_\nu \, \delta^{(d)}(x-y) \tag{12.79}$$

which is equivalent to

$$-\partial^2 D(x-y) \;=\; \delta^{(d)}(x-y) \tag{12.80}$$

since the contour C is arbitrary. The solution to Eq. (12.80) is unique, provided $D(x-y)$ decreases for large $x-y$, and recovers the propagator (11.4).

12.6 Loop-space Laplacian and regularization

The loop equation (12.59) is *not* yet entirely formulated in loop space. It is a d-vector equation, both sides of which depend explicitly on the point x which does not belong to loop space. The fact that we have a d-vector equation for a scalar quantity means, in particular, that Eq. (12.59) is overspecified.

A practical difficulty in solving Eq. (12.59) is that the area and path derivatives, $\delta/\delta\sigma_{\mu\nu}(x)$ and ∂_μ^x, which enter the LHS are complicated, generally speaking, noncommutative operators. They are intimately related to the Yang–Mills perturbation theory where they correspond to the non-Abelian field strength $F_{\mu\nu}$ and the covariant derivative ∇_μ. However, it is not easy to apply these operators to a generic functional $W(C)$ which is defined on elements of loop space.

A much more convenient form of the loop equation can be obtained by integrating both sides of Eq. (12.59) over dx_ν along the same contour C, which yields

$$\oint\limits_C \mathrm{d}x_\nu \, \partial_\mu^x \, \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \;=\; \lambda \oint\limits_C \mathrm{d}x_\mu \oint\limits_C \mathrm{d}y_\mu \, \delta^{(d)}(x-y) \, W(C_{yx}) \, W(C_{xy}) \, . \tag{12.81}$$

Now both the operator on the LHS and the functional on the RHS are scalars without labeled points and are well-defined in loop space. The operator on the LHS of Eq. (12.81) can be interpreted as an infinitesimal variation of elements of loop space.

Equations (12.59) and (12.81) are completely equivalent. A proof of equivalence of the scalar Eq. (12.81) and original d-vector Eq. (12.59) is based on the important property of Eq. (12.59), for which both sides are identically annihilated by the operator ∂_ν^x. It is a consequence of the identity (see Sect. 5.1)

$$\nabla_\mu \nabla_\nu \, \mathcal{F}_{\mu\nu} \;=\; -\frac{1}{2} \left[\mathcal{F}_{\mu\nu}, \mathcal{F}_{\mu\nu}\right] \;=\; 0 \tag{12.82}$$

in ordinary space. Owing to this property, the vanishing of the contour integral of some vector is equivalent to the vanishing of the vector itself, so that Eq. (12.59) can in turn be deduced from Eq. (12.81).

Equation (12.81) is associated with the second-order Schwinger–Dyson equation

$$-\int \mathrm{d}^d x \, \nabla_\mu F^a_{\mu\nu}(x) \, \frac{\delta}{\delta A^a_\nu(x)} \;\stackrel{\text{w.s.}}{=}\; \hbar \int \mathrm{d}^d x \, \mathrm{d}^d y \, \delta^{(d)}(x-y) \, \frac{\delta}{\delta A^a_\nu(y)} \frac{\delta}{\delta A^a_\nu(x)} \tag{12.83}$$

in the same sense as Eq. (12.59) is associated with Eq. (12.27). It is called "second order" since the RHS involves two variational derivatives with respect to A_ν.

The operator on the LHS of Eq. (12.81) is a well-defined object in loop space. When applied to regular functionals which do not have marked points, it can be represented, using Eqs. (12.54) and (12.55), in an equivalent form

$$\Delta \;\equiv\; \oint\limits_C \mathrm{d}x_\nu \, \partial^x_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} \;=\; \int\limits_{\sigma_0}^{\sigma_1} \mathrm{d}\sigma \int\limits_{\sigma-0}^{\sigma+0} \mathrm{d}\sigma' \frac{\delta}{\delta x_\mu(\sigma')} \frac{\delta}{\delta x_\mu(\sigma)} \,. \tag{12.84}$$

As was first pointed out by Gervais and Neveu [GN79b], this operator is nothing but a functional extension of the Laplace operator, which is known in mathematics as the Lévy operator.* Equation (12.81) can be represented in turn as an (inhomogeneous) functional Laplace equation

$$\Delta W(C) \;=\; \lambda \oint\limits_C \mathrm{d}x_\mu \oint\limits_C \mathrm{d}y_\mu \, \delta^{(d)}(x-y)\, W(C_{yx})\, W(C_{xy})\,. \tag{12.85}$$

We shall refer to this equation as the loop-space Laplace equation.

The form (12.85) of the loop equation is convenient for a nonperturbative ultraviolet regularization.

The idea is to start from the regularized version of Eq. (12.83), replacing the delta-function on the RHS by the kernel of the regularizing operator:

$$\delta^{ab}\delta^{(d)}(x-y) \;\stackrel{\text{reg.}}{\Longrightarrow}\; \left\langle y \left| \boldsymbol{R}^{ab} \right| x \right\rangle \;=\; \boldsymbol{R}^{ab}\, \delta^{(d)}(x-y) \tag{12.86}$$

with

$$\boldsymbol{R}^{ab} \;=\; \left(\mathrm{e}^{a^2\nabla^2/2}\right)^{ab}, \tag{12.87}$$

where ∇_μ is the covariant derivative in the adjoint representation. The regularized version of Eq. (12.83) is

$$-\int \mathrm{d}^d x\, \nabla_\mu F^a_{\mu\nu}(x) \frac{\delta}{\delta A^a_\nu(x)} \;\stackrel{\text{w.s.}}{=}\; \hbar \int \mathrm{d}^d x\, \mathrm{d}^d y \left\langle y \left| \boldsymbol{R}^{ab} \right| x \right\rangle \frac{\delta}{\delta A^a_\nu(y)} \frac{\delta}{\delta A^b_\nu(x)}. \tag{12.88}$$

To translate Eq. (12.88) in loop space, we use the path-integral representation (see Problem 5.5 on p. 91)

$$\left\langle y \left| \boldsymbol{R}^{ab} \right| x \right\rangle \;=\; \int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\, \mathrm{e}^{-\frac{1}{2}\int_0^{a^2} \mathrm{d}t\, \dot{r}^2_\mu(t)} \;\mathrm{tr}\left[t^a U(r_{yx}) t^b U(r_{xy})\right] \tag{12.89}$$

with

$$U(r_{yx}) \;=\; \boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_x^y \mathrm{d}r^\mu \mathcal{A}_\mu(r)}, \tag{12.90}$$

* See the book by Lévy [Lev51] and the review [Fel86].

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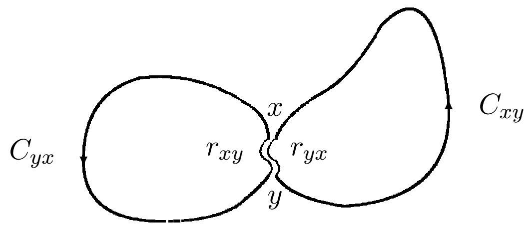

Fig. 12.7. Contours $C_{yx}r_{xy}$ and $C_{xy}r_{yx}$ which enter the RHSs of Eqs. (12.92) and (12.93).

where the integration is over regulator paths $r_\mu(t)$ from x to y, for which the typical length is $\sim a$. The conventional measure is implied in (12.89) so that

$$\int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^{a^2}\mathrm{d}t\,\dot{r}_\mu^2(t)}\;\mathrm{tr}\left[t^a t^b\right] \;=\; \delta^{ab}\frac{1}{(2\pi a^2)^{d/2}}\,\mathrm{e}^{-(x-y)^2/2a^2}. \tag{12.91}$$

Calculating the variational derivatives on the RHS of Eq. (12.88), using Eq. (12.89) and the completeness condition (11.6), we obtain as $N\to\infty$

$$\begin{aligned}&\int \mathrm{d}^d x\,\mathrm{d}^d y\,\left\langle y\left|\boldsymbol{R}^{ab}\right|x\right\rangle \frac{\delta}{\delta A_\nu^a(y)}\frac{\delta}{\delta A_\nu^b(x)}\Phi(C)\\ &\quad= \lambda\oint\limits_C dx_\mu\oint\limits_C dy_\mu \int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^{a^2}\mathrm{d}t\,\dot{r}_\mu^2(t)}\,\Phi(C_{yx}r_{xy})\,\Phi(C_{xy}r_{yx})\,,\end{aligned} \tag{12.92}$$

where the contours $C_{yx}r_{xy}$ and $C_{xy}r_{yx}$ are depicted in Fig. 12.7. Averaging over the gauge field and using the large-N factorization, we arrive at the regularized loop-space Laplace equation [HM89]

$$\begin{aligned}&\Delta W(C)\\ &\quad= \lambda\oint\limits_C \mathrm{d}x_\mu\oint\limits_C \mathrm{d}y_\mu \int\limits_{\substack{r_\mu(0)=x_\mu \\ r_\mu(a^2)=y_\mu}} \mathcal{D}r_\mu(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^{a^2}\mathrm{d}t\,\dot{r}_\mu^2(t)}\,W(C_{yx}r_{xy})\,W(C_{xy}r_{yx})\end{aligned} \tag{12.93}$$

which manifestly recovers Eq. (12.85) when $a\to 0$.

The constructed regularization is nonperturbative, while perturbatively it reproduces regularized Feynman diagrams. An advantage of this regularization of the loop equation is that the contours $C_{yx}r_{xy}$ and $C_{xy}r_{yx}$ on the RHS of Eq. (12.93) are both closed and do not have marked points if C does not have one. Therefore, Eq. (12.93) is written entirely in loop space.

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Remark on functional Laplacian

It is worth noting that the representation of the functional Laplacian on the RHS of Eq. (12.84) is defined for a wider class of functionals than Stokes functionals. The point is that the standard definition of the functional Laplacian from the book by Lévy [Lev51] uses solely the concept of the second variation of a functional $U[x]$, namely the term in the second variation which is proportional to $[\delta x_\mu(\sigma)]^2$:

$$\delta^2 U[x] \;=\; \frac{1}{2}\int_{\sigma_0}^{\sigma_1} \mathrm{d}\sigma\, [\delta x_\mu(\sigma)]^2\, U''_{xx}[x] + \cdots . \tag{12.94}$$

The functional Laplacian Δ is then defined by the formula

$$\Delta U[x] \;=\; \int_{\sigma_0}^{\sigma_1} \mathrm{d}\sigma\, U''_{xx}[x]\,. \tag{12.95}$$

Here $U[x]$ can be an arbitrary, not necessarily parametric invariant, functional. To emphasize this obstacle, we use the notation $U[x]$ for generic functionals which are defined on L_2 space in comparison with $U(C)$ for the functionals which are defined on elements of loop space. It is easier to deal with the whole operator Δ, rather than separately with the area and path derivatives.

The functional Laplacian is parametric invariant and possesses a number of remarkable properties. While a finite-dimensional Laplacian is a second-order operator, the functional Laplacian is of first order and satisfies the Leibnitz rule

$$\Delta\,(UV) \;=\; (\Delta U)\,V + U\,(\Delta V)\,. \tag{12.96}$$

The functional Laplacian can be approximated [Mak88] in loop space by a (second-order) partial differential operator in such a way as to preserve these properties in the continuum limit. This loop-space Laplacian can be inverted to determine a Green function $G\,(C, C')$ in the form of a sum over surfaces $S_{C,C'}$ connecting two loops:

$$G\left(C, C'\right) \;=\; \sum_{S_{C,C'}} \cdots\,, \tag{12.97}$$

which is analogous to the representation (1.102) of the Green function of the ordinary Laplacian. The standard perturbation theory can then be recovered by iterating Eq. (12.85) (or its regularized version (12.93)) in λ with the Green function (12.97).

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12.7 Survey of nonperturbative solutions

While the loop equations were proposed long ago, not much is known concerning their nonperturbative solutions except in two dimensions. We briefly list some of the available results.

It was shown [MM80] that the area law

$$W(C) \;\equiv\; \langle \Phi(C) \rangle \;\;\propto\;\; \mathrm{e}^{-K \cdot A_{\min}(C)} \tag{12.98}$$

satisfies the large-N loop equation for asymptotically large C. However, a self-consistency equation for K, which should relate it to the bare charge and the cutoff, was not investigated. In order to do this, one needs more detailed information concerning the behavior of $W(C)$ for intermediate loops.

The *free* bosonic Nambu–Goto string which is defined as a sum over surfaces spanned by C

$$W(C) \;\;=\;\; \sum_{S:\partial S = C} \mathrm{e}^{-K \cdot A(S)}\,, \tag{12.99}$$

with the action being the area $A(S)$ of the surface S, is *not* a solution for intermediate loops. Consequently, QCD does not reduce to this kind of string, as was expected originally in [GN79a, Nam79, Pol79]. Roughly speaking, the ansatz (12.99) is not consistent with the factorized structure on the RHS of Eq. (12.59).

Nevertheless, it was shown that if a free string satisfies Eq. (12.59), then the same interacting string satisfies the loop equations for finite N. Here "free string" means, as is usual in string theory, that only surfaces of genus zero are present in the sum over surfaces, while surfaces or higher genera are associated with a string interaction. The coupling constant of this interaction is $\mathcal{O}\big(N^{-2}\big)$.

A formal solution of Eq. (12.59) for all loops was found by Migdal [Mig81] in the form of a fermionic string

$$W(C) \;\;=\;\; \sum_{S:\partial S = C} \int \mathcal{D}\psi \, \mathrm{e}^{-\int \mathrm{d}^2\xi \left[\bar{\psi}\sigma_k \partial_k \psi + \bar{\psi}\psi\, m \sqrt[4]{g}\right]}, \tag{12.100}$$

where the world sheet of the string is parametrized by the coordinates ξ_1 and ξ_2 for which the two-dimensional metric is conformal, i.e. diagonal. The field $\psi(\xi)$ describes two-dimensional elementary fermions (elves) living in the surface S, and m denotes their mass. Elves were introduced to provide a factorization which now holds owing to some remarkable properties of two-dimensional fermions. For large loops, the internal fermionic structure becomes frozen, so that the empty string behavior (12.98) is recovered. For small loops, the elves are necessary for asymptotic freedom.

However, it is unclear whether or not the string solution (12.100) is practically useful for a study of multicolor QCD, since the methods of dealing with the string theory in four dimensions have not yet been developed.

A very interesting solution of the large-N loop equation on a lattice, found by Eguchi and Kawai [EK82], shows that the $SU(N)$ gauge theory on an infinite lattice and a unit hypercube are equivalent at $N = \infty$. With slight modifications this large-N reduction holds in the continuum theory as well, so that the space-time can be absorbed by the internal symmetry group. More concerning the large-N reduction will be said in Part 4.

12.8 Wilson loops in QCD$_2$

Two-dimensional QCD (QCD$_2$) has been popular since the paper by 't Hooft [Hoo74b] as a simplified model of QCD$_4$.

One can always choose the axial gauge

$$A_1 \;=\; 0\,, \tag{12.101}$$

so that the commutator in the non-Abelian field strength (5.14) vanishes in two dimensions. Therefore, there is no gluon self-interaction in this gauge and the theory looks, at first glance, like the Abelian one.

The Wilson loop average in QCD$_2$ can be calculated straightforwardly via the expansion (12.70) where only disconnected (free) parts of the correlators $G^{(n)}$ for even n should be left, since there is no interaction. Only the planar structure of color indices contributes at $N = \infty$. Diagrammatically, the diagrams of the type depicted in Figs. 12.6a and b are relevant for contours without self-intersections, while that in Fig. 12.6c should be omitted in two dimensions.

The color structure of the relevant planar diagrams can be reduced by virtue of the formula

$$\sum_a \left(t^a\right)^{ik} \left(t^a\right)^{kj} \;=\; N\delta^{ij}\,, \tag{12.102}$$

which is a consequence of the completeness condition (11.6) at large N. We have

$$\begin{aligned} W(C) \;=\; & 1 + \sum_k^\infty \,(-\lambda)^k \oint\limits_C \mathrm{d}x_1^{\mu_1} \oint\limits_C \mathrm{d}x_2^{\nu_1} \cdots \oint\limits_C \mathrm{d}x_{2k-1}^{\mu_k} \oint\limits_C \mathrm{d}x_{2k}^{\nu_k} \\ & \times \theta_{\mathrm{c}}(1,2,\ldots,2k)\, D_{\mu_1\nu_1}(x_1 - x_2) \cdots D_{\mu_k\nu_k}(x_{2k-1} - x_{2k})\,, \end{aligned} \tag{12.103}$$

where the points $x_1, \ldots, x_{2k}$ are still cyclic ordered along the contour. Similarly to Problem 5.2 on p. 89, we can exponentiate the RHS of

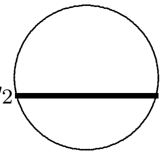

Fig. 12.8. Graphical representation of the contour integral on the LHS of Eq. (12.108) in the axial gauge. The bold line represents the gluon propagator (12.105) with $x_2 = y_2$ owing to the delta-function.

Eq. (12.103) to obtain finally

$$W(C) \;=\; \exp\left[-\frac{\lambda}{2}\oint_C \mathrm{d}x^\mu \oint_C \mathrm{d}y^\nu D_{\mu\nu}(x-y)\right]. \qquad (12.104)$$

This is the same formula as in the Abelian case if λ denotes e^2.

The propagator $D_{\mu\nu}(x,y)$ is, strictly speaking, the one in the axial gauge (12.101) which is given by

$$D_{\mu\nu}(x-y) \;=\; \frac{1}{2}\delta_{\mu 2}\delta_{\nu 2}\,|x_1-y_1|\,\delta^{(1)}(x_2-y_2)\,. \qquad (12.105)$$

However, the contour integral on the RHS of Eq. (12.104) is gauge invariant, and we can simply choose

$$D_{\mu\nu}(x-y) \;=\; \delta_{\mu\nu}\,D(x-y)\,. \qquad (12.106)$$

In two dimensions* we have

$$D(x-y) \;=\; \frac{1}{4\pi}\ln\frac{\ell^2}{(x-y)^2}\,, \qquad (12.107)$$

where ℓ is an arbitrary parameter with dimension of length. Nothing depends on it because the contour integral of a constant vanishes.

The propagator (12.106) is usually associated with the Feynman gauge. The explicit form (12.104) indicates that a contribution of diagrams with vertices, which are present in the Feynman gauge, vanishes in two dimensions.

The contour integral in the exponent on the RHS of Eq. (12.104) can be represented graphically as depicted in Fig. 12.8, where $x_2 = y_2$ owing

* In d dimensions

$$D(x-y) \;=\; \frac{1}{4\pi^{d/2}}\,\Gamma\left(\frac{d}{2}-1\right)\frac{1}{\left[(x-y)^2\right]^{d/2-1}}\,.$$

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to the delta-function in Eq. (12.105) and the bold line represents $|x_1 - y_1|$. This gives

$$\oint_C \mathrm{d}x^\mu \oint_C \mathrm{d}y^\nu D_{\mu\nu}(x-y) \quad = \quad A(C)\,, \tag{12.108}$$

where $A(C)$ is the area enclosed by the contour C. Finally, we obtain

$$W(C) \quad = \quad \mathrm{e}^{-\frac{\lambda}{2}A(C)} \tag{12.109}$$

for contours without self-intersections.

Therefore, the area law holds in two dimensions both in the non-Abelian and Abelian cases. This is, roughly speaking, because of the form of the two-dimensional propagator (12.107), which decreases with distance only logarithmically in the Feynman gauge.

Problem 12.9 Prove Eq. (12.109) in the Feynman gauge.

Solution To prove Eq. (12.108) in the Feynman gauge (12.106) and (12.107), we note that the area element in two dimensions can be represented by

$$\mathrm{d}\sigma^{\mu\nu}(x) \;\equiv\; \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \;=\; \varepsilon^{\mu\nu}\mathrm{d}^2x\,, \tag{12.110}$$

where $\varepsilon^{\mu\nu}$ is the antisymmetric tensor $\varepsilon^{12} = -\varepsilon^{21} = 1$. Therefore, the area can be represented by the double integral

$$A(C) \quad = \quad \frac{1}{2}\int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(x) \int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(y)\,\delta^{(2)}(x-y) \tag{12.111}$$

which goes along the surface $S(C)$ enclosed by the (nonintersecting) loop C.

Applying Stokes' theorem, we obtain

$$\begin{aligned}
\oint_C \mathrm{d}x^\mu \oint_C \mathrm{d}y^\mu D(x-y) \quad &= \quad \int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(x)\,\partial_\nu \oint_C \mathrm{d}y^\mu D(x-y) \\
&= \quad -\int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(x) \int\limits_{S(C)} \mathrm{d}\sigma^{\mu\rho}(y)\,\partial_\nu\partial_\rho D(x-y) \\
&= \quad -\frac{1}{2}\int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(x) \int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(y)\,\partial^2 D(x-y) \\
&= \quad \frac{1}{2}\int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(x) \int\limits_{S(C)} \mathrm{d}\sigma^{\mu\nu}(y)\,\delta^{(2)}(x-y)\,.
\end{aligned} \tag{12.112}$$

Using Eq. (12.111) we prove Eq. (12.109) in the Feynman gauge.

It is worth noting that Eq. (12.112) is based only on Stokes' theorem and holds for contours with arbitrary self-intersections. In contrast, Eq. (12.111) itself is valid only for nonintersecting loops.

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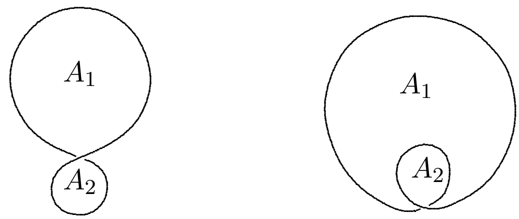

(a) (b)

Fig. 12.9. Contours with one self-intersection: A_1 and A_2 denote the areas of the proper windows. The total area enclosed by the contour in (a) is $A_1 + A_2$. The areas enclosed by the exterior and interior loops in (b) are $A_1 + A_2$ and A_2, respectively, while the total area of the surface with the folding is $A_1 + 2A_2$.

The difference between the Abelian and non-Abelian cases shows up for the contours with self-intersections.

We first note that the simple formula (12.108) does *not* hold for contours with arbitrary self-intersections.

The simplest contours with one self-intersection are depicted in Fig. 12.9. There is nothing special about the contour in Fig. 12.9a. Equation (12.108) still holds in this case with $A(C)$ being the total area, $A(C) = A_1 + A_2$.

The Wilson loop average for the contour in Fig. 12.9a coincides both for the Abelian and non-Abelian cases and equals

$$W(C) \quad = \quad \mathrm{e}^{-\frac{\lambda}{2}(A_1+A_2)}. \tag{12.113}$$

This is nothing but the exponential of the total area.

For the contour in Fig. 12.9b, we obtain

$$\oint\limits_C \mathrm{d}x^\mu \oint\limits_C \mathrm{d}y^\nu D_{\mu\nu}(x-y) \quad = \quad A_1 + 4A_2\,. \tag{12.114}$$

This is easy to understand in the axial gauge where the ends of the propagator line can lie both on the exterior and interior loops, or one end at the exterior loop and the other end on the interior loop. These cases are illustrated by Fig. 12.10. The contributions of the diagrams in Figs. 12.10a–d are A_1+A_2, A_2, A_2, and A_2, respectively. The result given by Eq. (12.114) is obtained by summing over all four diagrams.

For the contour in Fig. 12.9b, the Wilson loop average is

$$W(C) \quad = \quad \mathrm{e}^{-\frac{\lambda}{2}(A_1+4A_2)} \tag{12.115}$$

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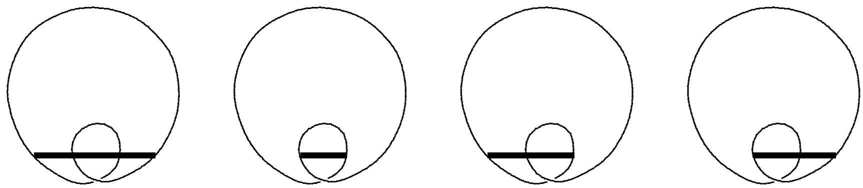

Fig. 12.10. Three type of contribution in Eq. (12.114). The ends of the propagator line lie both on (a) exterior and (b) interior loops, or (c), (d) one end on the exterior loop and another end on the interior loop.

in the Abelian case and

$$W(C) \quad = \quad (1-\lambda A_2)\,\mathrm{e}^{-\frac{\lambda}{2}(A_1+2A_2)} \tag{12.116}$$

in the non-Abelian case at $N=\infty$. They coincide only to order λ as they should.

The difference to the next orders is because only the diagrams with one propagator line connecting the interior and exterior loops are planar and, therefore, contribute in the non-Abelian case. Otherwise, the diagram is nonplanar and vanishes as $N \to \infty$.

Note that the exponential of the total area $A(C) = A_1 + 2A_2$ of the surface with the folding, which is enclosed by the contour C, appears in the exponent for the non-Abelian case. The additional pre-exponential factor could be associated with the entropy of folding the surface.

The Wilson loop averages (12.113) and (12.116) in QCD_2 at large N as well as those for contours with arbitrary self-intersections, which have a generic form

$$W(C) \quad = \quad P(A_1,\ldots,A_n)\,\mathrm{e}^{-\frac{\lambda}{2}A(C)}, \tag{12.117}$$

where P is a polynomial of the areas of individual windows and $A(C)$ is the total area of the surface with foldings, were first calculated in [KK80] by solving the two-dimensional loop equation and in [Bra80] by applying the non-Abelian Stokes theorem. The lattice version is given in [KK81].

Problem 12.10 Demonstrate that Eq. (12.104) satisfies the Abelian loop equation

$$\partial^x_\mu \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \quad = \quad \lambda \oint\limits_C \mathrm{d}y_\nu\, \delta^{(d)}(x-y)\, W(C)\,. \tag{12.118}$$

Solution The calculation is the same as in Problem 12.8 on p. 268. In $d=2$ one can alternatively use [OP81] the expression on the RHS of Eq. (12.112).

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Problem 12.11 Obtain Eqs. (12.113) and (12.116) for the contours with one self-intersection by solving the loop equation (12.59).

Solution Let us multiply Eq. (12.59) in $d = 2$ by $\varepsilon_{\rho\nu}$ and integrate over $\mathrm{d}x^\rho$ along a small (open) piece C' of the contour C including the point of self-intersection. We obtain

$$\varepsilon_{\rho\nu}\int\limits_{C'} \mathrm{d}x_\rho\, \partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \;=\; \lambda\varepsilon_{\rho\nu}\int\limits_{C'} \mathrm{d}x_\rho \oint\limits_{C} \mathrm{d}y_\nu\, \delta^{(2)}(x-y)\, W(C_{yx})\, W(C_{xy})\,. \tag{12.119}$$

The RHS of Eq. (12.119) can be calculated analogously to the known representation for the number of self-intersections of a loop in two dimensions. For the case of one self-intersection, we have

$$\varepsilon_{\rho\nu}\int\limits_{C'} \mathrm{d}x_\rho \oint\limits_{C} \mathrm{d}y_\nu\, \delta^{(2)}(x-y)\, W(C_{yx})\, W(C_{xy}) \;=\; W(C_1)\, W(C_2)\,, \tag{12.120}$$

where C_1 and C_2 denote, respectively, the upper and lower loops in Fig. 12.9a or the exterior and interior loops in Fig. 12.9b.

The LHS of Eq. (12.119) can be transformed as

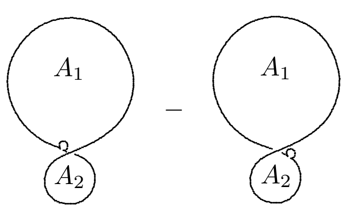
$$\varepsilon_{\rho\nu}\int\limits_{C'} \mathrm{d}x_\rho\, \partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W(C) \;=\; \int\limits_{C'} \mathrm{d}x_\nu\, \partial_\nu^x \frac{\delta}{\delta\sigma(x)} W(C)\,, \tag{12.121}$$

where $\delta/\delta\sigma(x)$ denotes the variational derivative with respect to the "scalar" area

$$\delta\sigma(x) \;=\; \frac{1}{2}\varepsilon_{\mu\nu}\delta\sigma^{\mu\nu}(x)\,. \tag{12.122}$$

The integrand on the RHS of Eq. (12.121) is a total derivative and the contour integral reduces to the difference of the Ω-variations at the end points of the contour C', which would vanish if there were no self-intersections. The RHS of Eq. (12.119) also vanishes if no self-intersections are present, so $W(C)$ is determined in this case by Eq. (12.118) rather than Eq. (12.119).

For the contour in Fig. 12.9a, this gives



$$= \left(-\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C)\,. \tag{12.123}$$

The Ω-variation of the contour on the LHS represents the variational derivative. The minus sign in front of $\partial/\partial A_1$ on the RHS is because adding the Ω-variation in the first term on the LHS decreases the area A_1, while that in the second term increases A_2. Then, for the contour in Fig. 12.9a, Eq. (12.119) takes the form

$$\left(-\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C) \;=\; \lambda W(C_1)\, W(C_2)\,. \tag{12.124}$$

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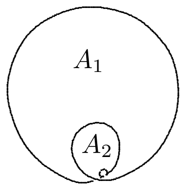
For the contour in Fig. 12.9b, we obtain quite similarly



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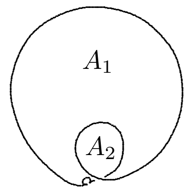

$$= \left(2\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C)\,. \tag{12.125}$$

Now adding the Ω-variation in the first term on the LHS increases A_1 and decreases A_2, while that in the second term decreases A_1. Equation (12.119) takes the form

$$\left(2\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C) \quad = \quad \lambda W(C_1)\, W(C_2)\,. \tag{12.126}$$

The RHSs of Eqs. (12.124) and (12.126) are known since C_1 and C_2 have no self-intersections so that Eq. (12.109) holds for $W(C_1)$ and $W(C_2)$. Finally, Eqs. (12.124) and (12.126) take the explicit form [KK80]

$$\left(-\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C) \quad = \quad \lambda\, \mathrm{e}^{-\frac{\lambda}{2}(A_1+A_2)} \tag{12.127}$$

and

$$\left(2\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) W(C) \quad = \quad \lambda\, \mathrm{e}^{-\frac{\lambda}{2}(A_1+2A_2)}, \tag{12.128}$$

respectively. Their solution is given uniquely by Eqs. (12.113) and (12.116).

It is worth noting that the linear Abelian loop equation (12.118) can be written for the contours in Figs. 12.9a and b as

$$\left(-\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) \ln W(C) \quad = \quad \lambda\,, \tag{12.129}$$

$$\left(2\frac{\partial}{\partial A_1} - \frac{\partial}{\partial A_2}\right) \ln W(C) \quad = \quad \lambda\,. \tag{12.130}$$

The operators on the LHSs are always the same for the non-Abelian and Abelian loop equations, which is a general property, but the RHSs differ generically: Eqs. (12.127) and (12.129) for the contour in Fig. 12.9a coincide, while Eqs. (12.128) and (12.130) for the contour in Fig. 12.9b differ. The solution to Eq. (12.130) is given by (12.115).

Problem 12.12 Prove Eq. (12.120) for the contours with one self-intersection.

Solution Let the intersection correspond to the values s_1 and s_2 of the parameter s, i.e. $x_\mu(s_1) = x_\mu(s_2)$. Noting that only the vicinities of s_1 and s_2 contribute

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to the integral on the LHS of Eq. (12.120), we obtain

$$\begin{aligned}
&\varepsilon_{\rho\nu} \int_{C'} \mathrm{d}x_\rho \oint_C \mathrm{d}y_\nu \, \delta^{(2)}(x-y)\, W(C_{yx})\, W(C_{xy}) \\
&\quad = \quad \varepsilon_{\rho\nu}\, \dot{x}_\rho(s_1)\, \dot{x}_\nu(s_2) \int \mathrm{d}s \int \mathrm{d}t \, \delta^{(2)}((s-s_1)\dot{x}(s_1) - (t-s_2)\dot{x}(s_2)) \\
&\qquad \quad \times W\big(C_{x(s_2)x(s_1)}\big)\, W\left(C_{x(s_1)x(s_2)}\right) \\
&\quad = \quad \frac{\varepsilon_{\rho\nu}\, \dot{x}_\rho(s_1)\, \dot{x}_\nu\,(s_2)}{\sqrt{\dot{x}^2_\mu(s_1)\, \dot{x}^2_\nu(s_2) - \left(\dot{x}_\mu\,(s_1)\, \dot{x}_\mu\,(s_2)\right)^2}} W\big(C_{x(s_2)x(s_1)}\big)\, W\big(C_{x(s_1)x(s_2)}\big) \\
&\quad = \quad W\big(C_{x(s_2)x(s_1)}\big)\, W\big(C_{x(s_1)x(s_2)}\big)
\end{aligned} \tag{12.131}$$

which is precisely the RHS of Eq. (12.120).

Remark on the string representation

A nice property of QCD_2 at large N is that the exponential of the area enclosed by the contour C emerges* for the Wilson loop average $W(C)$. This is as it should for the Nambu–Goto string (12.99). However, the additional pre-exponential factors (such as that in Eq. (12.116)) are very difficult to interpret in string language. They may become negative for large loops, which is impossible for a bosonic string. This demonstrates explicitly in $d = 2$ the statement of the previous section that the Nambu–Goto string is not a solution of the large-N loop equation. An appropriate string representation of two-dimensional large-N QCD was constructed by Gross and Taylor [GT93].

12.9 Gross–Witten transition in lattice QCD_2

The lattice gauge theory on a two-dimensional lattice is defined by the partition function (6.31) with $d = 2$:

$$Z_{\mathrm{2D}}(\beta) \quad = \quad \int \prod_x \prod_{\mu=1,2} \mathrm{d}U_\mu(x)\, \mathrm{e}^{-\beta S[U]}, \tag{12.132}$$

where the action is given by Eq. (6.16).

A specific property of two dimensions is that the number of lattice sites is equal to the number of plaquettes. For this reason, we can always perform a gauge transformation such that the link variables are chosen to be equal to unity along one of the axes, say

$$U_1(x) \;=\; 1\,. \tag{12.133}$$

* This is not true, as has already been discussed, in the Abelian case for contours with self-intersections.

Hence, the partition function (12.132) factorizes:

$$Z_{\rm 2D} \;=\; (Z_{\rm 1p})^{N_p}\,, \tag{12.134}$$

where N_p denotes the number of plaquettes of the lattice and $Z_{\rm 1p}$ is the one-matrix integral

$$Z_{\rm 1p}(\beta) \;=\; \int {\rm d}U\, {\rm e}^{\beta\left(\frac{1}{N}{\rm Re\,tr}\,U-1\right)}. \tag{12.135}$$

In other words, the plaquette variables of the lattice gauge theory can be treated in two dimensions as being independent.

The correct interpretation of Eq. (12.135) is that it is the partition function of the one-plaquette model, i.e. the lattice gauge theory on a single plaquette. This is consistent with the gauge invariance.

The unitary one-matrix model (12.135) can be easily solved in the large-N limit using loop equations.

We first introduce the "observables" for the one-matrix model:

$$W_n \;=\; \left\langle \frac{1}{N}\,{\rm tr}\,U^n \right\rangle_{\rm 1p}, \tag{12.136}$$

where the average is taken with the same weight as in Eq. (12.135). The interpretation of W_n in the language of the single-plaquette model is that these are the Wilson loop averages for contours which go along the boundary of n stacked plaquettes.

In order to derive the loop equation for the one-matrix model, we proceed quite analogous to the derivation of the loop equation in the lattice gauge theory (given in Problem 12.6 on p. 265).

Let us consider the obvious identity

$$0 \;=\; \langle {\rm tr}\; t^a U^n \rangle_{\rm 1p}\,, \tag{12.137}$$

and perform the (infinitesimal) change

$$U \to U\left(1 - {\rm i}t^a\epsilon^a\right), \qquad U^\dagger \to \left(1 + {\rm i}t^a\epsilon^a\right)U^\dagger \tag{12.138}$$

of the integration variable on the RHS of Eq. (12.137). Since the Haar measure is invariant under the change (12.138), we finally obtain

$$\left.\begin{aligned} \frac{\beta}{2N^2}\left(W_{n-1} - W_{n+1}\right) \;&=\; \sum_{k=1}^{n} W_k W_{n-k} \qquad \text{for } n \geq 1\,, \\ W_0 \;&=\; 1\,, \end{aligned}\right\} \tag{12.139}$$

where* $\beta = N^2/\lambda$ and $\lambda \sim 1$ as $N \to \infty$.

* In contrast to the previous section, here we include the factor of a^2 in the definition of λ to make it dimensionless.

Equation (12.139) has the following exact solution:

$$W_1 \;=\; \frac{1}{2\lambda}; \qquad W_n \;=0 \qquad \text{for } n \geq 2\,, \tag{12.140}$$

which reproduces the strong-coupling expansion. The leading order of the strong-coupling expansion turns out to be exact at $N=\infty$.

However, the solution (12.140) cannot be the desired solution at any values of the coupling constant. Since W_k are (normalized) averages of unitary matrices, they must obey

$$W_n \;\leq\; 1\,, \tag{12.141}$$

which is not the case for W_1, given by Eq. (12.140), at small enough values of λ.

In order to find all solutions to Eq. (12.139), let us introduce the generating function

$$f(z) \;\equiv\; \sum_{n=0}^{\infty} W_n z^n \tag{12.142}$$

and rewrite Eq. (12.139) as the quadratic equation

$$f z - \frac{1}{z}(f-1) + W_1 \;=\; 2\lambda\left(f^2 - f\right). \tag{12.143}$$

A formal solution to Eq. (12.143) is

$$f(z) \;=\; -\frac{1-2\lambda z - z^2}{4\lambda z} + \frac{\sqrt{\left(1+2\lambda z+z^2\right)^2 + 4z^2\left(2\lambda W_1 - 1\right)}}{4\lambda z}\,, \tag{12.144}$$

where the positive sign of the square root is chosen to satisfy $f(0)=1$.

The RHS of Eq. (12.144) depends on an unknown function $W_1(\lambda)$, which must guarantee $f(z)$ to be a holomorphic function of the complex variable z within the unit circle $|z|<1$. This is a consequence of the inequality (12.141) which stems from the unitarity of U.

There exist two solutions for which $f(z)$ is holomorphic inside the unit circle: the strong-coupling solution given for $\lambda \geq 1$ by Eq. (12.140) and the weak-coupling solution given for $\lambda \leq 1$ by

$$W_1 \;=\; 1 - \frac{\lambda}{2}\,. \tag{12.145}$$

A comparison with Eq. (7.1) for $d=2$ shows that the leading order of the weak-coupling expansion is now exact. Therefore, $f(z)$ is given by two *different* analytic functions for $\lambda > 1$ and $\lambda < 1$.

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At the point $\lambda = 1$, a phase transition occurs as was discovered by Gross and Witten [GW80] who first solved lattice QCD_2 in the large-N limit. This phase transition is of the third order since both the first and second derivatives of the partition function are continuous at $\lambda = 1$. The discontinuity resides only in the third derivative. This phase transition is pretty unusual from the point of view of statistical mechanics where phase transitions usually occur in the limit of an infinite volume (otherwise the partition function is analytic in temperature). Now the Gross–Witten phase transition occurs even for the single-plaquette model (12.135) in the large-N limit. In other words, the number of degrees of freedom is now infinite owing to the internal symmetry group rather than an infinite volume.

Finally, we mention that since plaquette variables are independent in lattice QCD_2, the Wilson loop average for a nonintersecting lattice contour C takes the form

$$W(C) \;=\; (W_1)^A\,, \tag{12.146}$$

where A is the area (in the lattice units) enclosed by the contour C. W_1 in this formula is given by Eq. (12.140) in the strong coupling phase ($\lambda \geq 1$) and Eq. (12.145) in the weak-coupling phase ($\lambda \leq 1$).

The continuum formula (12.109) can be recovered for small λ from Eq. (12.146) as follows:

$$W(C) \;=\; \left(1 - \frac{\lambda a^2}{2}\right)^{A/a^2} \;\xrightarrow{a\to 0}\; \mathrm{e}^{-\frac{\lambda}{2}A}\,, \tag{12.147}$$

where we have restored the a-dependence as is prescribed by the dimensional analysis.

The solution of $N = \infty$ lattice QCD_2 by the loop equations, which is described in this section, was given in [PR80, Fri81].

Problem 12.13 Calculate the density of eigenvalues for the matrix U in the one-matrix model (12.135).

Solution Let us reduce U to the diagonal form

$$U \;=\; \mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\alpha_1}, \ldots, \mathrm{e}^{\mathrm{i}\alpha_j}, \ldots, \mathrm{e}^{\mathrm{i}\alpha_N}\right). \tag{12.148}$$

The density of eigenvalues (or the spectral density), $\rho(\alpha)$, is then defined as a fraction of the eigenvalues which lie in the interval $[\alpha, \alpha + \mathrm{d}\alpha]$. In other words, introducing the continuum variable $x = j/N$ ($0 \leq x \leq 1$) in the large-N limit, we have

$$\rho(\alpha) \;=\; \frac{\mathrm{d}x}{\mathrm{d}\alpha} \;\geq\; 0 \tag{12.149}$$

which obeys the obvious normalization

$$\int_{-\pi}^{\pi} \mathrm{d}\alpha\, \rho(\alpha) \;=\; \int_0^1 \mathrm{d}x \;=\; 1\,. \tag{12.150}$$

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Given $\rho(\alpha)$, we can calculate W_n by

$$W_n = \int_{-\pi}^{\pi} \mathrm{d}\alpha \, \rho(\alpha) \cos n\alpha \,. \tag{12.151}$$

It is now clear from the definition (12.136) and (12.142) that

$$f(z) = \int_{-\pi}^{\pi} \mathrm{d}\alpha \, \rho(\alpha) \frac{1}{1 - z\,\mathrm{e}^{-\mathrm{i}\alpha}} \,. \tag{12.152}$$

Choosing $z = \exp(\mathrm{i}\omega)$, we rewrite Eq. (12.152) as

$$f\left(\mathrm{e}^{\mathrm{i}\omega}\right) = \frac{1}{2} + \frac{\mathrm{i}}{2} \int_{-\pi}^{\pi} \mathrm{d}\alpha \, \rho(\alpha) \cot \frac{\omega - \alpha}{2} \,. \tag{12.153}$$

The discontinuity of this analytic function at $\omega = \alpha \pm \mathrm{i}0$ then determines $\rho(\alpha)$.

Using the explicit solution (12.144), we formally find

$$\rho(\alpha) = \frac{1}{2\lambda\pi} \sqrt{\left(\cos\alpha + \lambda + \sqrt{1 - 2\lambda W_1}\right)\left(\cos\alpha + \lambda - \sqrt{1 - 2\lambda W_1}\right)}. \tag{12.154}$$

For W_1 given by Eqs. (12.140) and (12.145) for the strong- and weak-coupling phases, we finally obtain

$$\rho(\alpha) = \frac{1}{2\pi}\left(1 + \frac{1}{\lambda}\cos\alpha\right) \qquad \text{for } \lambda \geq 1 \,, \tag{12.155}$$

$$\rho(\alpha) = \frac{1}{\lambda\pi} \cos\frac{\alpha}{2} \sqrt{\lambda - \sin^2\frac{\alpha}{2}} \qquad \text{for } \lambda \leq 1 \tag{12.156}$$

for the strong- and weak-coupling solutions, respectively. Note that (12.155) is nonnegative for $\lambda \geq 1$ as it should be because of the inequality (12.149). For $\lambda < 1$, the strong-coupling solution (12.155) becomes negative somewhere in the interval $[-\pi, \pi]$ which cannot happen for a dynamical system. This is the reason why the other solution (12.156) is realized for $\lambda < 1$. It has the support on the smaller interval $[-\alpha_{\mathrm{c}}, \alpha_{\mathrm{c}}]$, where $0 < \alpha_{\mathrm{c}} < \pi$ is determined by the equation

$$\sin^2\frac{\alpha_{\mathrm{c}}}{2} = \lambda \tag{12.157}$$

which always has a solution for $\lambda < 1$. The weak-coupling spectral density (12.156) is nonnegative for $\lambda \leq 1$.

For small λ, $\alpha_{\mathrm{c}} = 2\sqrt{\lambda}$ so that

$$\rho(\alpha) = \frac{1}{2\lambda\pi} \sqrt{4\lambda - \alpha^2} \,. \tag{12.158}$$

As $\lambda \to 0$, $\rho(\alpha) \to \delta(\alpha)$ and U freezes, modulo a gauge transformation, near a unit matrix. This guarantees the existence of the continuum limit of QCD_2.

The spectral densities (12.155) and (12.156) were first calculated [GW80] by a direct solution of the saddle-point equation at large N.