6

Bosonization

In one space dimension there are obviously no rotations and hence no angular momentum. This raises the possibility of equivalence relations between scalar fields and fields of higher tensorial structure like spinors, vectors etc. However, spinors and scalars seem to be distinct even in two dimensions due to their different statistics. An equivalence between these two types of fields should therefore incorporate the identification of operators, made out of scalars, that are anti-commuting and vice versa. It is well known that a bilinear of fermi fields is a commuting field, but it is less obvious how to construct a field that obeys the Fermi–Dirac statistics from scalars. This is precisely what the bosonization procedure does.

Coleman [63] and Mandelstam [159] introduced the concept of bosonization. Their construction is now referred to as the "abelian bosonization". An anti-commuting Fermi field, constructed from the exponential of a boson, was given explicitly by Mandelstam [159].

The fact that the theories of a free massless scalar and a free Dirac fermion are equivalent can be proven by showing that they fall into the same representation of the affine current algebra and the Virasoto algebra. The bosonic–fermionic duality can also be further elevated to the free massive theories and also to interacting ones.

It turns out that the original abelian bosonization is not convenient to accommodate color (or flavor) degrees of freedom and hence is inconvenient to address systems like QCD_2. A breakthrough in that direction was achieved by Witten, in his non-abelian bosonization [224].[1]

The equivalence enables one to use, as convenient, either the fermionization of scalar fields or the bosonization of fermions. The latter is useful in several cases. For instance in the case of duality between the Thirring model [205] and the sine-Gordon model,[2] which will be discussed in Section 6.2, the bosonization takes the form of a strong–weak duality. For strong fermionic interactions one finds a weak bosonic coupling. In applications to gauge theories (Section 9) it will be shown that the one loop anomaly behavior is encoded in classical bosonized theory. In QCD_2, as will be discussed in Section 9.3.2, the bosonic version of

[1] This paper discusses Majorana fermions. The construction for Dirac fermions was done in [7].
[2] This was proved by Coleman [63].

the theory admits a separation between the color and flavor degrees of freedom, which is very useful in describing the low energy color singlet states.

We start this chapter by introducing the set of rules that span abelian bosonization, including the rules for mass terms and the equivalence of the interacting Thirring model and the sine-Gordon model. We then describe Witten's non-abelian bosonization of Majorana fermions. This is further generalized to the case of massless Dirac fermions. We discuss the subtleties of the massive case and present two methods of handling the non-abelian bosonization of massive Dirac fields.[3] We then discuss in detail the action formulation of the chiral bosonization both abelian and non-abelian. We then depart from the applications that will be found to be relevant to QCD_2 and present topics in bosonization which are more relevant to conformal field and string theories like the bosonization of ghost fields and the Wakimoto bosonization [213]. We do not discuss bosonization on higher Riemann surfaces. The interested reader can consult for instance [211] and [84].

The topic of bosonization in two-dimensional field theories has been reviewed in several papers and books, like that of Stone [202]. Here we mainly follow the review of Frishman and Sonnenschein [101] for the basic ingredients, and update it to include more recent topics.

6.1 Abelian bosonization

6.1.1 Bosonization of a free massless Dirac fermion

Both the theory of a free massless real scalar field and the theory of a free massless Dirac field are conformal field theories invariant under affine Lie algebra. Recall from Chapter 2 that the former theory is defined by the action,

$$
\begin{aligned}
S &= \int \mathrm{d}^2 x \mathcal{L} = \frac{1}{8\pi} \int \mathrm{d}^2 x \partial_\nu \hat{\phi} \bar{\partial}^\nu \hat{\phi} \\
&= \frac{1}{4\pi} \int \mathrm{d}^2 \xi \partial_\xi \hat{\phi} \partial_{\bar{\xi}} \hat{\phi} = \frac{1}{4\pi} \int \mathrm{d}^2 z \partial \hat{\phi} \bar{\partial} \hat{\phi}.
\end{aligned} \tag{6.1}
$$

The solution of the equation of motion takes the form,

$$
\hat{\phi}(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}). \tag{6.2}
$$

The theory has holomorphically (and anti-holomorphically) conserved currents,

$$
J(z) = i\partial\phi(z) \quad \bar{J}(z) = -i\bar{\partial}\bar{\phi}(\bar{z}), \tag{6.3}
$$

and similarly holomorphic (and anti-holomorphic) energy-momentum tensors,

$$
\begin{aligned}
T(z) &= -\frac{1}{2} : \partial\phi\partial\phi := -\frac{1}{2} : J(z)J(z) : \\
\bar{T}(\bar{z}) &= -\frac{1}{2} : \bar{\partial}\bar{\phi}\bar{\partial}\bar{\phi} := -\frac{1}{2} : \bar{J}(\bar{z})\bar{J}(\bar{z}) :,
\end{aligned} \tag{6.4}
$$

[3] A bosonization prescription for the mass term in the flavored case was suggested in [75] and [99].

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which admit a Virasoro algebra with $c = 1$ and affine Lie algebra with level $k = 1$.

Recall also that the theory of a free massless Dirac field with the action,

$$S = \frac{1}{4\pi}\int \mathrm{d}^2 z(\psi^\dagger\bar{\partial}\psi + \tilde{\psi}^\dagger\partial\tilde{\psi}) \tag{6.5}$$

admits conserved currents,

$$J(z) = \psi^\dagger\psi \quad \bar{J}(\bar{z}) = \tilde{\psi}^\dagger\tilde{\psi}, \tag{6.6}$$

and its energy-momentum tensor can be expressed as a bilinear of the currents using the Sugawara construction,

$$T(z) = -\frac{1}{2}[\psi^\dagger\partial\psi - \partial\psi^\dagger\psi] = -\frac{1}{2}:\psi^\dagger\psi\psi^\dagger\psi: = -\frac{1}{2}:J(z)J(z):. \tag{6.7}$$

The correponding level of the affine algebra and of the Virasoro anomaly are again $k = 1$, and $c = 1$, respectively.

Due to the uniqueness of the irreducible unitary $k = 1$ representation of the affine Lie algebras, and the fact that the infinite-dimensional algebraic structure fully determines the theories, we conclude that *in two space-time dimensions the theories of massless free scalar field and Dirac field are equivalent*.

The equivalence implies that every operator of one theory should have a partner in the other theory, in such a way that the OPEs of these dual operators should be identical. We have just realized such correspondence for the currents and energy-momentum tensor, namely,

$$\begin{aligned} J_b(z) &= \partial\phi(z) \quad \leftrightarrow \quad J_f(z) =:\psi^\dagger\psi(z):, \\ T_b(z) &= -\frac{1}{2}:\partial\phi\partial\phi: \;\leftrightarrow\; T_f(z) = -\frac{1}{2}[\psi^\dagger\partial\psi - \partial\psi^\dagger\psi], \end{aligned} \tag{6.8}$$

and similarly for the anti-holomorphic counterparts.

For completeness we now redescribe the currents using the "old" terminology of vector and axial currents. The vector current reads,

$$J^\mu_V =: \bar{\psi}\gamma^\mu\psi := -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\phi. \tag{6.9}$$

This identification of J^μ leads automatically to a conserved current,

$$\partial_\mu J^\mu_V = 0, \tag{6.10}$$

independent of the equations for ϕ. This is a "topological" conservation, connected with choosing the "vector conservation" scheme. In the applications to follow, we will demand more freedom in the scheme choice of interacting theories, in particular the possibility to have a vector current anomaly. The bosonization procedure will therefore be somewhat modified. The modification will correspond to a change of regularization scheme.

The overall coefficient of the current is such that the fermion number charge,

$$Q = \int_{-\infty}^{\infty} \mathrm{d}x j_0(x) = 1, \tag{6.11}$$

for the ψ-field. In addition to the "topologically" conserved vector current, the bosonic theory has an axial current, which is equivalent to the fermionic axial current,

$$J_A^\mu =: \bar{\psi}\gamma^\mu\gamma^5\psi := \frac{1}{\sqrt{\pi}}\partial^\mu\phi. \tag{6.12}$$

The bosonic current is the Neother current associated with the invariance of the bosonic action under the global shift $\delta\phi = \epsilon$. The holomorphic and anti-holomorphic conserved currents discussed above are (in real coordinates) nothing but the left and right chiral currents $J_\pm^\mu = J_v^\mu \pm J_A^\mu$, which correspond to shifts with $\epsilon(x_+)$ and $\epsilon(x_-)$. Using the commutation relation (8.4) the ALA reads,

$$\left[J_\pm(x_\pm), J_\pm(x'_\pm)\right] = \frac{2i}{\pi}\delta'(x_\pm - x'_\pm). \tag{6.13}$$

This is the same algebra as that of the fermionic chiral currents. The Sugawara construction in this terminology reads,

$$T_{\pm\pm} = \pi : J_\pm J_\pm : . \tag{6.14}$$

These obey the Virasoro algebra,

$$\left[T_\pm(x_\pm), T_\pm(x'_\pm)\right] = 2i\left(T_\pm(x_\pm) + T_\pm(x'_\pm)\right)\delta'(x_\pm - x'_\pm) - \frac{i}{6\pi}\delta'''(x_\pm - x'_\pm), \tag{6.15}$$

which is identical to that of the fermionic energy-momentum tensor.

The equivalence of the bosons and the fermion bilinears is not only mathematical. The fermion Fock-space contains those bosons as physical states. The reason for this is that in one space dimension a massless field can move either to the left or to the right. A Dirac fermion and its anti-particle having together zero fermionic charge and moving in the same direction will never separate. They are therefore indistinguishable from a free massless boson. This picture changes when masses are introduced, and the above relations will be approached at momenta high compared to the mass scale (including high off-mass shell).

A natural question to ask is which operator of the bosonic picture corresponds to the basic Dirac ferion? Since the latter is in fact a combination of a left chiral spinor and a right one, we would like to determine the "bosonized" Weyl fermion $\psi(z)$. It is a holomorphic function of conformal dimension 1/2, that transforms under the affine Lie transformation with a unit charge, namely, $\psi(z) \to \mathrm{e}^{i\epsilon(z)}\psi(z)$. Due to the fact that under the same transformation the scalar field transforms as $\phi(z) \to \phi(z) + \mathrm{e}(z)$ we are led to look for a candidate which is an exponential in the scalar field $\mathrm{e}^{i\alpha\phi(z)}$. We now use $T(z)$ given in (6.4) to compute the confomal dimension of $: \mathrm{e}^{i\alpha\phi(z)} :$ as follows,

$$T(z) : \mathrm{e}^{i\alpha\phi(w)} := \frac{\frac{\alpha^2}{2} : \mathrm{e}^{i\alpha\phi(w)} :}{(z-w)^2} + \ldots, \tag{6.16}$$

where ... stands for non-singular terms, Hence the conformal dimension is $\frac{\alpha^2}{2}$. Thus we conclude that the following equivalence should hold,

$$\psi(z) \leftrightarrow \mathrm{e}^{i\phi(z)}, \quad \psi^\dagger(z) \leftrightarrow \mathrm{e}^{-i\phi(z)}. \tag{6.17}$$

To confirm this bosonization rule we compute the OPEs in both descriptions and verify that they are indeed identical,

$$\begin{aligned}
\psi(z)\psi^\dagger(-z) &= \frac{1}{2z} + :\psi(0)\psi^\dagger(0): + 2z : [\psi(0)\partial\psi^\dagger(0): - \partial\psi(0)\psi^\dagger(0))] + \mathcal{O}(z^2)\\
\psi(z)\psi^\dagger(-z) &= \frac{1}{2z} + J(0) + 2zT(0) + \mathcal{O}(z^2)\\
: \mathrm{e}^{i\phi(z)} :: \mathrm{e}^{-i\phi(-z)} : &= \frac{1}{2z} + J(0) + 2zT(0) + \mathcal{O}(z^2)\\
: \mathrm{e}^{i\phi(z)} :: \mathrm{e}^{-i\phi(-z)} : &= \frac{1}{2z} + i\partial\phi(0) + 2z(\partial\phi\partial\phi(0)) + \mathcal{O}(z^2).
\end{aligned} \tag{6.18}$$

The bosonic version of the fermion ψ was originally proposed by Mandelstam. His formulation was done in terms of real coordinates and cannonical quantization. For completeness we now also present the "old" construction and proof of equivalence. The bosonized chiral fermion in the latter formulation takes the form,[4]

$$\begin{aligned}
\psi_{\mathrm{L}} &= \sqrt{\frac{c\mu}{2\pi}} : \exp\left(-i\sqrt{\pi}\left(\int_{-\infty}^{x} \mathrm{d}\xi[\pi(\xi) + \phi(x)] \right)\right) :\\
\psi_{\mathrm{R}} &= \sqrt{\frac{c\mu}{2\pi}} : \exp\left(-i\sqrt{\pi}\left(\int_{-\infty}^{x} \mathrm{d}\xi[\pi(\xi) - \phi(x)] \right)\right) :,
\end{aligned} \tag{6.19}$$

where $\pi(x) = \dot{\phi}(x)$ is the conjugate momentum of $\phi(x)$, c is a constant. A computation yields $c = \frac{1}{2}\mathrm{e}^{\gamma} \sim 0.891$, where γ is the Euler constant. The normal ordering denoted by : : is performed with respect to the scale μ.

The equal time commutation relations of the ϕ-field,

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y),$$

imply, upon using the formula $\mathrm{e}^A\mathrm{e}^B = \mathrm{e}^{[A,B]}\mathrm{e}^B\mathrm{e}^A$ (for $[A,B]$ a c-number) the canonical anti-commutation relations for the ψ field,

$$\left\{\psi^\dagger_{\mathrm{L,R}}(x,t), \psi_{\mathrm{L,R}}(y,t)\right\} = \delta(x-y). \tag{6.20}$$

The fermion field ψ is therefore, an inherently non-local functional of the scalar field. However fermion bilinears, such as the currents discussed above or the mass terms that will be described in the next section, are local functions.

So far we have addressed the map for massless theories. Let us now discuss the bosonization of a fermion mass bilinear operator. The mass term which mixes

[4] A generalization of this bosonization to a set of N fermions was done by Halpern [121].

the left and right chiral componenets of the Dirac fermion takes the following well-knows form,

$$m_f[\tilde{\psi}^\dagger(\bar{z})\psi(z) + \psi^\dagger(z)\tilde{\psi}(\bar{z})] = m_f(\psi_\mathrm{L}^\dagger\psi_\mathrm{R} + \psi_\mathrm{R}^\dagger\psi_\mathrm{L}). \tag{6.21}$$

Again for completeness we write down the expression both in the complex coordinates as well as in real coordinates.

Using the bosonization rules for chiral fermions (6.17) (to be justified below), we deduce the map of the fermion bilinear to the equivalent bosonic operator,

$$m_f[: \mathrm{e}^{i\bar{\phi}(\bar{z})} :: \mathrm{e}^{i\phi(z)} : + : \mathrm{e}^{-i\phi(z)} :: \mathrm{e}^{-i\bar{\phi}(\bar{z})} :] = m_f\mu : \cos(\hat{\phi}(z,\bar{z}) :, \tag{6.22}$$

where we have made use of $\hat{\phi}(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z})$ and of the fact that there is no non-trivial OPE between $\phi(z)$ and $\bar{\phi}(\bar{z})$. Note that we write down the bosonic equivalent of the mass term operator in the context of the massless theory and hence the factorization to holomorphic and anti-holomorphic parts of the scalar field holds. Once we identify this operation relation we will then use it to add a fermion mass term to the bosonized action. The additional parameter which has a dimension of mass μ is the normal ordering scale.

The derivation of the bosonized mass term in the "old language" is somewhat more involved. We will return to this after we address the bosonization duality between the fermionic Thirring model and the bosonic sine-Gordon model.

We now summarize the equivalence relations between the bosonic and fermionic operators of the free theories, in both the "modern" complex coordinate formulation, as well as the "old" formulation in terms of real coordinates:

Operator	Fermionic	Bosonic
$J(z)$	$: \psi^\dagger\psi(z) :$	$i\partial\phi(z)$
$\bar{J}(\bar{z})$	$: \tilde{\psi}^\dagger\tilde{\psi}(\bar{z}) :$	$-i\bar{\partial}\phi(\bar{z})$
$T(z)$	$-\frac{1}{2} : [\psi^\dagger\partial\psi - \partial\psi^\dagger\psi] :$	$-\frac{1}{2} : \partial\phi\partial\phi(z) :$
$\bar{T}(\bar{z})$	$-\frac{1}{2} : [\tilde{\psi}^\dagger\partial\tilde{\psi} - \partial\tilde{\psi}^\dagger\tilde{\psi}]:$	$-\frac{1}{2} : \bar{\partial}\phi\bar{\partial}\phi(\bar{z}) :$
$\mathrm{fermion_L}$	$\psi(z)$	$: \mathrm{e}^{i\phi(z)} :$
$\mathrm{fermion_R}$	$\tilde{\psi}(\bar{z})$	$: \mathrm{e}^{i\phi(\bar{z})} :$
mass term	$\tilde{\psi}^\dagger(\bar{z})\psi(z) + \psi^\dagger(z)\tilde{\psi}(\bar{z})$	$\mu : \cos\hat{\phi}(z,\bar{z}) :$

Bosonization in "modern" complex formulation.

6.2 Duality between the Thirring model and the sine-Gordon model

The Thirring model is a fermionic theory with a current–current interaction, given by the Lagrangian density,

$$\mathcal{L} = i\bar{\psi}\,\partial\!\!\!/\psi - \frac{1}{2}gJ^\mu J_\mu,$$

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Operator	Fermionic	Bosonic
$J_+(x^+)$	$:\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{L}}:$	$\partial_+\phi$
$J_-(x^-)$	$:\psi_{\mathrm{R}}^\dagger\psi_{\mathrm{R}}:$	$\partial_-\phi$
$T_{++}(x^+)$	$-\frac{1}{2}:[\psi_{\mathrm{L}}^\dagger\partial\psi_{\mathrm{L}}-\partial\psi_{\mathrm{L}}^\dagger\psi_{\mathrm{L}}]:$	$-\frac{1}{2}:\partial_+\phi\partial_+\phi(x^+):$
$T_{--}(x^-)$	$-\frac{1}{2}:[\psi_{\mathrm{R}}^\dagger\partial\psi_{\mathrm{R}}-\partial\psi_{\mathrm{R}}^\dagger\psi_{\mathrm{R}}]:$	$-\frac{1}{2}:\partial_-\phi\partial_-\phi(x^+):$
$\mathrm{fermion}_{\mathrm{L}}$	$\psi_{\mathrm{L}}(x^+)$	$\sqrt{\frac{c\mu}{2\pi}}:\exp\left(-i\sqrt{\pi}\left(\int_{-\infty}^{x}\mathrm{d}\xi\pi(\xi)+\phi(x)\right)\right):$
$\mathrm{fermion}_{\mathrm{R}}$	$\psi_{\mathrm{R}}(x^-)$	$\sqrt{\frac{c\mu}{2\pi}}:\exp\left(-i\sqrt{\pi}\left(\int_{-\infty}^{x}\mathrm{d}\xi\pi(\xi)-\phi(x)\right)\right):$
mass term	$\psi_{\mathrm{L}}^\dagger(x^+)\psi_{\mathrm{R}}(x^-)$ $+\psi_{\mathrm{R}}^\dagger(x^+)\psi_{\mathrm{L}}(x^-)$	$\mu:\cos\hat{\phi}(x^+,x^-):$

Bosonization in "old" formulation.

where $J_\mu = :\bar{\psi}\gamma_\mu\psi:$. The model is exactly solvable and meaningful for $g > -\pi$. The corresponding equation of motion reads,

$$i\,\partial\!\!\!/\psi(x) = g\gamma_\mu J^\mu(x)\psi(x). \tag{6.23}$$

The theory is invariant under vector and axial $U(1)$ global transformations. The corresponding conserved currents are,

$$J_\mu^V = J^\mu = :\bar{\psi}\gamma_\mu\psi: \quad J_\mu^A = \epsilon_{\mu\nu}J^{V\,\nu}. \tag{6.24}$$

The model can be studied by means of the operator product expansion on the light-cone [76]. The fermionic bilinears of the model are expressed as a function of the current, and the expressions obtained turn out to be very natural in the light of the bosonization procedure, which we now describe.

We start with the following generalization of the bosonization formula (6.19):

$$\psi_{\mathrm{L}} = \sqrt{\frac{c\mu}{2\pi}}:\exp\left(-i\sqrt{\pi}\left(\frac{2\sqrt{\pi}}{\beta}\int_{-\infty}^{x}\mathrm{d}\xi\pi(\xi)+\frac{\beta}{2\sqrt{\pi}}\phi(x)\right)\right):$$
$$\psi_{\mathrm{R}} = \sqrt{\frac{c\mu}{2\pi}}:\exp\left(-i\sqrt{\pi}\left(\frac{2\sqrt{\pi}}{\beta}\int_{-\infty}^{x}\mathrm{d}\xi\pi(\xi)-\frac{\beta}{2\sqrt{\pi}}\phi(x)\right)\right):. \tag{6.25}$$

The meaning of the new parameter β will be clarified shortly. In a similar manner to the derivation of (6.20) we can verify that the equal-time anti-commutation relations are still obeyed. Furthermore one can show that the Dirac operator built from (6.25) obeys the equation of motion (6.23) provided that the bosonic field $\phi(x)$ obeys the equation of motion of the sine-Gordon model discussed in Section 5.3, namely,

$$\partial_\mu\partial^\mu\phi(x) + \frac{\mu^2}{\beta}:\sin(\beta\phi(x)):= 0, \tag{6.26}$$

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and that the parameter β is related to the coupling constant g in the following way,

$$\frac{\beta^2}{4\pi} = \frac{1}{1+\frac{g}{\pi}}. \tag{6.27}$$

From this last relation it follows that the special value $\beta^2 = 4\pi$ corresponds to $g = 0$ and hence a free Dirac fermion. Indeed as we shall see below for that value of β the sine-Gordon potential translates into the bosonized mass term. It is interesting to note a remarkable property of (6.27) which relates the coupling constants of the Thirring model and its bosonic equivalent, the sine-Gordon model. The weak coupling of one theory is the strong coupling of the other. This property often occurs in bosonized theories and hints at the usefulness of the method in dealing with theories for strong coupling, where perturbative methods fail.

The bosonization dictionary of the vector fermion number current is the following,

$$J^V_\mu = :\bar{\psi}\gamma_\mu\psi: \leftrightarrow J^\mu = -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi. \tag{6.28}$$

This expression differs from the bosonized current of the free Dirac theory (6.9) in its normalization factor. We immediately realize that for the special value $\beta^2 = 4\pi$ we precisely reproduce (6.9). The normalization factor can be determined from the assignment of the fermion number charge of a soliton that should be equal to the charge of the field ψ. Recall from Section 5.3 that the classical sine-Gordon model admits a finite energy soliton solution. It is time-independent and interpolates between adjacent wells of the scalar potential. In quantum theory this classical solution becomes a particle. The static soliton solution is given by (5.26),

$$\phi = \frac{4}{\beta}\tan^{-1}[\exp\mu(x-x_0)],$$

where x_0 is the "center" of the soliton. Substituting this into the integral of the current we find the fermion number of this solution to be,

$$Q = \frac{\beta}{2\pi}[\phi(\infty) - \phi(-\infty)] = 1. \tag{6.29}$$

Thus we see that indeed the normalization factor in the vector current (6.28) is the right one.

The relation (6.28) implies that the level in the affine Lie algebra will be $\frac{\beta^2}{4\pi}$, as compared to 1 for the free case.

Let us address again the issue of the bosonization of the fermion mass bilinear (6.22). The definition of the mass term in the "old formulation", as that of the current, requires some care due to the appearance of the products of operators at the same point. In fact when x approaches y one gets the following OPEs,

$$
\begin{aligned}
\psi_{\mathrm{R}}^{\dagger}(x)\psi_{\mathrm{L}}(y) &= \frac{c\mu}{2\pi}|c\mu(x-y)|^{\delta} : \mathrm{e}^{-i\beta\phi} : \\
\psi_{\mathrm{L}}^{\dagger}(x)\psi_{\mathrm{R}}(y) &= \frac{c\mu}{2\pi}|c\mu(x-y)|^{\delta} : \mathrm{e}^{i\beta\phi} :
\end{aligned} \tag{6.30}
$$

with $\delta = -\frac{g}{2\pi}(1+\frac{\beta^2}{4\pi})$. The proper fermion mass term will therefore be defined by,

$$
\lim_{y\to x}\int_{-\infty}^{\infty} \mathrm{d}x\ |c\mu(x-y)|^{-\delta} m\bar{\psi}(x)\psi(y) = \frac{c\mu}{\pi} m \int_{-\infty}^{\infty} \mathrm{d}x : \cos\beta\phi(x) : .
$$

With μ chosen such that $m = \frac{\mu\pi}{c\beta^2}$, the mass term transforms in the bosonic language to,

$$
\Delta\mathcal{L} = \frac{\mu^2}{\beta^2} : \cos\beta\phi : .
$$

The normal ordering is with respect to μ.[5]

6.3 Witten's non-abelian bosonization

The non-abelian bosonization introduced by Witten is a set of rules assigning bosonic operators to fermionic ones, in a theory of free fermions invariant under a global non-abelian symmetry.[6] Originally the fermions considered were Majorana fermions and the corresponding global symmetry was $O(N)$. The bosonic operators are not expressed in terms of free bosonic fields as in abelian bosonization, but rather in terms of interacting group elements. In particular, bosonic expressions can be written for the energy-momentum tensor, various chiral currents, the mass term and the complete action.

The generalization to the case of N_f Dirac fermions was introduced in [112] and [7].

6.3.1 Bosonization of Majorana fermions

Let us start with N free Majorana fermions governed by the action,

$$
S_{\psi} = \frac{i}{2}\int \mathrm{d}^2x \sum_{k=1}^{N}(\psi_{\mathrm{L}k}\partial_+\psi_{\mathrm{L}k} + \psi_{\mathrm{R}k}\partial_-\psi_{\mathrm{R}k})
$$

where $\psi_{\mathrm{L}}, \psi_{\mathrm{R}}$ are left and right Weyl–Majorana spinor fields, $\partial_{\pm} = \frac{1}{\sqrt{2}}(\partial_0 \pm \partial_1)$ and $k = 1, \ldots, N$. The corresponding bosonic action is the Wess–Zumino–Witten

[5] The flavored Thirring model was studied in [73].
[6] A related approach is presented in [179].

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(WZW) action discussed in Chapter 4:

$$S_b[u] = \frac{1}{16\pi}\int \mathrm{d}^2 x Tr(\partial_\mu u \partial^\mu u^{-1}) \\ + \frac{1}{24\pi}\int_{\mathrm{B}} \mathrm{d}^3 y \varepsilon^{ijk} Tr(u^{-1}\partial_i u)(u^{-1}\partial_j u)(u^{-1}\partial_k u), \tag{6.31}$$

where u is a matrix in $O(N)$ whose elements are bosonic fields. The second term, the Wess–Zumino (WZ) term, is defined on the ball B whose boundary Σ is taken to be the Euclidean two-dimensional space-time. Now, since $\pi_2[O(N)] = 0$, a mapping u from a two-dimensional sphere S into the $O(N)$ manifold can be extended to a mapping of the solid ball B into $O(N)$. The WZ term however is well defined only modulo a constant. It was normalized so that if u is a matrix in the vector representation of $O(N)$ the WZW term is well defined modulo $WZ \to WZ + 2\pi$. The source of the ambiguity is that $\pi_3[O(N)] \simeq Z$, namely there are topologically inequivalent ways to extend u into a mapping from B into $O(N)$.

Note that $O(2)$ is an exception, as $\pi_3[O(2)] = 0$.

Note that the equivalence is between a fermionic theory expressed in terms of an N-dimensional fermion in the vector representation of $O(N)$ and a group element which is an $N \times N$ matrix. Nevertheless, as will be shown below the two theories are fully equivalent.

Both the theory of N free Majorana fermions and the WZW model of (15.2) are invariant under ALA transformations of $O_{\mathrm{L}}(N) \times O_{\mathrm{R}}(N)$. The latter take the following forms for the bosonic and fermionic theories:

$$\begin{aligned} &u \to g(z)u \quad \psi_i \to [g(z)]_i^j \psi(z)_j \\ &u \to uh(\bar{z}) \quad \tilde{\psi}_i \to [h(\bar{z})]_i^j \tilde{\psi}(\bar{z})_j, \end{aligned} \tag{6.32}$$

where $g(z) \in O_L(N)$ and $h(\bar{z}) \in O_R(N)$. The corresponding currents in both pictures satisfy the ALA at level $k = 1$.

The two theories are also invariant under the conformal transformations,

$$z \to f(z) \quad \bar{z} \to \bar{f}(\bar{z}). \tag{6.33}$$

The associated Virasoro central charges of the two descriptions are identical, as follows,

$$c_f = N \times \frac{1}{2} \quad c_b = \frac{k[\dim O(N)]}{k+N-2} = \frac{1/2N(N-1)}{1+N-2} = \frac{N}{2}. \tag{6.34}$$

For the fermions it is just N times the central charge of a single Majorana fermion, whereas for the bosonized version we make use of the fact that the dual Coexter number of $O(N)$ is $N-2$. The conformal invariance of the action (15.2) can be also shown by realizing that the corresponding β function vanishes. If one generalizes (15.2) by taking a coupling $\frac{1}{4\lambda^2}$ as a coefficient of the first term and $\frac{k}{24\pi}$ of the WZ term (k integer), the β function associated with λ is given at the

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one loop level (in the sense of expanding around u = 1), by,

$$\beta \equiv \frac{d\lambda^2}{d\ln\Lambda} = -\frac{(N-2)\lambda^2}{4\pi}\left[1-\left(\frac{\lambda^2 k}{4\pi}\right)^2\right],$$

namely (15.2) is at a fixed point for $\lambda^2 = \frac{4\pi}{k}$ and hence exhibits conformal invariance there. By showing that the energy-momentum tensor obeys the Virasoro algebra, one can show that this property is in fact exact.

To summarize, the dictionary that translates the ALA currents and the energy momentum tensor of the fermionic theory, into the bosonic one and vice versa, is given by,

Operator	Fermionic	Bosonic
$J_{ij}(z)$	$:\psi_i\psi_j(z):$	$\frac{iN}{4\pi}[u^{-1}\partial u]_{ij}(z)$
$\bar{J}_{ij}(\bar{z})$	$:\tilde{\psi}_i\tilde{\psi}_j(\bar{z}):$	$\frac{iN}{4\pi}[u\bar{\partial}u^{-1}]_{ij}(\bar{z})$
$T(z)$	$-\frac{1}{2}\sum_{i=1}^N[\psi_i\partial\psi_i-\partial\psi_i\psi_i]:$	$-\frac{1}{2(N-1)}:J^aJ^a(z):$
$\bar{T}(\bar{z})$	$-\frac{1}{2}\sum_{i=1}^N:[\tilde{\psi}\partial\tilde{\psi}_i-\partial\tilde{\psi}_i\tilde{\psi}_i]:$	$-\frac{1}{2(N-1)}:\bar{J}^a\bar{J}^a:(\bar{z})$

6.3.2 Bosonization of Dirac fermions

The bosonic picture for the theory of N free massless Dirac fermions is built from a boson matrix $g \in SU(N)$ and a real boson ϕ. The bosonized action now has the form,

$$\begin{aligned} S[g,\phi] = &\frac{1}{8\pi}\int d^2x Tr(\partial_\mu g\partial^\mu g^{-1}) \\ &+\frac{1}{12\pi}\int_B d^3y\varepsilon^{ijk}Tr(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g) \\ &+\frac{1}{2}\int d^2x\partial_\mu\phi\partial^\mu\phi. \end{aligned} \quad (6.35)$$

Note the difference of factor two between the WZW action associated with the $SU(N)$ and the $O(N)$.

Here again both theories are conformal invariant with an identical Virasoro central charge,

$$c_f = N\times 1 \quad c_b = \frac{k\dim SU(N)}{k+N}+1 = \frac{N^2-1}{1+N}+1 = N. \quad (6.36)$$

ALA transformation with respect to global $SU_{\mathrm{L}}(N)\times SU_{\mathrm{R}}(N)\times U(1)$,

$$\begin{aligned} &g\to h(z)u, \quad \phi\to\phi+a(z); \quad \psi_i\to[g(z)]_i^j\psi(z)_j \\ &g\to u\bar{h}(\bar{z}), \quad \phi\to\phi+\bar{a}(\bar{z}); \quad \tilde{\psi}_i\to[h(\bar{z})]_i^j\psi(\bar{z})_j, \end{aligned} \quad (6.37)$$

leave the actions of both pictures invariant.

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One way to prove the equivalence of the fermionic and bosonic theories now, for N free massless Dirac fermions and the $k=1$ WZW theory on $U(N)$ group manifold, is by showing that the generating functionals of the current Green functions of the two theories are the same. For the fermions we have,

$$e^{-iW_\psi(A_\mu)} = \int (d\psi_+ d\psi_- d\bar{\psi}_+ d\bar{\psi}_-) e^{i\int d^2x \bar{\psi} i \not{D} \psi}, \tag{6.38}$$

where $D_\mu = \partial_\mu + iA_\mu$, $A_\mu = A_\mu^A(\frac{1}{2}T^A) + A_\mu^{(1)} \times 1$ and $(\frac{1}{2}T^A)$ generators of $SU(N)$. The term $W_\psi(A_\mu)$ was calculated by Polyakov and Wiegmann in a regularization scheme which preserves the global chiral $SU(N_\mathrm{L}) \times SU(N_\mathrm{R})$ symmetry and the local $U(1)$ diagonal symmetry, leading to,

$$W_\psi(A_\mu) = S[\tilde{A}] + S[\tilde{B}] + \frac{1}{4\pi N}\int d^2x A_\mu^{(1)} A^{\mu(1)}, \tag{6.39}$$

where $\tilde{A}, \tilde{B} \subset SU(N)$ are related to the gauge fields A_μ^A by $iA_+^A = (\tilde{A}^{-1}\partial_+ \tilde{A})^A$, $iA_-^A = (\tilde{B}^{-1}\partial_- \tilde{B})^A$.

In the bosonic theory one calculates,

$$\begin{aligned} e^{-iW_B(A_\mu^A)} &= \int [du] e^{iS[u] + i\int d^2x (J_-^B A_+^B + J_+^B A_-^B)} \\ e^{-iW_B(A_\mu^{(1)})} &= \int [d\phi] e^{\frac{i}{2}\int d^2x [(\partial\phi)^2 + (J_- A_+^{(1)} + J_+ A_-^{(1)})]} \end{aligned} \tag{6.40}$$

where $J_+^B A_-^B$ and $J_+ A_-^{(1)}$ are the appropriate parts of $\frac{i}{4\pi}Tr[(g^{-1}\partial_+ g)A_-]$, and similarly for the $(-\,+)$ case and with $A_\pm^{(1)} = Tr(A_\pm)$. These functional integrals can be performed exactly, leading to,

$$W_B(A_\mu^A) = S[\tilde{A}] + S[\tilde{B}] \quad W_B(A^{(1)}) = \frac{1}{4\pi N}\int d^2x A_\mu^{(1)} A^{\mu(1)}.$$

Thus the bosonic current Green functions are identical to those of the fermionic theory, the latter regulated in the way mentioned above.

6.3.3 The bosonization of a mass bilinear of Dirac fermions

A further bosonization rule has to be invoked for the mass bilinear. For a theory with a $U(N)$ symmetry group the rule is,

$$\psi_+^{\dagger l}\psi_{-j} = \tilde{c}\mu N_\mu g_j^l e^{-i\sqrt{\frac{4\pi}{N}}\phi}, \tag{6.41}$$

where N_μ denotes normal ordering at mass scale μ. The fermion mass term $m_q\bar{\psi}^i\psi_i$ is therefore,

$$m'^2 N_\mu \int d^2x Tr(g + g^\dagger),$$

where $m'^2 = m_q\tilde{c}\mu$, m_q is the quark mass, and c is the same constant as in (6.19). It is straightforward to show that the above bosonic operator transforms

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correctly under the $U(N)_{\mathrm{L}} \times U(N)_{\mathrm{R}}$ chiral transformations. On top of that it has the correct total dimension,

$$\Delta = \Delta_g + \Delta_\phi = \left(\frac{N-1}{N} + \frac{1}{N}\right) = 1, \tag{6.42}$$

where $\Delta_g = \frac{N-1}{N}$ and $\Delta_\phi = \frac{1}{N}$ are the dimensions associated with the $SU(N)$ and $U(1)$ group factors, respectively. Moreover in Section 4.4 it was explicitly shown that the four-point function,

$$G(z_i, \bar{z}_i) = <g(z_1, \bar{z}_1) g^{-1}(z_2, \bar{z}_2) g^{-1}(z_3, \bar{z}_3) g(z_4, \bar{z}_4)>, \tag{6.43}$$

is given by,

$$G(z_i, \bar{z}_i) = [(z_1 - z_4)(z_2 - z_3)(\bar{z}_1 - \bar{z}_4)(\bar{z}_2 - \bar{z}_3)]^{-\Delta_g} G(x, \bar{x}), \tag{6.44}$$

where $G(x, \bar{x})$ is a function of the harmonic quotients,

$x = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$ and $\bar{x} = \frac{(\bar{z}_1-\bar{z}_2)(\bar{z}_3-\bar{z}_4)}{(\bar{z}_1-\bar{z}_4)(\bar{z}_3-\bar{z}_2)}$ only, and in the free case is,

$$G(x, \bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{\frac{1}{N}} \times \left[I_1 \frac{1}{x} + I_2 \frac{1}{1-x}\right] \left[\bar{I}_1 \frac{1}{\bar{x}} + \bar{I}_2 \frac{1}{1-\bar{x}}\right], \tag{6.45}$$

where $I_1, I_2, \bar{I}_1, \bar{I}_2$ are group invariant factors. This result for the correlation function, combined with the $U(1)$ part gives an expression identical to that for the fermionic bilinears. Moreover the result can be generalized to an n-point function.

6.3.4 Bosonization of Dirac fermions with color and flavor

In his pioneering work on non-abelian bosonization Witten also proposed a prescription for bosonizing Majorana fermions which carry both N_{F} "flavors" as well as N_{C} "colors", namely transform under the group $[O(N_{\mathrm{F}}) \times O(N_{\mathrm{C}})]_L \times [O(N_{\mathrm{F}}) \times O(N_{\mathrm{C}})]_R$. The action for free fermions is

$$S_\psi = \frac{i}{2} \int \mathrm{d}^2 x (\psi_{-ai} \partial_+ \psi_{-ai} + \psi_{+ai} \partial_- \psi_{+ai}),$$

where now $\mathrm{a} = 1, \ldots, N_{\mathrm{C}}$ and $\mathrm{i} = 1, \ldots, N_{\mathrm{F}}$ are the color and flavor indices, respectively. The equivalent bosonic action is,

$$\tilde{S}[g, h] = \frac{1}{2} N_{\mathrm{C}} S[g] + \frac{1}{2} N_{\mathrm{F}} S[h]. \tag{6.46}$$

The bosonic fields g and h take their values in $O(N_{\mathrm{F}})$ and $O(N_{\mathrm{C}})$, respectively and $S[u]$ is the WZW action given in (15.2).

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The bosonization dictionary for the currents was shown to be,

$$J_{+ij} =: \psi_{+ai}\psi_{+aj} := \frac{iN_{\mathrm{C}}}{2\pi}(g^{-1}\partial_+ g)_{ij} \quad J_{-ij} =: \psi_{-ai}\psi_{-aj} := \frac{iN_{\mathrm{C}}}{2\pi}(g\partial_- g^{-1})_{ij} \tag{6.47}$$

$$J_{+ab} =: \psi_{+ai}\psi_{+bi} := \frac{iN_{\mathrm{F}}}{2\pi}(h^{-1}\partial_+ h)_{ab} \quad J_{-ab} =: \psi_{-ai}\psi_{-bi} := \frac{iN_{\mathrm{F}}}{2\pi}(h\partial_- h^{-1})_{ab}, \tag{6.48}$$

where : : stands for normal ordering with respect to fermion creation and annihilation operators. As for the bosonic expressions for the currents, regularization is obtained by subtracting the appropriate singular parts.

In terms of the complex coordinates $z = \xi_1 + i\xi_2$, $\bar{z} = \xi_1 - i\xi_2$ (where ξ_1 and ξ_2 are complex coordinates spanning C^2, and the Euclidian plane ($\xi_1 \to x$, $\xi_2 \to -t$) and Minkowski space-time ($\xi_1 \to x, \xi_2 \to -it$) can be obtained as appropriate real sections), one can express the currents as

$$J(z)_{ij} \equiv \pi J_{-ij} = \frac{iN_{\mathrm{C}}}{2}(g\partial_z g^{-1})_{ij} \quad \bar{J}(\bar{z})ij \equiv \pi J_{+ij} = \frac{iN_{\mathrm{C}}}{2}(g^{-1}\partial_{\bar{z}} g)_{ij},$$

and similarly for the flavored currents.

In a complete analogy the theory of $N_{\mathrm{F}} \times N_{\mathrm{C}}$ Dirac fermions can be expressed in terms of the bosonic fields g, h, $\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_{\mathrm{F}}N_{\mathrm{C}}}}\phi}$ now in $SU(N_{\mathrm{F}}), SU(N_{\mathrm{C}})$ and $U(1)$ group manifolds respectively. The corresponding action is now,

$$S[g,h,\phi] = N_{\mathrm{C}}S[g] + N_{\mathrm{F}}S[h] + \frac{1}{2}\int \mathrm{d}^2x\partial_\mu\phi\partial^\mu\phi. \tag{6.49}$$

This action is derived simply by substituting $ghe^{-i\sqrt{\frac{4\pi}{N_{\mathrm{C}}N_{\mathrm{F}}}}\phi}$ instead of u in (6.31).

As for the equivalence between the bosonic and fermionic theories, we note that in both theories the commutators of the various currents have the same current algebra, and the energy-momentum tensor is the same when expressed in terms of the currents. But the situation changes when mass terms are introduced (see next section). The bosonization rules for the color and flavor currents are obtained from (6.47) and (6.48) by replacing the Weyl–Majorana spinors with Weyl ones, and in addition we have the $U(1)$ current,

$$J^{(1)}(z) \equiv \sqrt{\pi}J_-^{(1)} = :\psi_{-ai}^\dagger\psi_{-ai} := \sqrt{\frac{N_{\mathrm{F}}N_{\mathrm{C}}}{\pi}}\partial_{-\phi}$$
$$\bar{J}^{(1)}(\bar{z}) \equiv \sqrt{\pi}J_+^{(1)} = :\psi_{+ai}^\dagger\psi_{+ai} := \sqrt{\frac{N_{\mathrm{F}}N_{\mathrm{C}}}{\pi}}\partial_{+\phi}. \tag{6.50}$$

The affine Lie algebras are given by,

$$\left[J_n^A, J_m^B\right] = if^{ABC}J_{n+m}^C + \frac{i}{2}kn\delta^{AB}\delta_{n+m,0},$$

where $J^A = Tr(T^AJ)$, T^A the matrices of $SU(N_{\mathrm{C}})$, $k = N_{\mathrm{F}}$ for the colored currents and $J(z)$ is expanded in a Laurent series as $J(z) = \sum z^{-n-1}J_n$. A similar

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expression will apply for the flavor currents with T^I the matrices of $SU(N_\mathrm{F})$, and the central charge $k = N_\mathrm{C}$ instead of N_F. The commutation relation for $\bar{J}(\bar{z})$ will have the same form.

Generalizing the case of $SU(N) \times U(1)$ to our case, the Sugawara form for the energy-momentum tensor of the WZW action is given by,

$$\begin{aligned} T(z) = \frac{1}{2\kappa_\mathrm{C}} \sum_A : J^A(z) J^A(z) : + \frac{1}{2\kappa_\mathrm{F}} \sum_I : J^I(z) J^I(z) : \\ + \frac{1}{2\kappa} : J^{(1)}(z) J^{(1)}(z) :, \end{aligned} \tag{6.51}$$

where the dots denote normal ordering with respect to n ($n > 0$ meaning annihilation). The κs are constants yet to be determined. In terms of the affine Lie generators this can be written as,

$$\begin{aligned} L_n = \frac{1}{2\kappa_\mathrm{C}} \sum_{m=-\infty}^{\infty} : J_m^A J_{n-m}^A : + \frac{1}{2\kappa_\mathrm{F}} \sum_{m=-\infty}^{\infty} : J_m^I J_{n-m}^I : \\ + \frac{1}{2\kappa} \sum_{m=-\infty}^{\infty} : J_m^{(1)} J_{n-m}^{(1)} : \end{aligned} \tag{6.52}$$

Now, by applying the last expression on any primary field ϕ_l we can get a set of infinitely many "null vectors" of the form,

$$\begin{aligned} \chi_l^n = \Bigg[L_n - \frac{1}{2\kappa_\mathrm{C}} \sum_{m=n}^{0} : J_m^A J_{n-m}^A : \\ - \frac{1}{2\kappa_\mathrm{F}} \sum_{m=n}^{0} : J_m^I J_{n-m}^I : - \frac{1}{2\kappa} \sum_{m=n}^{0} : J_m^{(1)} J_{n-m}^{(1)} : \Bigg] \phi_l, \end{aligned}$$

for any $n \leq 0$ (for $n > 0$ holds immediately). Since each of these vectors must certainly be a primary field, $L_m \chi^n = J_m^A \chi^n = J_m^I \chi^n = J_m \chi^n = O$, which holds trivially for $m > 0$. When checking for $m \leq 0$, it leads to expressions for the various κ, for the central charge c of the Virasoro Algebra, and for the dimensions of the primary fields $\Delta_l = \Delta_{l+} + \Delta_{l-}$, in terms of $N_\mathrm{C}, N_\mathrm{F}$ and the group properties of the primary fields,

$$\begin{aligned} \kappa_\mathrm{C} = \kappa_\mathrm{F} &= \frac{1}{2}(N_\mathrm{C} + N_\mathrm{F}), \quad \kappa = N_\mathrm{F} N_\mathrm{C} \\ c &= \frac{N_\mathrm{C}(N_\mathrm{F}^2 - 1)}{(N_\mathrm{C} + N_\mathrm{F})} + \frac{N_\mathrm{F}(N_\mathrm{C}^2 - 1)}{(N_\mathrm{C} + N_\mathrm{F})} + 1 = N_\mathrm{F} N_\mathrm{C} \\ \Delta_{l\pm} &= \frac{(c_{l\pm}^2)^\mathrm{F}}{(N_\mathrm{F} + N_\mathrm{C})} + \frac{(c_{l\pm}^2)^\mathrm{C}}{(N_\mathrm{F} + N_\mathrm{C})} + \frac{(c_{l\pm}^2)^{(1)}}{N_\mathrm{C} N_\mathrm{F}} \end{aligned} \tag{6.53}$$

where $(c_{l\pm}^2)^\mathrm{C}$ is the eigenvalue of the $SU_{\mathrm{R,L}}(N_\mathrm{C})$ second Casimir operator in the representation of the primary field ϕ_l, namely $(\frac{1}{2}T^A)(\frac{1}{2}T^A) = (c_l^2)^\mathrm{C} I$, and similarly for the flavor group. In the cases of $SU(N_\mathrm{C})$ and $SU(N_\mathrm{F})$ the discussion applies to Δ_{l+} or Δ_{l-} separately, with C^2_{l+} and C^2_{l-}, respectively. Note that the expressions for $\kappa_{\rm F}$ and $\kappa_{\rm C}$ of equation (6.53) are an immediate generalization of the case of the group $SU(N)$ with the central term equal to one. There the factor was $N+1$, the N being the second Casimir of the adjoint representation, and the 1 being the central term.

The equivalence of the bosonic and fermionic Hilbert spaces was demonstrated by showing that the two theories have the same current algebra (affine Lie algebra), and that the energy-momentum tensor can be constructed from the currents in a Sugawara form. Goddard *et al.* [110] showed that a necessary and sufficient condition for such a construction of the fermionic $T(z)$, in a theory with a symmetry group G, is the existence of a larger group $G \subset G'$ such that G'/G is a symmetric space with the fermions transforming under G just as the tangent space to G'/G does. Based on this theorem they found all the fermionic theories for which an equivalent WZW bosonic action can be constructed. The cases stated above fit in this category. Note in passing that this does not hold for cases where the symmetry group includes more non-abelian group factors, like for example $SU(N_{\rm A}) \times SU(N_{\rm F}) \times SU(N_{\rm C}) \times U(1)$.

The prescription equation (6.49) described above , for the bosonic action that is equivalent to that of colored and flavored Dirac fermions, is by no means unique. In fact it will be shown that this prescription will turn out to be inconvenient once mass terms are introduced. Another scheme, based on the WZW theory of $U(N_{\rm F}N_{\rm C})$ will be recommended.

6.3.5 Bosonization of mass bilinears in the product scheme

A natural question here is how to generalize the rule (6.41) to Majorana fermions with action (6.46), and its analog for the case of $SU(N_{\rm F}) \times SU(N_{\rm C}) \times U(1)$ given in (6.49). We call the latter the **product scheme**. The bosonization rule for the latter case is,

$$\psi_+^{\dagger ai}\psi_{-bj} = \tilde{c}\mu N_\mu g^i_j h^a_b \mathrm{e}^{-i\sqrt{\frac{4\pi}{N_{\rm F}N_{\rm C}}}\phi}. \tag{6.54}$$

Consequently, the bosonic form of the fermion mass term $m_q\bar{\psi}^{ia}\psi_{ia}$ is,

$$m'^2 N_\mu \int \mathrm{d}^2x (TrgTrh + Trh^\dagger Trg^\dagger)\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_{\rm F}N_{\rm C}}}\phi}, \tag{6.55}$$

with $m'^2 = m_q\tilde{c}\mu$. Once again the bosonic operator (6.54) has the correct chiral transformations and the proper dimension,

$$\Delta = \Delta_g + \Delta_h + \Delta_\phi = \frac{N_{\rm F}^2-1}{N_{\rm F}(N_{\rm F}+N_{\rm C})} + \frac{N_{\rm C}^2-1}{N_{\rm C}(N_{\rm C}+N_{\rm F})} + \frac{1}{N_{\rm C}N_{\rm F}} = 1.$$

Unfortunately, the explicit calculation of the four-point function reveals a discrepancy between the fermionic and bosonic terms in (6.54). This can actually be understood directly. Since g and h are fields defined on entirely independent group manifolds, then (ignoring for a moment the $U(1)$ factor) the four-point function of the mass term can be written as,

$$<g(z_1,\bar{z}_1)g^{-1}(z_2,\bar{z}_2)g^{-1}(z_3,\bar{z}_3)g(z_4,\bar{z}_4)>$$
$$<h(z_1,\bar{z}_1)h^{-1}(z_2,\bar{z}_2)h^{-1}(z_3,\bar{z}_3)h(z_4,\bar{z}_4)> .$$

This expression differs from the corresponding fermionic Green's function, as it includes independent "contractions" for the g and h factors, whereas in the fermionic correlation function the flavor and color contractions are correlated. Moreover, the expressions for the bosonic Green's function involve hypergeometric functions, and do not resemble the case of free fermions, which is a product of poles.

6.3.6 Bosonization of the $U(N_F \times N_C)$ WZW action

It is clear from the previous discussion that the bosonization prescription for our case needs alteration. A priori there can be two ways out, either modifying the rule for the bosonization of the mass bilinear or using a different bosonic scheme altogether. As for the first approach, (6.54) preserves the proper chiral transformation laws under the product group $SU(N_{\rm F}) \times SU(N_{\rm C}) \times U(1)$ as well as the correct dimension, and therefore the number of possible modifications is very limited. For example, one might think of multiplying the expression in (6.54) by an operator which is a chiral singlet under the above group, with zero dimension. We do not know of such a modification. Therefore we are going to try a different bosonic theory than (6.49). The symmetry of the free fermionic theory can actually be taken as $U_L(N_{\rm F} \times N_{\rm C}) \times U_R(N_{\rm F} \times N_{\rm C})$ rather than $[SU(N_{\rm F}) \times SU(N_{\rm C}) \times U(1)]_L \times [SU(N_{\rm F}) \times SU(N_{\rm C}) \times U(1)]_R$. The natural bosonic action is hence a WZW theory of $u \subset U(N_{\rm F} N_{\rm C})$ and with $k = 1$. The currents are now,

$$J(z)_{\alpha\beta} = \frac{i}{2}(u\partial_z u^{-1})_{\alpha\beta} \quad \bar{J}(\bar{z})_{\alpha\beta} = \frac{i}{2}(u^{-1}\partial_{\bar{z}} u)_{\alpha\beta},$$

with α, β running from 1 to $N_{\rm F} \times N_{\rm C}$. The expressions for the flavor and color currents can be obtained by appropriate traces, over color for the flavor currents and over flavor for the color currents.

The mass bilinear is now,

$$\psi^\dagger_{+\alpha}\psi_{-\beta} = \tilde{c}\mu N_\mu u_{\alpha\beta},$$

where now the $U(1)$ term is absorbed into u.

Clearly the requirement for Sugawara construction of T, for proper chiral transformations of all the operators and for a correct dimension for the mass bilinear are fulfilled. Since now the flavor and color degrees of freedom are attached to the same bosonic field, the previous "contraction problem" in the n point functions is automatically resolved. Moreover as stated above the four-point function and in fact any Green's function will now reproduce the results of the fermionic calculation.

The currents constructed from u obey the Affine Lie algebra with $k=1$. The color currents, for instance, are $J^A = Tr(T^A J)$, where T^A are expressed as $(N_C N_F) \times (N_C N_F)$ matrices defined by $\lambda^A \otimes 1$, with λ^A the Gell–Mann matrices in color space and 1 stands for a unit $N_F \times N_F$ matrix. The central charge is $k = N_F$. The same arguments will apply for the flavor currents, now with $k = N_C$. The central charge for the $U(1)$ current is $N_C N_F$.

To see the difference between the present theory and the previous one let us express u in terms of $(N_F N_C) \times (N_F N_C)$ matrices $\tilde{g}, \tilde{h}$ and $\tilde{l}$ in $SU(N_F), SU(N_C)$ and the coset-space,

$$SU(N_F \times N_C)/\{SU(N_F) \times SU(N_C) \times U(1)\}$$

respectively, through,

$$u = \tilde{g}\tilde{h}\tilde{l}\mathrm{e}^{-i\sqrt{\frac{4\pi}{N_C N_F}}\phi}.$$

Using the formula for expressing an action of the form $S[AgB^{-1}]$ we get

$$\begin{aligned}
S[u] &= S[\tilde{g}\tilde{h}\tilde{l}] + \frac{1}{2}\int \mathrm{d}^2 x \partial_\mu \phi \partial^\mu \phi \\
S[\tilde{g}\tilde{h}\tilde{l}] &= S[\tilde{g}] + S[\tilde{l}] + S[\tilde{h}] + \frac{1}{2\pi}\int \mathrm{d}^2 x Tr(\tilde{g}^\dagger \partial_+ \tilde{g}\tilde{l}\partial_- \tilde{l}^\dagger + \tilde{h}^\dagger \partial_+ \tilde{h}\tilde{l}\partial_- \tilde{l}^\dagger).
\end{aligned}$$

We can now choose $\tilde{l} = l$ so that $l\partial_- l^\dagger$ will be spanned by the generators that are only in $SU(N_F \times N_C)/\{SU(N_F) \times SU(N_C) \times U(1)\}$. This can be achieved by taking $\tilde{u} = \tilde{g}\tilde{h}\tilde{l}$, which is u but without the $U(1)$ part, and then taking for $\tilde{h} = h \otimes 1$ a solution of the equation $\partial_- h h^\dagger = \frac{1}{N_F} Tr_F[(\partial_- \tilde{u})\tilde{u}^\dagger]$, and similarly for g with $\frac{1}{N_C} Tr_C$. These are also the conditions that the flavor currents should be expressed in terms of $\tilde{g}$ and the color currents in terms of $\tilde{h}$. For this choice, the mixed term in the above action, the term involving products of $\tilde{l}s$ with $\tilde{g}s$ or $\tilde{h}s$, is zero, and so the new action is,

$$S[u] = N_C S[g] + N_F S[h] + \frac{1}{2}\int \mathrm{d}^2 x (\partial_\mu \phi \partial^\mu \phi + S[l]).$$

Note that l is still an $SU(N_C N_F)$ matrix, while g and h are expressed as $SU(N_F)$ and $SU(N_C)$ matrices respectively, but the matrix l involves only products of color and flavor matrices (not any of them separately).

6.4 Chiral bosons

So far we have discussed the bosonization of a Dirac fermion via abelian bosonization, and N Dirac fermions using the $U(N)$ WZW model, or N Majorana fermions with an $SO(N)$ WZW model. What about the bosonization of chiral left or right Weyl fermions? In the fermionic language it is trivial to write an action of a fermion with one given chirality. It is also easy to factorize a scalar field into its left $\phi_{\rm L}(x^-)(\phi(z))$ and right moving $\phi_{\rm R}(x^+)(\bar{\phi}(\bar{z}))$ parts since the solution of equation of motion of a scalar field in real and complex coordinates takes the form

$$\phi(x_+, x_-) = \phi_{\rm L}(x^-) + \phi_{\rm R}(x^+) \quad \textit{or} \quad \phi(z, \bar{z}) = \phi(z) + \bar{\phi}(\bar{z}). \tag{6.56}$$

However, as will be discussed shortly it turns out that it is quite subtle to write down an action of a chiral boson which is equivalent to that of a left or a right chiral fermion. Once we establish an action for a chiral boson, the question is how can one couple it to abelian and non-abelian gauge fields?

In this section we will construct two seemingly independent constructions of the action of a chiral boson. In fact, it will turn out that one formulation is a special case of the other. We start with Siegel's formulation which is based on the coupling of a scalar field to fictitious gravity in a light-cone gauge [194] and then we describe a manifestly non-Lorentz invariant action [92] which is a special case of the former.[7] In [36], [199], and in [100] the two formulations were related and further generalizations were discussed. We follow the latter paper here.

Chiral bosons play an important role in string theories and in particular chiral bosons on Riemann surfaces of any genus. Here we will not enter into discussions on these constructions and describe chiral bosons only on a two-dimensional flat Minkowsky space-time.

6.4.1 Chiral boson via coupling to fictitious "light-cone gravity"

A scalar field in two space-time dimensions couples to two-dimensional gravity via the following well-known action,

$$\mathcal{L} = \frac{1}{2}\sqrt{g}g_{\alpha\beta}\partial^\alpha\phi\partial^\beta\phi, \tag{6.57}$$

where $g_{\alpha\beta}$ is the two-dimensional metric. In the " light-cone gravitational gauge" the metric has the form,

$$g^{++} = 0 \quad g^{+-} = 1 \quad g^{--} = 2\lambda. \tag{6.58}$$

In this gauge the action (6.57) reads,

$$\mathcal{L} = \partial_+\phi\partial_-\phi + \lambda(\partial_-\phi)^2. \tag{6.59}$$

Since we have fixed only part of the local symmetries of (6.57) it is straightforward to realize that the last action is still invariant under the following local transformation

$$\delta\phi = \epsilon^-\partial_-\phi \quad \delta\lambda = -\partial_+\epsilon^- + \epsilon^-\partial_-\lambda - \partial_-(\epsilon^-\lambda) \tag{6.60}$$

[7] The anomalies of the system were analyzed in [129].

which is the transformation of the scalar field under a combination of x^- coordinate transformation and a Weyl rescaling,

$$\delta x^- = \epsilon^-(x) \quad \delta g_{\alpha\beta} = -g_{\alpha\beta}\partial_-\epsilon^-. \tag{6.61}$$

It is important to emphasize that we are in fact considering a flat two-dimensional Minkowski space-time and 2λ is the $^{--}$ component of a fictitious metric. The action (6.59) is further invariant under the global shift symmetry $\phi \to \phi + a$. We denote the corresponding current as the axial current defined in (6.12) which takes the following form,

$$J^+_{(ax)} = \partial_-\phi \quad J^-_{(ax)} = \partial_+\phi + 2\lambda\partial_-\phi. \tag{6.62}$$

One can also define $J_{-(ax)} = J^+_{(ax)}$ and $J_{+(ax)} = J^-_{(ax)} - 2\lambda J_{-(ax)}$, namely using the fictitious metric to raise and lower indices. In addition we have the topological vector conserved current ${J_{(v)}}_\mu = \epsilon_{\mu\nu}\partial^\nu\phi$. The vector and axial currents defined here are those defined in (6.9), times a factor of $-\sqrt{\pi}$. Following from these two currents we can obviously also define the left and right conserved currents $J_{l/r} = \frac{1}{2}(J_{(v)} \pm J_{(ax)})$.

The equations of motion derived by variations with respect to λ and to ϕ are,

$$\delta\lambda : \quad (\partial_-\phi)^2 = 0 \quad \delta\phi : \partial_+\partial_-\phi + \partial_-(\lambda\partial_-\phi) = 0. \tag{6.63}$$

These equations imply classically the chiral nature of the boson, namely a left moving boson $\phi(x^+)$ and the conservation of the axial current. Note that unlike an ordinary scalar field the chiral scalar action (6.59), admits only the "holomorphic" conservation of the left current but not the "anti-holomorphic" conservation of the right one namely,

$$\partial_- {J_{(l)}}^- = 0 \quad \partial_+ {J_{(r)}}^- = -\partial_- {J_{(r)}}^- \neq 0, \tag{6.64}$$

which implies that there is only left affine symmetry but not a right one.

So far we have discussed the classical system. Quantum mechanically it turns out that the symmetry (6.60) is anomalous. There are several ways to verify this anomaly. Probably the easiest way is to realize the resemblance of the action (6.57) to that of the bosonic string. It is well known that the latter is consistent only in 26 dimensions and not, as our case seems to be, in one dimension. Technically this follows from the fact that the ghost system associated with the fixing of the fictitious diffeomorphism and Weyl invariance have a Virasoro anomaly (see Section 6.5.1) equal to -26. In fact following the discussion in Section 6.5.1 we know that there is another way to cancel this anomaly and that is to add to the action a background charge term of the following form,

$$\mathcal{L} = \mathcal{L}_{cl} + qR^{(2)}\phi = \mathcal{L}_{cl} + q\partial_-^2\lambda\phi = \mathcal{L}_{cl} - q\lambda\partial_-^2\phi, \tag{6.65}$$

where q is the background charge and $R^{(2)}$ is the fictitious scalar curvature. This modification of the action yields a modified energy-momentum tensor as

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will be discussed in Section 6.5.1, of the form,

$$T^{\phi}_{--} = -\frac{1}{2}(\partial_- \phi)^2 - q\partial^2_- \phi. \tag{6.66}$$

To fix the gauge associated with the symmetry (6.60) one introduces a (b, c) ghost system (see Section 6.5) that contributes $c = -26$ to the Virasoro anomaly. The modified energy-momentum tensor (6.66) contributes $c = 1 + 6\pi q^2$ so that to cancel the anomaly one has to take the background charge $q = \sqrt{\frac{25}{6\pi}}$ and hence the quantum mechanical action of the chiral boson is (6.65) with the value of q just quoted.

There are several possible methods to quantize this action. One approach is to follow the BRST quantization of the bosonic string. In this procedure one uses the Nielpotent operator associated with the Noether charge that corresponds to the BRST symmetry

$$Q_{\mathrm{BRST}} = \int \mathrm{d}xc \left(T^{\phi}_{--} + \frac{1}{2} T^{\mathrm{ghost}}_{--} \right), \tag{6.67}$$

to construct the space of physical states. The latter furnish the cohomology of Q_{BRST}, namely,

$$Q_{\mathrm{BRST}} |\mathrm{phys}\rangle = 0 \quad |\mathrm{phys}\rangle \neq Q_{\mathrm{BRST}} |\mathrm{state}\rangle\ . \tag{6.68}$$

For the vanishing ghost sector this implies that,

$$T^{\phi}_{--}{}^{(+)} |\mathrm{phys}\rangle = 0 \quad \rightarrow \quad a(k)|\mathrm{phys}\rangle = 0 \ for \ k > 0, \tag{6.69}$$

where $T^{\phi}_{--}{}^{(+)}$ is the positive frequency part of T^{ϕ}_{--} and $a(k)$ is the annihilation operator of momentum k. Thus the space of physical states is made of only left-moving $(k > 0)$ states.

Once the local symmetry (6.60) has been made non-anomalous, one can safely choose any gauge fixing. In particular we can take $\lambda = 0$ while keeping the construction of the physical states discussed above, or fixing $\lambda = -1$. As will be seen in the following section this gauge will turn out to be convenient and rather than addressing the issues of coupling to abelian and non-abelian gauge fields of the action (6.59) we will do it instead with the gauge fixed action.

Another approach of quantizing the system is based on the implementation of Dirac brackets. Starting from (6.59) and its corresponding Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\left[\frac{(\pi + \lambda\phi')^2}{1 + \lambda} + (1 - \lambda)(\phi')^2 \right], \tag{6.70}$$

we realize that the conjugate momentum of λ, π_λ vanishes. This is a primary constraint $\chi_1 = \pi_\lambda = 0$. Requiring that this constraint is preserved in time, namely, $\dot{\chi}_1 = \{\pi_\lambda(x), H\} = 0$ we find a secondary constraint $\tilde{\chi}_2 = \frac{(\pi-\phi')^2}{1+\lambda} = 0$. The Poisson bracket of χ_1 and $\tilde{\chi}_2$ vanish and hence they are a first-class constraint. However, if we replace $\tilde{\chi}_2$ with its classical equivalent constraint $\chi_2 = \pi - \phi'$ then the latter is a second-class constraint. If we add the additional constraint in the

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form of gauge fixing $\chi_3 = \lambda(x) - \lambda_0(x)$ with $\lambda_0(x)$ a given function, than all the constraints are second order with the constraint algebra,

$$c_{ij}(x,y) = \{\chi_i(x), \chi_j(y)\} \quad c_{22} = -2\delta'(x-y) \quad c_{13} = -\delta(x-y). \tag{6.71}$$

The next step is to define the Dirac bracket,

$$\{F(x), G(y)\}_D = \{F(x), G(y)\} - \int \mathrm{d}z\mathrm{d}w \{F(x), \chi_i(z)\} c_{ij}^{-1}(z,w)\{\chi_j(w), G(y)\}, \tag{6.72}$$

where $c_{ij}(x,z)c_{jk}^{-1}(z,y) = \delta_{ik}(x-y)$. The Dirac bracket rather than the Poisson bracket is then elevated to the commutator in the quantum theory $[\ ] = i\{\ \}$. Using this prescription one finds the desired result,

$$[\phi(x), \phi(y)] = \frac{1}{4i}\epsilon(x-y). \tag{6.73}$$

Implementing the constraint quantization in the path integral formulation, one has,

$$\begin{aligned} Z(J) = &\int [\mathrm{d}\phi][\mathrm{d}\pi][\mathrm{d}\lambda][\mathrm{d}\pi_\lambda]\delta(\chi_1)\delta(\chi_2)\delta(\chi_3) \\ &\times \sqrt{\mathrm{Det}[C_{ij}(x,y)]}\mathrm{e}^{i\int \mathrm{d}^2x(\pi\dot{\phi} + \pi_\lambda\dot{\lambda} - \mathcal{H} - J\phi)}. \end{aligned} \tag{6.74}$$

Using the δ functions and since $\mathrm{Det}[C]$ is field independent we find,

$$Z[J] = \int [\mathrm{d}\phi]\mathrm{e}^{i\int \mathrm{d}^2x\tilde{\mathcal{L}} - J\phi} \quad \tilde{\mathcal{L}} = \dot{\phi}\phi' - (\phi')^2. \tag{6.75}$$

Thus we see that this procedure yields the action (6.59) in the gauge $\lambda = -1$ which will be the topic of the following section.

6.4.2 Non manifestly Lorentz invariant classical action

Let us start with the following non-local action,

$$\mathcal{L} = \int \mathrm{d}z\mathrm{d}y\rho(z)\epsilon(z-y)\dot{\rho}(y) - \int \mathrm{d}x\rho^2(x), \tag{6.76}$$

where $\rho(x)$ is a local bosonic field. The system can also be described in terms of the non-local bosonic field,

$$\phi(x) = \frac{1}{2}\int \mathrm{d}y\epsilon(x-y)\rho(y), \quad \phi'(x) = \rho(x), \tag{6.77}$$

with a local Lagrangian density,

$$\mathcal{L} = \phi'\dot{\phi} - \phi'^2 = \partial_-\phi(\partial_+\phi - \partial_-\phi). \tag{6.78}$$

As we said the Lagrangian density is in fact (6.59) in the gauge $2\lambda = -1$. The classical equation of motion which corresponds to (6.76) is,

$$\partial_-\phi' = 0. \tag{6.79}$$

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This equation has a general solution $\partial_-\phi = g(t)$, however by requiring $\partial_-\phi = 0$ at the spatial boundaries, we set $g(t) = 0$ and get the chiral solution $\partial_-\phi = 0$.

Even though the action (6.76) is not manifestly Lorentz invariant, it is easy to see that it is invariant under time translation $\delta\phi = \epsilon\dot{\phi}$, space translation $\delta\phi = \epsilon\phi'$, and the unconventional Lorentz transformation $\delta\phi = (t+x)\phi'$. For the above invariance transformations to exist we assume vanishing surface terms. The associated Noether charges are $H = \int \phi'^2,\ P = -H$ and $M = \int \mathrm{d}x[(x+t)(\phi')^2]$. The system is, in fact, invariant under yet another unusual Lorentz transformation, $\delta(\partial_-\phi) = x^+\partial_+(\partial_-\phi) - x^-\partial_-(\partial_-\phi) - \partial_\phi$.

In addition, equation (6.76) is invariant under the global axial transformation $\phi \to \phi + \alpha$. The associated current $j^\mu_{(ax)}$ and the vector current which as was discussed in Section 6.1 is "topologically conserved" are given by

$$
\begin{aligned}
j_{(ax)_+} &= \partial_+\phi - 2\partial_-\phi \qquad j_{(ax)_-} = \partial_-\phi \\
j_{(v)_+} &= \partial_+\phi \qquad j_{(v)_-} = -\partial_-\phi.
\end{aligned}
\tag{6.80}
$$

As usual from the vector and the axial currents we can write down the left and right currents $J(lr) = \frac{1}{2}[j_{(v)} \pm j_{(ax)}]$, respectively. They have the following expressions,

$$
J_{(l)_-} = 0, \quad J_{(l)_+} = \phi; \quad J_{(r)_-} = -\partial_-\phi, \quad J_{(r)_+} = \partial_-\phi. \tag{6.81}
$$

Note however that only the left current is holomorphically conserved namely $\partial_- J_{(l)_+} = 0$, while the right current is not antiholomorphically conserved. This property is related to the invariance of the Lagrangian under $\delta\phi = \alpha(x^+)$ and not under $\delta\phi = \alpha(x^-)$. As will be explained below only the left $U(1)$ affine Lie algebra current exists in the quantum theory. Similarly, the Lagrangian (6.76) is invariant under the conformal transformations $\delta\phi = \epsilon(x^+)\phi'$ and $\delta\phi = \epsilon(t)\partial_-\phi$. The associated Sugawara type Noether currents are,

$$
\begin{aligned}
T_{(l)_{++}} &= (\phi')^2 = J_{(l)_+}^2 \qquad \partial_- T_{(l)_{++}} = 0 \\
T_{(r)_{11}} &= (\partial_-\phi)^2 = J_{(r)_1}^2 \qquad \partial_1 T_{(r)_{++}} = 0.
\end{aligned}
\tag{6.82}
$$

Quantization of the model

By treating the system as a constrained system, we now invoke its canonical and path-integral quantization. We repeat here the Dirac bracket procedure discussed above. The constraint $\chi = \pi - \phi' = 0$, is a second-class constraint since $\{\chi(x), \chi(y)\} = -2\delta(x-y)$. The Hamiltonian density of the system takes the following form $\mathcal{H} = \mathcal{H}_c + v(\phi, \pi)\chi$, where $\mathcal{H}_c = \phi'^2 = \pi^2 = \pi\phi'$. Using this form for $\mathcal{H}$ it is easy to verify that the condition $\dot{\chi} = \{\chi, \mathcal{H}\} = 0$, does not lead to further constraints but fixes $v(\phi, \pi)$. The passage to the quantum theory is performed by passing from the Dirac brackets rather than the usual Poisson bracket to the commutator via,

$$-i[F(x),G(y)] = \{F(x),G(y)\} - \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \{F(x),G(\xi_1)\} \\ \times\Big[-\frac{1}{4}\epsilon(\xi_1-\xi_2)\{F(\xi_2),G(y)\}. \tag{6.83}$$

Following this definition, the operator algebra for π and ϕ takes the form,

$$\begin{aligned}[]
[\phi(x),\phi(y)] &= \frac{1}{4i}\epsilon(x-y),\\
[\pi(x),\phi(y)] &= \frac{1}{2i}\delta(x-y),\\
[\pi(x),\pi(y)] &= \frac{1}{2i}\delta(x-y).
\end{aligned} \tag{6.84}$$

One also finds that, for an arbitrary operator $F(\phi,\pi)$, $[\chi(x),F(y)] = 0$ so that the constraint $\pi(x) - \phi'(x) = 0$, is now realized at the operator level. For example the Hamiltonian density can now be expressed in the three forms of $\mathcal{H}_c$ mentioned above. The system is solved by Fourier transforming,

$$\begin{aligned}
\phi(x) &= \int_{-\infty}^{0} \frac{\mathrm{d}k}{2\sqrt{\pi k}}[a_k \mathrm{e}^{-ikx} + a_k^\dagger \mathrm{e}^{ikx}]\\
\pi(x) &= \phi'(x),
\end{aligned} \tag{6.85}$$

with the usual algebra,

$$[a(k),a^\dagger(k')] = \delta(k-k'). \tag{6.86}$$

Note that only $k \le 0$ appears in the decomposition of $\phi(x)$, which expresses the chiral nature of the field. The single-particle Hilbert space is then a continuum of states with energy $E = |k|,\ k \le 0$. Hence the Hamiltonian formalism has correctly implemented the chirality constraint $\partial_-\phi = 0$. Furthermore, this property can also be deduced from the Hamiltonian equation of motion $\dot\phi = i[H,\phi] = \phi'$. Note that to get the chiral solution $\partial_-\phi = 0$ as a solution of the equation of the motion, we had to assume chiral boundary conditions. Here it looks at first that the chirality property was derived with no assumptions, but in passing from (6.83) to (6.84 we assumed that $(\pi - \phi')(x = \infty) = -(\pi - \phi')(x = -\infty)$, so together with choosing zero surface terms we in fact assumed chiral boundary conditions.

For the path integral quantization of the system we use the method developed for Hamiltonian systems with constraints. The generating functional is given by,

$$\begin{aligned}
Z[J] &= \int \mathrm{d}\phi \mathrm{d}\pi \delta(\chi) \mathrm{e}^{\int \mathrm{d}^2x(\pi\dot\phi - \mathcal{H}_c - J\phi)}\\
&= \int \mathrm{d}\phi \mathrm{e}^{\int \mathrm{d}^2x(\mathcal{L} - J\phi)},
\end{aligned} \tag{6.87}$$

where a normalization by $Z[O]^{-1}$ is implied. The Lagrangian density that emerges in (6.87) is clearly in the original form of equation (6.78). The functional integral (6.87) is not specified completely until we include $\partial_-\phi = 0$ on the boundary; thus we are in the same situation as in the canonical quantization.

Using the commutation relations (6.84), it is straightforward to verify that the Noether charges H, P, M, respectively generate the transformations $\delta_T\phi$, $\delta_S\phi$ and $\delta_M\phi$ given above. It can easily be shown that they satisfy the Poincare algebra,

$$[H,P] = 0 \quad [M,H] = iP \quad [M,P] = iH. \tag{6.88}$$

Abelian bosonization of a chiral fermion

Recall (Section 3.8) that the action of one left-handed complex chiral fermion,

$$S_{f_+} = \int \mathrm{d}^2 x \psi^\dagger \partial_- \psi, \tag{6.89}$$

is classically invariant under both an affine chiral transformation $\delta\psi = i\epsilon(x^+)\psi$ and the conformal transformation $\delta\psi = i\epsilon^+(x^+)\partial_+\psi$. The associated Noether currents were shown to have the following quantum form,

$$J_+ =: \psi^\dagger\psi \quad T_{++} = i : \psi^\dagger \partial_+ \psi := \pi : J_+ J_+ :, \tag{6.90}$$

obey the left $U(1)$ affine Lie and Virasoro algebras, respectively with the well-known central charges $k = 1$ and $c = 1$.

It is now straightforward to realize that the $J_{(l)} = \phi'$ and $T_{(l)} = (\phi')^2$ have the same $k = 1$ and $c = 1$ central charges. Using the operator algebra we can now evaluate the commutators of the chiral current and of the energy-momentum tensor. The results are as follows

$$\begin{aligned}
[J_{(l)}(x), J_{(l)}(y)] &= [\phi'(x), \phi'(y)] = \frac{i}{2}\delta'(x-y), \\
[T_{(l)}(x), T_{(l)}(y)] &= [: (\phi'(x))^2, (\phi'(y))^2] = i(T_{(l)}(x) + T_{(l)}(y))\delta'(x-y) \\
&\quad - \frac{i}{24\pi}\delta'''(x-y).
\end{aligned} \tag{6.91}$$

From the general discussion in Sections 2.4 and 3.3 it follows now that these commutation relations correspond to central charges of $c = k = 1$. Hence our bosonic theory furnishes the same irreducible representation of the affine Lie and Virasoro algebras as one free left-handed chiral fermion. Since for $k = 1$ the affine Lie algebra has a unique irreducible unitary representation the two theories on flat two-dimensional space-time are therefore equivalent. Below it will be shown that the anomalies of the bosonized theory in coupling to gauge and gravitational background are the same as for the fermionic theory.

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Combining left and right chiral bosons

The canonical quantization procedure of the last section can be repeated for a right chiral boson. Let us rename the operators for the $\partial_- \phi = 0$ case as $\phi_{\rm L}$ and $\pi_{\rm L}$ and call the corresponding operators for the $\partial_+ \phi = 0$ case as $\psi_{\rm R}$ and $\pi_{\rm R}$. The Lagrangian density for the right moving field has the form $\mathcal{L}_{\rm R} = \partial_+ \phi(\partial_+ \phi - \partial_- \phi)$ Then the combined Lagrangian,

$$\mathcal{L} = \mathcal{L}_{\rm L} + \mathcal{L}_{\rm R} \tag{6.92}$$

and the corresponding Hamiltonian density,

$$\mathcal{H} = (\pi_{\rm L})^2 + (\pi_{\rm R})^2 \tag{6.93}$$

describe a single free massless scalar defined by,

$$\phi = \phi_{\rm L} + \phi_{\rm R} \quad \pi = \pi_{\rm L} + \pi_{\rm R} \tag{6.94}$$

with a Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\phi'^2. \tag{6.95}$$

Using the commutation relations of the chiral bosons (6.84) we find as expected that,

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \quad [\phi(x), \pi(y)] = i\delta(x-y). \tag{6.96}$$

Unsurprisingly the Noether charges associated with the space-time translation and Lorentz transformation, $H = H_{\rm R} + H_{\rm L}$, $P = P_{\rm R} + P_{\rm L}$ and $M = M_{\rm L} + M_{\rm R}$ obey the usual Poincare algebra. The (lr) $U(1)$ affine Lie algebra currents of the combined system are given by the left current of the left system and the right current of the right system, respectively. The central charges are $k = 1$ for the algebras in both sectors. Similarly, because of the Sugawara construction, $T_{++} = T_{++{\rm L}}$ and $T_{--} = T_{--{\rm R}}$ with c = 1 for the left and the right Virasoro algebras.

Partition function

We would now like to compare the one loop partition function of a chiral fermion and that of our chiral boson. We therefore pass to a two-dimensional space-time domain with $1 \geq x \geq 0$. The mode expansion for ϕ previously given by equation (6.85) now takes the form,

$$\phi(x,t) = \phi_0 + p(x+t) + \sum_{n>0} \frac{1}{\sqrt{2n}} \left[a_n^\dagger {\rm e}^{2\pi i n x^+} + a_n {\rm e}^{-2\pi i n x^+}\right]. \tag{6.97}$$

The one loop partition function which corresponds to this mode expansion is

$$Z(\tau) = {\rm Tr}[{\rm e}^{-\tau_2 H + i\tau_1 P}] = {\rm Tr}[q^{\frac{H}{\pi}}] \tag{6.98}$$

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where $q = \mathrm{e}^{i\tau\pi}$ and we use the fact that $H = -P$. Let us first calculate the contribution to the trace of the oscillation modes,

$$\mathrm{Tr}[q^{\frac{H}{\pi}}] = \mathrm{Tr}[q^{(\sum_n n a_n^\dagger a_n - \frac{1}{12})}] = \prod_{n=1} q^{-\frac{1}{12}}[1-q^{2n}]^{-1} = \eta^{-1}(\tau), \tag{6.99}$$

where the factor $-\frac{1}{12}$ is the output of the normal ordering and η is the Dedekin η function (see eqn. (2.48)). The Hamiltonian of the zero modes is $H = \frac{1}{4\pi}p^2$. If we now take the bosonic momenta to lie on a shifted lattice such that the eigenvalues of p are $2\pi(m+\alpha)$ and introduce the twist operator $g = \mathrm{e}^{i\beta p}$, we get the zero mode partition function to be equal to the Riemann theta function $\Theta[(\alpha\beta)](\tau|0)$ and so that altogether the full partition function is given by,

$$Z(\tau) = \frac{\Theta[(\alpha\beta)](\tau|0)}{\eta(\tau)}. \tag{6.100}$$

This corresponds to the partition function of a Weyl fermion with the boundary conditions $\psi(x+1,t) = -\mathrm{e}^{2\pi i\alpha}\psi(x,t); \psi(x+\mathrm{Re}\tau, t+\mathrm{Im}\tau) = -\mathrm{e}^{-2\pi i\beta}\psi(x,t)$.

6.4.3 Coupling to abelian gauge fields

There are several ways to couple an abelian gauge field to a chiral boson corresponding to the various regularization schemes in the fermionic theory. We start by analyzing the bosonization in the vector conserving regularization scheme. The vector current is coupled to an abelian gauge field via,

$$\mathcal{L}_{(V)} = \mathcal{L}_0 + ({J_{(V)}}_- A_+ + {J_{(V)}}_+ A_-), \tag{6.101}$$

where $\mathcal{L}_0$ is the uncoupled Lagrangian density and V stands for the vector conserving scheme. The vector current is still obviously conserved, but the divergence of the axial current,

$$\partial_- {J_{(\mathrm{ax})}}_+ + \partial_+ {J_{(\mathrm{ax})}}_- = \partial_- A_+ - \partial_+ A_- = \epsilon^{\mu\nu}F_{\mu\nu}, \tag{6.102}$$

is now equal to the anomaly deduced in the fermionic theory from the one loop diagram.

Next we discuss the bosonization in the left-right scheme. For that purpose a term bilinear in the gauge fields has to be added to the ${J_{(l)}}_+ {A_{(l)}}_-$ term. The Lagrangian then takes the form:

$$\mathcal{L}_{(\mathrm{LR})} = \mathcal{L}_0 + {J_{(l)}}_+ {A_{(l)}}_- + \frac{\sqrt{2}}{4}{A_{(l)}}_- {A_{(l)}}_1, \tag{6.103}$$

where (LR) indicates the left-right scheme. The divergence of the left current which is derived from $\frac{\partial\mathcal{L}_{(\mathrm{L,R})}}{\partial A_\pm}$ now has the form,

$$\partial_-({J_{(l)}}^-)_{(\mathrm{L,R})} + \partial_+({J_{(l)}}^+)_{(\mathrm{L,R})} = \frac{1}{4}(\partial_- A_+ - \partial_+ A_-) = \frac{1}{4}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{6.104}$$

The Lagrangian (6.103) is therefore really the bosonized fermionic action regularized in the left-right scheme. Obviously, a similar prescription for the (V) and (LR) schemes can be applied to the right chiral boson. It is straightforward to show that the vector as well as the left-right actions are invariant under "curved space-time" Lorentz transformations as discussed above.

6.4.4 Chiral WZW and coupling to non-abelian gauge fields

The non-abelian bosonization of N Majorana or N Dirac fermions using WZW theories of $SO(N)$ and $U(N)$, respectively was described in Section 6.3. For the non-abelian bosonization of left chiral fermions we propose to generalize the action (6.78) into an action that describes a map to the group manifold of the form,

$$S_+[u] = \frac{\sqrt{2}}{4\pi}\int \mathrm{d}^2x Tr(\partial_- u\partial_1 u^{-1}) + S_{WZ}. \tag{6.105}$$

In a similar manner to the abelian case this is a WZW action coupled to fictitious chiral gravity in the gauge $h_{++} = -1$. In fact we can consider a generalization of this action to the so-called k-level chiral WZW namely $S_{+k}[U] = kS_+[u]$. The equation of motion that follows from the variation of (6.105) with respect to the variation of u can be expressed as,

$$\partial_-(u^{-1}\partial_1 u) = 0 \quad or \quad \partial_1(u\partial_- u^{-1}) = 0, \tag{6.106}$$

where each form can be obtained from the other.

The global, chiral transformations $u \to Au$ and $u \to uB^{-1}$ where $A, B \in G$ leave the action (6.105) invariant. However, out of the invariance under the two affine Lie algebra transformation of the original WZW action $u \to uB^{-1}(x^+)$ and $u \to A(x^-)u$, only the first survives. As for the abelian case the invariance under the second transformation is lost. The Noether currents associated with the left-right transformations of (6.105) are,

$$\begin{aligned}
J_{(l)\,-} &= 0 \quad J_{(\ell)\,+} = \frac{i\sqrt{2}k}{2\pi}u^{-1}\partial_1 u \\
J_{(r)\,-} &= \frac{ik}{2\pi}u\partial_- u^{-1} \quad J_{(r)\,+} = -\frac{ik}{2\pi}u\partial_- u^{-1}.
\end{aligned} \tag{6.107}$$

The conservation of the left and right currents follow here simply from the equations of motion. However, unlike the ordinary WZW action only the left current is holomorphically conserved a $\partial_- J_{(l)} = 0$; whereas $\partial_+ J_{(r)} \neq 0$. Obviously this is a manifestation of the invariance of the action only under the left affine Lie algebra transformation $\delta u = -iu\epsilon(x^+)$, discussed above. The left current transforms as follows $\delta J_{(l)} = [i\epsilon(x^+), J_{(l)}] + \frac{\sqrt{2}k}{2\pi}\epsilon$, leading to the $O(N)(U(N))$ affine Lie algebra with central charge equal to k.

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The action $S_{+k}[u]$ is invariant under the left affine transformation $\delta u = \epsilon(x^+)u$. The corresponding energy momentum tensor has again a Sugawara form and Virasoro central charge which are,

$$T_{(l)} = \frac{2\pi}{(c_2+k)}\sum_a : J^a_{(l)} J^a_{(l)} : \quad c = \frac{k\text{dim}G}{c_2+k}, \tag{6.108}$$

where as usual $J_{(l)} = J^a_{(l)} T^a$ and T^a are hermitian matrices representing the algebra of the group, c_2 is the second Casimir operator in the adjoint representation and dim$G$, is the dimension of the group.

The coupling of non-abelian gauge fields to a chiral WZW action, is a straightforward generalization of the coupling of the abelian gauge fields. Again there are several ways to couple gauge fields corresponding to the various regularization schemes in the fermionic theory. The bosonized action (for k = 1) related to the vector conserving regularization scheme is given by,

$$\begin{aligned} S_V[u, A_-, A_+] &= S_+[u] + \int \mathrm{d}^2x \mathrm{Tr}[J^-_{(v)} A_- + J^+_{(v)} A_+] \\ &\quad - \frac{\sqrt{2}}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}[u^{-1} A_1 u A_- - A_- A_1], \end{aligned} \tag{6.109}$$

where $J_{(v)}$ and $J_{(\text{ax})}$ are constructed from the left and right currents of (6.107) in the usual way. Using the equation of motion one finds the conservation of the vector current and the anomalous divergence of the axial current,

$$D_\mu (J^\mu_{(v)})_V = 0 \quad D_\mu (J^\mu_{(\text{ax})})_V = \frac{1}{\pi}\epsilon^{\mu\nu} F_{\mu\nu} \tag{6.110}$$

The coupling of a left non-abelian gauge field that corresponds to the fermionic description in the left-right regularization scheme is given by,

$$S_{(\text{LR})}[u, A_-, A_1] = S_+[u] + \frac{\sqrt{2}}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}[(u^{-1}\partial_1 u + \frac{1}{2\pi}A_1)A_-]. \tag{6.111}$$

The associated current divergence is,

$$D_\mu (J^\mu_{(l)})_{\text{LR}} = D_- J^-_{l\ (\text{LR})} + D_+ J^+_- l_{(\text{LR})} = \frac{1}{\pi}\epsilon^{\mu\nu} F_{\mu\nu}. \tag{6.112}$$

This expression for the anomalous divergence of the left current is identical to the result of the loop calculation in the fermionic version regularized in the left-right scheme.

6.5 Bosonization of systems of operators of high conformal dimension

Can bosonization, the equivalence map between systems of dimension half fermions and dimension zero bosons, be extended also to systems made out of higher-dimensional operators? In particular can one map theories of odd and even fields of higher-dimensional operators to theories built from dimension zero

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scalar fields. In this section it will be shown that indeed such a map exists for two families of theories, one with anti-commuting fields and the other with commuting ones with arbitrary integer and half-integer dimensions.

The bosonization of the b, c and β, γ systems was introduced in [94]

6.5.1 The bosonization of the "b,c" free CFT

We first briefly describe the system. Consider a system built from a pair of anti-commuting fields b and c which is described by the action,

$$S_{b,c} = \frac{1}{2\pi} \int \mathrm{d}^2 z b \bar{\partial} c. \tag{6.113}$$

It is easy to see that this action is invariant under conformal transformation $z \to a^{-1} z$ $b \to a^{\lambda} b$ and $c \to a^{1-\lambda} c$, namely the b and c fields have conformal dimensions,[8]

$$h_b = \lambda \quad h_c = (1 - \lambda). \tag{6.114}$$

The classical equations of motion,

$$\bar{\partial} b(z, \bar{z}) = 0 \quad \bar{\partial} c(z, \bar{z}) = 0, \tag{6.115}$$

imply that both fields are holomorphic, namely $b(z), c(z)$. Similar to the derivation of the operator equations of motion using the path integral (1.55) for the scalar field we find for the (b, c) system that,

$$\bar{\partial} b(z) c(0) = 2\pi \delta^2(z, \bar{z}). \tag{6.116}$$

This is compatible with the OPE,

$$b(z)c(w) =: b(z)c(w) : + \frac{1}{z - w} \quad c(z)b(w) =: c(z)b(w) : + \frac{1}{z - w}. \tag{6.117}$$

The OPEs $b(z)b(w)$ and $c(z)c(w)$ do not have singular parts when w it brought to z. To compute the energy-momentum tensor we use the Noether procedure. We vary the b and c fields as follows,

$$\begin{aligned} \delta b &= \bar{\epsilon} \partial b + \lambda (\partial \bar{\epsilon}) b, \\ \delta c &= \bar{\epsilon} \partial c + (1 - \lambda)(\partial \bar{\epsilon}) b. \end{aligned} \tag{6.118}$$

For holomorphic $\bar{\epsilon}(z)$ this is a symmetry transformation. Taking now $\bar{\epsilon}(z, \bar{z})$ we read the energy-momentum tensor from the variation of the action as follows,

$$\delta S_{b,c} = -\frac{1}{2\pi} \int \mathrm{d}^2 z \bar{\partial} \bar{\epsilon} T = \frac{1}{2\pi} \int \mathrm{d}^2 \bar{\partial} \bar{\epsilon} [(\partial b) c - \lambda \partial (bc)]. \tag{6.119}$$

[8] We denote by λ the dimension of b as is common in the literature. Obviously it has nothing to do with λ used in the previous section on chiral bosons.

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Thus the energy-momentum tensor is,

$$T = (\partial b)c - \lambda\partial(bc). \tag{6.120}$$

It is straightforward to verify that the OPE of T with b and with c indeed yields the variation (6.118). From the OPE $T(z)T(w)$ we read the Virasoro anomaly associated with the (b, c) system,

$$c = -3(2\lambda - 1)^2 + 1. \tag{6.121}$$

The action (6.113) is also invariant under the fermion number transformation,

$$b \to \mathrm{e}^{i\alpha(z)}b \quad c \to \mathrm{e}^{-i\alpha(z)}c. \tag{6.122}$$

Using again the Noether procedure we find that the corresponding conserved fermion number current is,

$$j =: bc : \quad \bar{\partial}j = 0. \tag{6.123}$$

From the basic OPE of $b(z)$ and $c(w)$ one finds that the OPE of the energy-momentum tensor and the fermion number current reads,

$$T(z)j(w) = \frac{1-2\lambda}{(z-w)^3} + \frac{j(w)}{(z-w)^2} + \frac{\partial j(w)}{(z-w)}. \tag{6.124}$$

The $(b(z), c(z))$ conformal field theory is fully holomorphic. Needless to say that one can similarly write down an anti-holomorphic system $\bar{b}(\bar{z}), \bar{c}(\bar{z})$. In fact in Section 2.12 we have already discussed a special case of the b, c family. For $\lambda = \frac{1}{2}$ we get the Weyl spinor ψ of dimension $\frac{1}{2}$ such that the Virasoro anomaly of the system is $c = 1$. Another "famous" case is that of the b and c ghosts associated with the covariant fixing of two-dimensional diffeomorphism. In this case $\lambda = 2$ the dimensions of b and c are 2 and -1, respectively and the corresponding Virasoro anomaly is -26.

Now we raise again the question, can one describe the b, c system in terms of a scalar CFT? Since the $\psi, \psi^\dagger$ is a special case of the b, c system and since we have already developed the bosonization rules for Dirac and Weyl spinors (see Section 6.1.1) we start with a similar ansatz for the bosonic version of b and c, namely,

$$b(z) \leftrightarrow : \mathrm{e}^{i\phi(z)} : \quad c(z) \leftrightarrow : \mathrm{e}^{-i\phi(z)} :, \tag{6.125}$$

Comparing (6.18) and (6.117) it is evident that indeed this map reproduces the algebra of the (b, c) system. It is also easy to realize that the fermion number current has the following bosonic equivalent,

$$: b(z)c(z) : \leftrightarrow i\partial\phi(z). \tag{6.126}$$

What is left to determine is whether the energy-momentum tensors of the two theories and correspondingly the dimensions of the fields match. Obviously the free scalar action (6.1) which is the bosonic dual of the action of the Dirac

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operator cannot describe the (b,c) systems which are a family of CFTs. We also need to identify a family characterized by the parameter λ of scalar field theories. A simple way to achieve this is to realize that the energy-momentum of the general (b,c) system can be written in terms of the spin 1/2 fermion as follows,

$$T^{b,c} = T^{\psi} - \left(\lambda - \frac{1}{2}\right)\partial(bc). \tag{6.127}$$

Thus following (6.126) the scalar energy momentum has to have the form,

$$T^{\phi}_{\lambda} = T^{\phi} - \left(\lambda - \frac{1}{2}\right)\partial^2\phi. \tag{6.128}$$

This is the energy-momentum of a scalar field with a background charge, or the linear dilaton theory with $q = -i(\lambda - \frac{1}{2})$. The central charge of these theories was shown to be,

$$c^{\phi} = 1 + 12q^2 = 1 - 3(2\lambda - 1)^2 = c^{b,c}. \tag{6.129}$$

Moreover using the fact that the dimension of an operator $:\mathrm{e}^{ik\phi(z)}:$ was shown to be $\frac{k^2}{2} + ikq$ it is easy to check that,

$$h_{e^{i\phi}} = \lambda = h_b \quad h_{e^{-i\phi}} = 1 - \lambda = h_c, \tag{6.130}$$

which verifies that the operators mapped by the bosonization indeed have the same conformal dimensions.

The anti-commutative nature of the b,c system is obeyed also by their bosonic duals, just as for the case of spin 1/2 fields, namely,

$$:\mathrm{e}^{i\phi(z)}::\mathrm{e}^{i\phi(w)}:= \mathrm{e}^{-[\phi(z),\phi(w)]}:\mathrm{e}^{i\phi(w)}::\mathrm{e}^{i\phi(z)}:= -:\mathrm{e}^{i\phi(w)}::\mathrm{e}^{i\phi(z)}:, \tag{6.131}$$

at equal times, namely $|z| = |w|$ since for that case $[\phi(z),\phi(w)] = \pm i\pi$.

6.5.2 *The bosonization of the β,γ system*

The b,c system is built from anti-commuting fields which as we have just seen are describable in terms of a scalar field. In a complete analogy one would suspect that a similar bosonization also holds for a system built from commuting fields with the same structure of action. As we shall see shortly this is indeed the case. The so-called β,γ system defined by the action (6.132),

$$S_{\beta,\gamma} = \frac{1}{2\pi}\int \mathrm{d}^2 z\beta\bar{\partial}\gamma. \tag{6.132}$$

The fact that now the building blocks are commuting introduces of course a sign change with respect to the b,c system when one interchanges the fields, namely,

$$\beta(z)\gamma(w) = -\frac{1}{z-w} \quad \gamma(z)\beta(w) = \frac{1}{z-w}. \tag{6.133}$$

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The energy-momentum tensor is the same as (6.120) when replacing b and c with β and γ, respectively. For the Virasoro anomaly one has just to reverse the sign of that of the b, c system. A distinguished member of this family of conformal field theories is the case of $\lambda = 3/2$ which describes the ghost system associated with the gauge fixing of the superdiffeomorphism. In this case the $c = 11$. As is well known this combined with $c = -26$ requires a contribution to the Virasoro anomaly of 15 of the non-ghost fields which requires ten dimensions for superstring theories.

The bosonization of this commuting system is slightly more involved than that of the b, c system. It turns out that now one has to invoke two scalar fields ϕ and χ with the OPEs,

$$\phi(z)\phi(w) = -\ln(z-w) + \dots \quad \chi(z)\chi(w) = \ln(z-w) + \dots, \tag{6.134}$$

where $\dots$ stands for non-singular terms. The corresponding scalar theories have a background charge of $(\lambda' - 1/2)$ for ϕ and $i/2$ for χ so that the energy-momentum tensor of the full bosonic reads,

$$T_{\phi,\chi} = -\frac{1}{2}\partial\phi\partial\phi + \frac{1}{2}\partial\chi\partial\chi + \frac{1}{2}(1-2\lambda')\partial^2\phi + \frac{1}{2}\partial^2\chi, \tag{6.135}$$

which yields the desired Virasoro anomaly $c = 1 + 3(2\lambda' - 1)^2 = -c_{b,c}$. The bosonic operators that correspond to β and γ are,

$$\beta(z) \leftrightarrow \mathrm{e}^{-\phi+\chi}\partial\chi \quad \gamma \leftrightarrow \mathrm{e}^{\phi-\chi}, \tag{6.136}$$

which have the conformal dimensions $h_\beta = \lambda'$ and $h_\gamma = 1 - \lambda'$.

6.5.3 The Wakimoto bosonization

One application of the β, γ system enables us to transform the WZW model into a theory expressed in terms of free fields. Consider a combined system that includes a $(\lambda' = 1, 0)$ β, γ system and an additional scalar field with a background charge $-\frac{i}{\sqrt{2(k+2)}}$. This system is described by the following Lagrangian,

$$\mathcal{L}_{\mathrm{Wak}} = \beta\bar\partial\gamma + \bar\beta\partial\bar\gamma + \partial\varphi\bar\partial\varphi. \tag{6.137}$$

Alternatively, as was discussed in Section 6.4 one can add to this Lagrangian density also a term of the form $-\frac{i}{\sqrt{2(k+2)}}R^2\phi$. The corresponding energy-momentum tensor $T(z)$ is given by,

$$T(z) = -\beta\partial\gamma - \frac{1}{2}\partial\varphi\partial\varphi - \frac{i}{\sqrt{2(k+2)}}\partial^2\varphi, \tag{6.138}$$

and the non-trivial OPEs of these field are given by,

$$\gamma(z)\beta(w) = \frac{1}{z-w} + O(z-w) \quad \varphi(z)\varphi(w) = -\ln(z-w) + O(z-w). \tag{6.139}$$

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We define now the following holomorphic currents in terms of the β, γ and φ fields:

$$\begin{aligned}
J^+ &= \beta(z), \\
J^0 &= i\sqrt{2(k+2)}\partial\varphi(z) + 2 :\gamma\beta(z):, \\
J^- &= -i\sqrt{2(k+2)} :\partial\varphi\gamma:(z) - k\partial\varphi(z) - :\beta\gamma\gamma:(z).
\end{aligned} \tag{6.140}$$

Using the OPEs it is straightforward to determine the OPEs of the currents,

$$\begin{aligned}
J^+(z)J^-(w) &= \frac{k}{(z-w)^2} + \frac{J^0(z)}{(z-w)}, \\
J^0(z)J^+(w) &= \frac{2J^+}{(z-w)}, \\
J^0(z)J^0(w) &= \frac{2k}{(z-w)^2}, \\
J^0(z)J^+(w) &= \frac{-2J^-}{(z-w)},
\end{aligned} \tag{6.141}$$

which are the OPEs of the $SU(2)$ affine current algebra, with level k. Furthermore the Sugawara energy-momentum tensor,

$$T(z) = \frac{1}{2(k+2)}\left[\frac{1}{2}J^0J^0 + J^+J^- + J^-J^+\right], \tag{6.142}$$

is identical to the energy-momentum tensor given in (6.138), and the associated Virasoro anomaly is,

$$c = 2 + 1 - 24\left(\frac{1}{4(k+2)}\right) = \frac{3k}{k+2}. \tag{6.143}$$

Since β has dimension one and γ dimension zero, their mode expansion takes the form,

$$\beta = \sum_n \beta_n z^{-n-1}, \quad \gamma = \sum_n \gamma_n z^{-n}. \tag{6.144}$$

Substituting this into the expressions of the currents one finds,

$$\begin{aligned}
J^+_n &= \beta_n, \\
J^0_n &= i\sqrt{2(k+2)}n\varphi_n + 2\sum_m :\beta_m\gamma_{m-n}:, \\
J^-_n &= -i\sqrt{2(k+2)}\sum_m :n\varphi_m\gamma_{n-m}: -kn\varphi_n - \sum_{lm} :\beta_l\gamma_m\gamma_{n-m-l}: .
\end{aligned} \tag{6.145}$$

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