BULL. AUSTRAL. MATH. SOC. 46L40
VOL. 33 (1986), 145-154.

AUTOMORPHISMS OF HOMOGENEOUS C^*-ALGEBRAS

KLAUS THOMSEN

For a homogeneous C^*-algebra we identify the quotient of the automorphism group by the locally unitary automorphisms as a subgroup of the homeomorphisms of the spectrum. We sharpen a known criterion on the spectrum that ensures that all locally unitary automorphisms of the algebra are inner.

In [7] Phillips and Raeburn proved the existence of two short exact sequences

$$1 \to \mathrm{Inn}(A) \to \mathrm{Aut}_{C_0(X)}(A) \overset{\phi}{\to} H^2(X,\mathbb{Z})$$

and

$$1 \to \mathrm{Aut}_{C_0(X)}(A) \to \mathrm{Aut}(A) \overset{\psi}{\to} \mathrm{Hom}_\delta(X)$$

for a separable continuous trace C^*-algebra A with spectrum X. Under the additional assumption that A is stable, they concluded that both ϕ and ψ are surjective. In this note we investigate what happens when A is n-homogeneous, $n \in \mathbb{N}$. Since a homogeneous C^*-algebra has continuous trace the interesting question is what can be said about the ranges of ϕ and ψ.

In Theorem 1 we identify the range of ψ as the subgroup of homeomorphisms of X which fix the isomorphism class of the canonical fibre-bundle defined by A, thus obtaining a complete analogue of Phillips and Raeburn's result for stable algebras.

For ϕ it is a priori known that the range is contained in the

Received 18 June 1985.

torsion subgroup of $H^2(X,\mathbb{Z})$ when X is compact. We sharpen this to the effect that ϕ maps into elements whose order divides n, whether X is compact or not. Combining these results it follows that the quotient group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$ can be identified in most cases.

Let A be a C^*-algebra with primitive ideal spectrum X. Assume X is Hausdorff in the Jacobson topology.

For each $x \in X$, let $A(x)$ denote the quotient C^*-algebra A/x, and for each $a \in A$, let $a(x)$ denote the image of a in $A(x)$.

Consider the disjoint union $B = \bigcup_{x \in X} A(x)$ with the obvious projection

$$p : B \to X .$$

Sets of the form

$$\{b \in B \mid p(b) \in U , \|b - a(p(b))\| < \varepsilon\}$$

where $U \subseteq X$ is open, $a \in A$ and $\varepsilon > 0$, constitute a base for a topology on B such that the triple (p,B,X) becomes a C^*-bundle [3]. For each $a \in A$ we can define a cross-section $f_a : X \to B$ by

$$f_a(x) = a(x) , x \in X .$$

By results of Fell [2] the map $A \ni a \to f_a$ defines a $*$-isomorphism of A onto the cross-sections of (p,B,X) which vanish in norm at infinity.

In the case that A is a n-homogeneous C^*-algebra, it follows from a result of Fell [2], that there is an open covering $\{V_i\}_{i \in I}$ of X and homeomorphisms

$$\phi_i : V_i \times M_n(\mathbb{C}) \to p^{-1}(V_i)$$

such that the maps $\phi_{i,x} \equiv \phi_i(x,\cdot)$ define $*$-isomorphisms of $M_n(\mathbb{C})$ onto $p^{-1}(x) = A(x), x \in V_i$. So in this case (p,B,X) is a locally trivial fibre bundle with group $\mathrm{Aut}(M_n(\mathbb{C}))$ and fibre space $M_n(\mathbb{C})$. This is the canonical fibre bundle associated with A.

Locally trivial fibre bundles over X with group $\mathrm{Aut}(M_n(\mathbb{C}))$ and fibre space $M_n(\mathbb{C})$ are classified by the cohomology set

https://doi.org/10.1017/S0004972700002975 Published online by Cambridge University Press

$$H^1(X, \mathrm{Aut}(M_n(\mathbf{C}))_c)$$

where $\mathrm{Aut}(M_n(\mathbf{C}))_c$ is the sheaf of germs of continuous $\mathrm{Aut}(M_n(\mathbf{C}))$-valued functions on X (see [4], pp. 37-41).

If A is a n-homogeneous C^*-algebra the corresponding element $\eta(A) \in H^1(X, \mathrm{Aut}(M_n(\mathbf{C}))_c)$ is represented by $\{V_i , \alpha_{ij}\}_{i\in I}$, where the functions

$$\alpha_{ij} : V_i \cap V_j \to \mathrm{Aut}(M_n(\mathbf{C}))$$

are given by $\alpha_{ij}^x = \phi_{i,x}^{-1} \phi_{j,x}$, $x \in V_i \cap V_j$.

The group of homeomorphisms of X , $\mathrm{Hom}(X)$, acts on $H^1(X, \mathrm{Aut}(M_n(\mathbf{C}))_c)$ in the obvious way, that is if $\{U_i , \beta_{ij}\}_{i\in J}$ represents an element η of $H^1(X, \mathrm{Aut}(M_n(\mathbf{C}))_c)$ and $\psi \in \mathrm{Hom}(X)$, then the action of ψ takes η to the element $\psi^*(\eta)$ represented by $\{\psi(U_i) , \beta_{ij} \circ \psi^{-1}\}_{i\in J}$.

Given an element $\eta \in H^1(X, \mathrm{Aut}(M_n(\mathbf{C}))_c)$, we let $\mathrm{Hom}_\eta(X)$ denote the subgroup of $\mathrm{Hom}(X)$ consisting of homeomorphisms that fixes η , that is

$$\mathrm{Hom}_\eta(X) = \{\psi \in \mathrm{Hom}(X) \mid \psi^*(\eta) = \eta\} .$$

If $\alpha \in \mathrm{Aut}(A)$, we let $\rho(\alpha)$ denote the homeomorphism on the primitive ideal spectrum X induced by α . Then ρ defines a homomorphism

$$\rho : \mathrm{Aut}(A) \to \mathrm{Hom}(X) .$$

If we let $LU(A)$ denote the locally unitary automorphisms of A , [8], we have

THEOREM 1. *For a n-homogeneous C^*-algebra A with primitive ideal spectrum X , we have an exact sequence of groups:*

$$1 \to LU(A) \to \mathrm{Aut}(A) \overset{\rho}{\to} \mathrm{Hom}_{\eta(A)}(X) \to 1 .$$

Proof. That $\ker \rho = LU(A)$ follows from ([9], Theorem 3.4) since $\ker \rho$ consists of the π-inner automorphisms. So it suffices to identify $\mathrm{Hom}_{\eta(A)}(X)$ as the range of ρ .

To prove $\mathrm{ran}(\rho) \subseteq \mathrm{Hom}_{\eta(A)}(X)$, we must show that

$$\{\rho(\alpha)(V_i) , \alpha_{ij} \circ \rho(\alpha)^{-1}\}_{i\in I}$$

represents $\eta(A)$ in $H^1(X,\mathrm{Aut}(M_n(\mathbb{C}))_c)$, where α is an arbitrary automorphism of A .

With the notation introduced above, α induces a map $\tilde{\alpha} : B \to B$ defined on $A(x) = p^{-1}(x)$ by

$$\tilde{\alpha}(a(x)) = \alpha(a)(\rho(\alpha)(x)) , a \in A , x \in X .$$

Then $\tilde{\alpha}$ defines a homeomorphism on B such that $p \circ \tilde{\alpha} = \rho(\alpha) \circ p$.

Let $M_{(i,j)} = V_i \cap \rho(\alpha)(V_j)$, $i,j \in I$. Define $\beta_{(i,j)} : M_{(i,j)} \to \mathrm{Aut}(M_n(\mathbb{C}))$ by

$$\beta^x_{(i,j)} = \phi^{-1}_{i,x} \circ \tilde{\alpha} \circ \phi_{j,\rho(\alpha)^{-1}(x)} , x \in M_{(i,j)} .$$

Then $\beta_{(i,j)}$ is continuous and

$$\alpha_{ik} = \beta_{(i,j)}(\alpha_{jl} \circ \rho(\alpha)^{-1})\beta^{-1}_{(k,l)}$$

on $M_{(i,j)} \cap M_{(k,l)}$. By the definition of $H^1(X,\mathrm{Aut}(M_n(\mathbb{C}))_c)$ this gives the desired conclusion, that is $\mathrm{ran}(\rho) \subseteq \mathrm{Hom}_{\eta(A)}(X)$.

Now let $\psi \in \mathrm{Hom}_{\eta(A)}(X)$. We will construct a *-automorphism α_ψ of A such that α_ψ induces ψ on X , that is $\rho(\alpha_\psi) = \psi$.

Since $\psi \in \mathrm{Hom}_{\eta(A)}(X)$ there is a common refinement $\{M_p\}_{p\in J}$ of $\{V_i\}_{i\in I}$ and $\{\psi(V_i)\}_{i\in I}$, functions $\tau,\sigma : J \to I$ such that

$$M_p \subseteq V_{\tau(p)} , M_p \subseteq \psi(V_{\sigma(p)}) , p \in J$$

and continuous maps $\beta_p : M_p \to \mathrm{Aut}(M_n(\mathbb{C}))$ such that

(1) $$\alpha_{\tau(p)\tau(q)} = \beta_p(\alpha_{\sigma(p)\sigma(q)} \circ \psi^{-1})\beta^{-1}_q \text{ on } M_p \cap M_q .$$

Let $f : X \to B$ be a cross-section. Define a family $\{f_i\}_{i\in I}$ of continuous maps

$$f_i : V_i \to M_n(\mathbb{C})$$

by $f_i(x) = \phi^{-1}_{i,x}(f(x))$, $x \in V_i$.

https://doi.org/10.1017/S0004972700002975 Published online by Cambridge University Press

Then

(2) $$f_i = \alpha_{ij}(f_j) \quad \text{on} \quad V_i \cap V_j \ .$$

Now define continuous functions $g_p : M_p \to M_n(\mathbb{C})$ by

$$g_p = \beta_p(f_{\sigma(p)} \circ \psi^{-1}) \ .$$

Using (1) and (2) one sees that

(3) $$\alpha_{\tau(p)\tau(q)}(g_q) = g_p \quad \text{on} \quad M_p \cap M_q \ .$$

Next define maps $\tilde{g}_i : V_i \to M_n(\mathbb{C})$ by

$$\tilde{g}_i(x) = \alpha^x_{i\tau(p)}(g_p(x)) \ , \ x \in M_p \cap V_i \ .$$

It follows from the cocycle relation of the α_{ij}'s and (3), that the $\tilde{g}_i$'s are well-defined and that $\alpha_{ij}(\tilde{g}_j) = \tilde{g}_i$ on $V_i \cap V_j$.

Hence we can define a cross-section

$$\alpha_\psi(f) : X \to B$$

by

$$\alpha_\psi(f)(x) = \phi_{i,x}(\tilde{g}_i(x)) \ , \ x \in V_i \ .$$

In this way we have defined a map $\alpha_\psi : A \to A$ which is clearly an injective *-homomorphism. We need to prove that α_ψ is surjective.

So let $h : X \to B$ be a cross-section and construct continuous maps $h_i : V_i \to M_n(\mathbb{C})$ as above such that

(4) $$\alpha_{ij}(h_j) = h_i \quad \text{on} \quad V_i \cap V_j \ .$$

Then consider the functions $f_i : V_i \to M_n(\mathbb{C})$ defined by

$$f_i(x) = \alpha^x_{i\sigma(p)}(\beta_p^{\psi(x)^{-1}}(h_{\tau(p)}(\psi(x))) \ , \ x \in \psi^{-1}(M_p) \cap V_i \ .$$

Using the cocycle relation of the α_{ij}'s together with (1) and (4) one sees that the f_i's are well-defined and that

$$\alpha_{ij}(f_j) = f_i \quad \text{on} \quad V_i \cap V_j \ .$$

https://doi.org/10.1017/S0004972700002975 Published online by Cambridge University Press

If f is the cross-section constructed from the f_i's , then $\alpha_\psi(f) = h$. Hence α_ψ is surjective, that is α_ψ is an automorphism of A .

Since $\alpha_\psi(f)(x) = 0$ if and only if $f(\psi^{-1}(x)) = 0$, we conclude that the homeomorphism of X induced by α_ψ is ψ , that is $\rho(\alpha_\psi) = \psi$. □

The proof of the surjectivity of ρ in the preceding argument is based on [7], proof of Theorem 2.22. All we have done is to make explicit the identifications used by Phillips and Raeburn.

Next we turn to the inclusion $\mathrm{Inn}(A) \subseteq LU(A)$ of the inner automorphisms $\mathrm{Inn}(A)$ of A into the locally unitary automorphisms. According to [7], a theorem of Knus [6] implies that $\mathrm{Inn}(A) = LU(A)$ when X is compact and $H^2(X,\mathbb{Z})$ is torsion-free. We prove a slightly stronger result which applies in situations where X is non-compact and in some situations where $H^2(X,\mathbb{Z})$ does contain torsion elements.

Let G be a discrete group and denote by $\mathrm{Inn}(G,A)$ and $LU(G,A)$ the inner and the locally unitary actions of G on A , respectively, [8]. Even though G need not be abelian we can consider the dual group $\hat{G}$ of characters on G . $\hat{G}$ is compact in the topology of point-wise convergence and we can construct the cohomology set $H^1(X,\hat{G}_c)$, where $\hat{G}_c$ denotes the sheaf of germs of $\hat{G}$-valued continuous functions on X . Since $\hat{G}$ is abelian $H^1(X,\hat{G}_c)$ is in fact an abelian group and we can consider this group as a pointed set with the trivial element as the base point. Note that also $\mathrm{Inn}(G,A)$ and $LU(G,A)$ are pointed sets with the trivial action as base point in both cases.

THEOREM 2. *Let A be an n-homogeneous C^*-algebra with primitive ideal spectrum X and G a discrete group. Then there is an exact sequence of pointed sets*

$$0 \to \mathrm{Inn}(G,A) \to LU(G,A) \overset{\mu}{\to} H^1(X,\hat{G}_c)$$

such that μ maps into the elements of $H^1(X,\hat{G}_c)$ whose order divides n .

Proof. Let $\alpha : G \to \mathrm{Aut}(A)$ be a locally unitary action. This means that, if $M(A)$ denotes the multiplier algebra of A , then we can choose

https://doi.org/10.1017/S0004972700002975 Published online by Cambridge University Press

an open cover $\{N_i\}_{i\in I}$ of X and maps

$$U_i : G \to M(A)$$

such that $\alpha_g(a)(x) = U_i(g)(x)a(x)U_i(g)(x)^*$, $a \in A$, $x \in N_i$, $g \in G$ and $G \ni g \mapsto U_i(g)(x)$ is a unitary representation of G for each $x \in N_i$. Note that we have tacitly extended the quotient map $A \to A(x)$ to the multiplier algebra $M(A)$. However, by shrinking the N_i's and multiplying the U_i's with suitable central elements of A , we can assume that $U_i(g) \in A$ for all $i \in I$, $g \in G$.

Define maps $\chi^g_{ij} : N_i \cap N_j \to \mathbb{T}$ by

(5) $$\chi^g_{ij}(x)1_x = U_i(g)(x)^*U_j(g)(x) \ , \ x \in N_i \cap N_j \ , \ g \in G \ .$$

Here 1_x denotes the unit in $A(x) \simeq M_n(\mathbb{C})$. Then $G \ni g \to \chi^g_{ij}(x)$ defines a character $\phi_{ij}(x)$, and, using the traces for instance, one sees that the corresponding map

$$\phi_{ij} : N_i \cap N_j \to \hat{G}$$

is continuous. Since $\phi_{ij}\phi_{jk} = \phi_{ik}$ on $N_i \cap N_j \cap N_k$ we conclude that $\{N_i,\phi_{ij}\}_{i\in I}$ defines an element $\mu(\alpha) \in H^1(H,\hat{G}_c)$.

It is not hard to see that the element $\mu(\alpha)$ depends only on the locally unitary action α and not on any of the choices we have made.

Let Det_x denote the determinant map on $A(x) \simeq M_n(\mathbb{C})$. Then (5) yields that

(6) $$(\chi^g_{ij}(x))^n = \overline{\mathrm{Det}_x(U_i(g)(x))}\ \mathrm{Det}_x(U_j(g)(x))$$

for all $g \in G$, $x \in N_i \cap N_j$.

Since Det_x is continuous, it follows that we have continuous maps $\phi_i : N_i \to \hat{G}$ given by

$$\phi_i(x)(g) = \mathrm{Det}_x(U_i(g)(x)) \ , \ x \in N_i \ , \ g \in G \ .$$

Then (6) tells us that $n\eta(A) = 0$ in $H^1(X,\hat{G}_c)$.

To complete the proof it suffices to show that $\mu(\alpha) = 0$ implies $\alpha \in \mathrm{Inn}(G,A)$.

But if $\mu(\alpha) = 0$, we can shrink the N_i's and assume that $\phi_{ij}(x) = \chi_j(x)^{-1}\chi_i(x)$, $x \in N_i \cap N_j$, where $\chi_i : N_i \to \hat{G}$ are continuous maps, $i \in I$.

Consider A as consisting of cross-sections of the canonical bundle associated with A .

Then define multipliers U_g of A by

$$(U_g a)(x) = \chi_i(g)(x) U_i(g)(x) a(x)$$

and

$$(aU_g)(x) = \chi_i(g)(x) a(x) U_i(g)(x) \quad , \quad g \in G \, , \; x \in N_i \, , \; a \in A \, .$$

Since $\chi_i(g)U_i(g) = \chi_j(g)U_j(g)$ over $N_i \cap N_j$, the U_g's are well-defined.

Then $G \ni g \mapsto U_g$ is a unitary representation of G as multipliers of A such that $\alpha_g = A\,d\,U_g$, $g \in G$. Hence we conclude that $\alpha \in \mathrm{Inn}(G,A)$. □

COROLLARY 3. *Let A be a n-homogeneous C^*-algebra with paracompact spectrum X. Then there is an exact sequence of groups*

$$1 \to \mathrm{Inn}(A) \to LU(A) \overset{\mu}{\to} H^2(X,\mathbb{Z})$$

such that μ maps into the elements of $H^2(X,\mathbb{Z})$ whose order divides n .

Proof. Apply Theorem 2 with $G = \mathbb{Z}$ and use that $H^2(X,\mathbb{Z}) \simeq H^1(X,\mathbb{T}_c)$. The only thing to check is that the map μ of Theorem 2 induces a group homomorphism $\mu : LU(A) \to H^2(X,\mathbb{Z})$ in this case. □

COROLLARY 4. *If A is a n-homogeneous C^*-algebra with paracompact spectrum X such that $H^2(X,\mathbb{Z})$ contains no nontrivial element of order dividing n , then*

$$\mathrm{Aut}(A)/\mathrm{Inn}(A) \simeq \mathrm{Hom}_{\eta(A)}(X) \, .$$

Proof. Combine Theorem 1 and Corollary 3. □

COROLLARY 5. *If X is a locally compact paracompact space and $n \in \mathbb{N}$ an integer such that no nontrivial element of $H^2(X,\mathbb{Z})$ has an order dividing n, then every automorphism α of $C_0(X,M_n(\mathbb{C}))$ is given by a pair (u,ψ), where u is a unitary in $C_b(X,M_n(\mathbb{C}))$ and ψ is a homeomorphism of X, that is*

$$\alpha(f)(x) = u(x)f(\psi(x))u(x)^* , \quad f \in C_0(X,M_n(\mathbb{C})) , \quad x \in X .$$

Proof. Combine Theorem 1 and 2 and use that the sequence of Theorem 1 splits in this case, together with the fact that $C_b(X,M_n(\mathbb{C}))$ is the multiplier algebra of $C_0(X,M_n(\mathbb{C}))$. □

In [5], Example (d), Kadison and Ringrose gave examples of π-inner (that is locally unitary) automorphisms of $C(PU(n), M_n(\mathbb{C}))$ which are not inner. $PU(n)$ is the projective unitary group $U(n)/\mathbb{T} \simeq \mathrm{Aut}(M_n(\mathbb{C}))$ and $H^2(PU(n), \mathbb{Z}) \simeq \mathbb{Z}_n$ for all $n \in \mathbb{N}$ [1]. It follows from Corollary 3 that all locally unitary automorphisms of $C(PU(n), M_m(\mathbb{C}))$ are inner whenever n and m are mutually prime.

References

[1] P.F. Baum and W. Browder, "The cohomology of quotients of classical groups", *Topology*, **3** (1965), 305-336.

[2] J.M.G. Fell, "The structure of algebras of operator fields", *Acta Math.*, **106** (1961), 233-280.

[3] J.M.G. Fell, "An extension of Mackey's method to Banach *-algebraic bundles", *Mem. Amer. Math. Soc.*, **90** (1969).

[4] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, (Springer Verlag, Berlin-Heidelberg-N.Y., 1966).

[5] R.V. Kadison and J.R. Ringrose, "Derivations and automorphisms of operator algebras", *Comm. Math. Phys.*, **4** (1967), 32-63.

[6] M.A. Knus, "Algèbres d'Azumaya et modules projectifs", *Comment Math. Helv.*, **45** (1970), 372-383.

[7] J. Phillips and Iain Raeburn, "Automorphisms of C^*-algebras and Second Čech Cohomology", *Indiana Univ. Math. J.*, 29 (1980), 799-822.

[8] J. Phillips and Iain Raeburn, "Crossed products by locally unitary automorphism groups and principal bundles", *J. Operator Theory*, 11 (1984), 215-241.

[9] M.J. Russell, "Automorphisms and derivations of continuous trace C^*-algebras", *J. London Math. Soc.*, (2)22 (1980), 139-145.

Matematisk Institut,
Aarhus Universitet,
Ny Munkegade,
DK - Aarhus C,
Denmark.