# HEREDITARY LOCAL RINGS

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To the memory of TADASI NAKAYAMA

## 1. Introduction

Many questions about free ideal rings ( = firs, cf. [5] and § 2 below) which at present seem difficult become much easier when one restricts attention to local rings. One is then dealing with hereditary local rings, and any such ring is in fact a fir (§ 2). Our object thus is to describe hereditary local rings. The results on firs in [5] show that such a ring must be a unique factorization domain; in § 3 we prove that it must also be rigid (cf. the definition in [3] and § 3 below). More precisely, for a semifir[1] $R$ with prime factorization rigidity is necessary and sufficient for $R$ to be a local ring.

§ 4 gives an example of a right fir (in fact a principal right ideal domain) with prime factorization, which is not left hereditary and hence is not a left fir. Since the example is of a local ring, this provides an example of a rigid unique factorization domain which is a semifir but not a fir.

The final section concerns the centre of a hereditary local ring. If this is not a field, then both the ring and its centre are discrete valuation rings. This improves a result of Northcott [8] who showed that the centre, if not a field, must be a 1-dimensional regular local ring. The actual result proved in § 5 is rather more general (apart from the stronger conclusion) in that the hypothesis is weaker: we do not require the existence of a central non-unit ($\neq 0$) but merely a 'large' non-unit, and in an integral domain every central non-unit $\neq 0$ is large.

## 2. Hereditary and semihereditary local rings

Throughout, all rings are associative with 1, and all modules are unital. We recall that a ring $R$ is said to be *p-trivial* ( = projective-trivial) if there is

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[1] Semifirs, called 'local firs' in [5] have been renamed here (by analogy with 'semi-hereditary') to avoid confusion with local rings.



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a projective $R$-module $P$ such that each finitely generated projective $R$-module has the form $P^n$ for a unique integer $n$. In particular, such a ring has invariant basis number (i.e. any two bases of a free $R$-module have the same number of elements). An integral domain (not necessarily commutative) with invariant basis number, in which all (finitely generated) right ideals are free, is called a *right (semi) fir;*[1] *left (semi) firs* are defined similarly. In [5] it was shown that every right semifir is a left semifir, so we may simply speak of semifirs. By a *local ring* we understand as usual a ring in which the sum of two non-units is a non-unit.

Since a local ring has a skew field as homomorphic image, it certainly has invariant basis number. Further, all projectives are free[2] and this shows that a local ring must be $p$-trivial. Now it was proved in [6] (Theorem 2) that any right (or left) semihereditary $p$-trivial ring $R$ is a total matrix ring over a semifir, $R \cong T_n$ say. If $n > 1$, the equation $1 = e_{11} + (1 - e_{11})$ in $R$ contradicts the fact that $R$ is local; hence $n = 1$ and $R \cong T$ is itself a semifir. Similarly, if $R$ is right hereditary (and local) it is a right fir ([6], Theorem 1) so we obtain

THEOREM 1. *A right or left semihereditary local ring is a semifir; a right hereditary local ring is a right fir.*

In particular this shows that a right hereditary local ring is an integral domain. We recall from [5] that a right fir satisfies the ascending chain condition for principal right ideals and hence that every non-unit has a left factor which is a prime ([5], proof of Theorem 2.8). Moreover, by Theorem 2.8 of [5], every left and right hereditary local ring is a unique factorization domain, (UFD) in the sense of [4]. We return to this result in the next section.

## 3. Rigid unique factorization domains

We begin with an obvious general remark.

LEMMA 1. *Let $R$ be a semihereditary local ring. Given $a, b \in R$ such that $aR \cap bR \neq 0$, then $bR \subseteq aR$ or $aR \subseteq bR$.*

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[2] This is only needed here for finitely generated projectives. The general result (projectives over a local ring are free) was first explicitly proved by Kaplansky [7], but it may be worth noting that this is an immediate consequence of Azumaya's form of the Krull-Schmidt theorem [1].

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*Proof.* By Theorem 1, $R$ is a semifir and hence $aR+bR=dR$, for some $d\in R$. This means that $a=da_0$, $b=db_0$ and $a_0u-b_0v=1$. Since $R$ is a local ring, $a_0$ or $b_0$ must be a unit, and accordingly, $bR\subseteq aR$ or $aR\subseteq bR$.

If $R$ is a left and right fir, then $R$ is a UFD, by Theorem 2.8 of [5]. More generally, this holds for a semifir in which each non-zero element not a unit can be written as a product of primes. The next theorem gives a criterion for such a ring to be local. We recall the following definition:

A (not necessarily commutative) UFD $R$ is said to be *rigid* if, for any two prime factorizations of an element $a$ of $R$:

$$a=p_1p_2\cdots p_r=q_1q_2\cdots q_r \tag{1}$$

there exist units $u_0, u_1, \ldots, u_r$ $(u_0=u_r=1)$ such that

$$q_i=u_{i-1}^{-1}p_iu_i \quad (i=12\cdots r).$$

THEOREM 2. *Let $R$ be a semifir with prime factorization* (*and hence a UFD*). *Then $R$ is rigid if and only if it is a local ring.*

By the preceding remarks the theorem applies in particular to rings which are left and right firs.

*Proof.* Let $R$ be a local ring and consider any two prime factorizations (1) of an element $a$. Then $p_1R\cap q_1R\neq 0$; since $p_1$ is a non-unit and $q_1$ a prime, we deduce from Lemma 1 that $p_1R\subseteq q_1R$, and by symmetry, $q_1R\subseteq p_1R$. Hence $p_1R=q_1R$, i.e. $q_1=p_1u_1$ for some unit $u_1\in R$. Cancelling $p_1$ in (1) we obtain

$$p_2\cdots p_r=u_1q_2\cdots q_r$$

and induction on $r$ shows $R$ to be rigid.

Conversely, let $R$ be rigid; we shall show that $R$ is a local ring by showing that the sum of a non-unit is a unit. Let $a$ be a non-unit and $u$ a unit; since $R$ is a UFD we can write $a=pa_1$ where $p$ is a prime. Now we have the identity

$$(1+pa_1)p=p(1+a_1p).$$

If $1+pa_1$ were a non-unit, then by rigidity $1+pa_1=pb$ for some $b\in R$, whence $1=p\,(b-a_1)$, which contradicts the fact that $p$ is a non-unit. Hence $1+a$ is a unit, as we wished to show.

Combining this result with Theorem 1, we obtain the

COROLLARY. *A (left or right) semihereditary local ring with prime factorization (and in particular, a left and right hereditary local ring) is a rigid UFD.*

## 4. A principal right ideal domain with prime factorization which is not left hereditary

Let $R$ be a UFD with invariant basis number, which is not a field, and consider $R^N$, the direct product of a countable number of copies of $R$. We propose to show that $R^N$, as right $R$-module, cannot be projective. Put $A^{(i)} \cong R$, $A_n = \prod_{n+1}^{\infty} A^{(i)}$, then $A_n \cong R^N$ for $n = 1, 2 \cdots$. If $R^N$ were projective, we would have an equation

$$A_1 \oplus B = \sum_I \oplus C_i \qquad (C_i \cong R) \tag{2}$$

where $i$ runs over some index set $I$. Denote by $f_i$ the projection onto $C_i$ and take $a \in R$, where $a$ is not zero or a unit. Then $\bigcap_n Ra^n = 0$ and hence $\bigcap_n C_i a^n = 0$ for each $i \in I$. It follows by a theorem of Chase [2] that there exist integers $m$, $n$ such that

$f_i(A_n a^m) = 0$ for all but a finite number of $i \in I$.

Since $f_i(A_n a^m) = f_i(A_n)a^m$ and $a^m$ annihilates no element of $C_i$, we have

$$f_i(A_n) = 0 \tag{3}$$

for all but a finite number of $i \in I$. Suppose that (3) holds except when $i = i_1, \ldots, i_k$ and write $C' = \sum_1^k C_{i_\nu}$, $C'' = \sum_{j \neq i_\nu} C_j$ then (2) may be rewritten as

$$A^{(1)} \oplus \cdots \oplus A^{(n-1)} \oplus A_n \oplus B = C' \oplus C'';$$

now (3) shows that $A_n \subseteq C'$ and it follows that $A_n$ is complemented in $C'$, i.e. there exists $D$ such that

$$A_n \oplus D = C'. \tag{4}$$

Since $A_n \cong R^N$, we have $R \oplus A_n \cong A_n$. Adding $R$ to (4) and remembering that $C' \cong R^k$, we obtain

$$R^{k+1} \cong R^k,$$

which is impossible in a ring with invariant basis number. Thus we have proved

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THEOREM 3. *If $R$ is any UFD with invariant basis number which is not a field, then $R^N$ is not projective.*

COROLLARY. *If $R$ is a semifir with prime factorization then $R^N$ is not projective.*

Let $k$ be any field with an endomorphism $\sigma$ which is not surjective[3], and consider the ring $R=k[[x;\ \sigma]]$ of all skew power series, consisting of all infinite series

$$\sum x^n\alpha_n \qquad (\alpha_n\in k),$$

with multiplication according to the rule

$$\alpha x = x\alpha^\sigma.$$

The mapping $\sigma$ can be extended to an endomorphism of $R$ by defining $x^\sigma = x$.

It is clear that $R$ is a local ring with maximal ideal $xR$ and is a principal right ideal domain (in fact $R$ is a discrete valuation ring in the sense defined below, § 5); moreover, $R$ has prime factorization and so is a rigid UFD. All these facts are easily verified; we shall now show that $R$ is not a left fir, thus in $R$ we have an example of a one-sided fir.

To show that $R$ is not a left fir we must find a left ideal which is not free; now by Theorem 3, Cor., $R^N$ is not projective, so it is enough to find a left ideal isomorphic to $R^N$. To obtain such an ideal we need only take a sequence of elements $(u_n)$ say, tending to zero, which is left $R$-free. For then $\sum a_n u_n \in R$ for all $(a_n)\in R^N$ and by freeness, the mapping

$$(a_n)\to\sum a_n u_n$$

is an isomorphism of left $R$-modules. Let $\rho$ belong to $k$ but not to $k^\sigma$. Then we assert that the sequence $(x\rho x^n)$ has the required properties. Clearly it tends to zero, and if

$$\sum c_i x\rho x^i = 0,$$

then by cancelling a power of $x$, if possible, we may assume that $c_0 \neq 0$. Then

$$\sum x^{i+1} c_i^{\sigma^{i+1}} \rho^{\sigma^i} = 0.$$

Hence $xc_0^\sigma\rho \in R^\sigma$, whereas not all coefficients of $xc_0^\sigma\rho$ lie in $k^\sigma$. This is a con-

[3] It must be injective, since $1^\sigma=1$ by definition.

tradiction, and it shows that the elements $x\rho x^n$ are left $R$-free, as asserted.

## 5. The centre of a hereditary local ring

In [4] it was shown that a rigid UFD which is commutative is necessarily a discrete valuation ring. In general we define a *discrete valuation ring* (DVR) as an integral domain $R$ (not necessarily commutative) with a prime $p$ such that every non-zero element of $R$ is of the form

$$p^r u \quad (r \geq 0,\ u \text{ a unit}).$$

When $R$ is commutative, this reduces to the usual definition. A general DVR is clearly a rigid UFD and it may be verified that a rigid UFD is a DVR if and only if any two primes are right associated.

An element $a$ of an arbitrary ring $R$ is said to be *large* if $aR$ is a large right ideal in $R$, i.e. if

$$aR \cap \mathfrak{c} \neq 0 \text{ for all non-zero right ideals } \mathfrak{c}.$$

It is easily verified that in any integral domain the set $S$ of large elements is closed under multiplication and left division. Moreover, in a semifir (or more generally in any weak Bezout ring) the set $S$ of large elements satisfies the generalized Ore condition, hence in this case the ring $R$ admits a ring of right fractions with respect to $S$, $RS^{-1}$ say, and this ring is again a semifir. We shall omit the (easy) proofs as we do not need these results. In fact, to adjoin inverses of large elements would be a retrograde step, as the next theorem shows.

THEOREM 4. *Let $R$ be a (left or right) semihereditary local ring with prime factorization. Then $R$ is a discrete valuation ring if and only if it contains a large non-unit.*

*Proof.* Let $R$ be a left or right semihereditary local ring with prime factorization. Then by Theorem 2, $R$ is a rigid UFD. Now assume that $R$ contains a large non-unit $c$, say. Given any prime $p \in R$, we have $cR \cap pR \neq 0$ and hence $c = pc'$. Write

$$c = p^k u, \tag{5}$$

where $k \geq 1$ is chosen maximal (clearly $k$ is bounded by the number of factors in a prime decomposition of $c$). Then $u$ must be a unit, for by (5), $p$ is large

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and so $pR \cap uR \neq 0$; if $u$ were not a unit, this would mean (by Lemma 1) that $u = pu'$, which, inserted in (5), contradicts the maximality of $k$. The argument applies to all primes of $R$; it shows that all primes are large and hence are right associated. Therefore $R$ is a discrete valuation ring. Conversely, in a DVR, the unique prime is a large non-unit.

The result applies in particular to non-zero centre elements (which in an integral domain are always large), to show that a left and right hereditary local ring with centre not a field is a DVR. But now it is no longer necessary to assume that the ring is a UFD. The precise result is

THEOREM 5. *Let R be a right hereditary local ring whose centre is not a field. Then R is a discrete valuation ring, and so is its centre.*

*Proof.* Let $c$ be a central element which is not zero or a unit. As before, we can show that for any prime $p$, we have $c = pc'$. Suppose that for every $n$ there exists $u_n \in R$ such that

(6) $$c = p^n u_n.$$

Since $c$ is central, $p^n u_n = u_n p^n$ and we have the strictly ascending sequence of right ideals

$$cR = u_0 R \subset u_1 R \subset \cdots$$

But by Theorem 1, $R$ is a right fir, and this satisfies the ascending chain condition for principal right ideals. Hence (6) cannot hold for all $n$, and this means that we can again choose $k$ in (5) to be maximal. Then $u$ must be a unit, and as before it follows that all primes are right associated. To complete the proof we need only show that every non-unit $\neq 0$ has a prime factorization. Let $a \in R$ and suppose that for each $n$ there exists $a_n$ such that

$$a = p^n a_n.$$

Taking $n = kr$ where $k$ is as in (5) and $r$ is arbitrary, we have

$$a = p^{kr} a_{kr} = (cu^{-1})^r a_{kr} = c^r u^{-r} a_{kr} = u^{-r} a_{kr} c^r,$$

hence

$$aR \subset u^{-1} a_k R \subset u^{-2} a_{2k} R \subset \cdots$$

which again gives a contradiction. This shows $R$ to be a DVR. If we now choose $c$, in the centre of $R$, so that $k$ in (5) takes its least positive value, then it is easily seen that every centre element has the form $c^n v$, where $v$ is a unit, again in the centre. Hence the centre of $R$ is also a DVR, and the proof is complete.

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