ON JOINT ESSENTIAL SPECTRA OF DOUBLY COMMUTING n-TUPLES OF p-HYPONORMAL OPERATORS

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Abstract. Let A be an operator on a Hillbert space with polar decomposition $A = U|A|$, let $\hat{A} = |A|^{1/2}U|A|^{1/2}$ and let $\hat{A} = V|\hat{A}|$ be the polar decomposition of $\hat{A}$. Write $\tilde{A}$ for the operator $\tilde{A} = |\hat{A}|^{1/2}V|\hat{A}|^{1/2}$. If $\mathbb{A} = (A_1, \cdots, A_n)$ is a doubly commuting n-tuple of p-hyponormal operators on a Hillbert space with equal defect and nullity, then $\tilde{\mathbb{A}} = \left(\tilde{A}_1, \cdots, \tilde{A}_n\right)$ is a doubly commuting n-tuple of hyponormal operators. In this paper we show that

$$\sigma_*(\mathbb{A}) = \sigma_*\left(\tilde{\mathbb{A}}\right),$$

where σ_* denotes σ_{Te} (Taylor essential spectrum), σ_{Tw} (Taylor-Weyl spectrum) and σ_{Tb} (Taylor-Browder spectrum), respectively.

1. Introduction. Let $B(\mathcal{H})$ denote the set of bounded linear operators on a complex Hillbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called *p-hyponormal* if $(A^*A)^p - (AA^*)^p \geq 0$ for $0 < p < 1$. Let $HU(p)$ denote the set of p-hyponormal operators with equal defect and nullity. Hence for $A \in HU(p)$, we may assume that the operator U in a polar decomposition $A = U|A|$ is unitary. Aluthge[**1**] introduced an operator $\hat{A}$ defined as follows. Let A have the polar decomposition $A = U|A|$ and let $\hat{A} = |A|^{1/2}U|A|^{1/2}$. Let $\hat{A}$ have the polar decomposition $\hat{A} = V|\hat{A}|$. The operator $\tilde{A}$ is then defined by $\tilde{A} = |\hat{A}|^{1/2}V|\hat{A}|^{1/2}$. If A is p-hyponormal, $0 < p < 1/2$, then $\hat{A}$ is $(p + 1/2)$-hyponormal and $\tilde{A}$ is hyponormal [**1**, Corollary 3]. These relationships of A and $\tilde{A}$ (or $\hat{A}$) have been useful tools to study p-hyponormal operators [**1**],[**2**],[**4**],[**9**],[**10**]. The Löwner inequality [**14**, p.15] implies that if A is p-hyponormal, then A is q-hyponormal for $0 < q \leq p$. Thus we may assume henceforth, without loss of generality, that $0 < p < 1/2$. For $A \in HU(p)$, Duggal [**9**] proved the following result.

Proposition 1.1. *If A is an operator in $HU(p)$, then*

$$\sigma(A) = \sigma\left(\tilde{A}\right), \sigma_p(A) = \sigma_p\left(\tilde{A}\right), \sigma_\pi(A) = \sigma_\pi\left(\tilde{A}\right), \sigma_e(A) = \sigma_e\left(\tilde{A}\right),$$

and

$$\sigma_w(A) = \sigma_w\left(\tilde{A}\right),$$

where $\sigma(A), \sigma_p(A), \sigma_\pi(A), \sigma_e(A)$, and $\sigma_w(A)$ denote the spectrum, the point spectrum, the approximate point spectrum, the essential spectrum, and the Weyl spectrum of A, respectively.

An n-tuple $\mathbb{A} = (A_1, \cdots, A_n)$ of operators is said to be *doubly commuting* if $A_iA_j = A_jA_i$ and $A_i^*A_j = A_jA_i^*$, for every $i \neq j$. The spectral properties of doubly commuting n-tuples of operators in $HU(p)$ have been first considered by Muneo Chō [**4**]. Duggal [**10**] partially extended Proposition 1.1 to doubly commuting n-tuples of operators in $HU(p)$. The aim of this paper is to continue and complete an extension of Proposition 1.1: we show equalities of the Taylor essential spectra, the Taylor-Weyl spectra, and the Taylor-Browder spectra of doubly commuting n-tuples $\mathbb{A}$ and $\tilde{\mathbb{A}}$, respectively. It is my pleasure to thank Professor Woo Young Lee, Professor Muneo Chō and Professor Bhaggy P. Duggal for helpful suggestions.

2. Taylor essential spectra of n-tuples of operators in $HU(p)$. For a commuting n-tuple of operators in $B(\mathcal{H})$ we shall denote the *joint point spectrum*, the *joint left spectrum*, the *joint right spectrum*, the *joint Harte spectrum*, and the *Taylor spectrum* by $\sigma_p(\mathbb{A})$, $\sigma^\ell(\mathbb{A})$, $\sigma^r(\mathbb{A})$, $\sigma_H(\mathbb{A})$, and $\sigma_T(\mathbb{A})$, respectively (see [**6**] or [**11**] for the definitions of these joint spectra). Let $K(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$ and $C(\mathcal{H}) = B(\mathcal{H})/K(\mathcal{H})$ be the Calkin algebra with the canonical map $\pi : B(\mathcal{H}) \to C(\mathcal{H})$. In a similar fashion we define joint essential spectra using $\pi(\mathbb{A}) = (\pi(A_1), \cdots, \pi(A_n))$ instead of $\mathbb{A}$. The *Taylor essential spectrum*, denoted by $\sigma_{Te}(\mathbb{A})$, of $\mathbb{A}$ is defined by

$$\sigma_{Te}(\mathbb{A}) = \sigma_T(\pi(\mathbb{A})),$$

and the *joint left essential spectrum*, denoted by $\sigma_e^\ell(\mathbb{A})$, of $\mathbb{A}$ is defined by

$$\sigma_e^\ell(\mathbb{A}) = \sigma^\ell(\pi(\mathbb{A})).$$

Similarly, the *joint right essential spectrum*, denoted by $\sigma_e^r(\mathbb{A})$, of $\mathbb{A}$ is defined by $\sigma_e^r(\mathbb{A}) = \sigma^r(\pi(\mathbb{A}))$. Duggal [**10**] showed that Harte essential spectra of $\mathbb{A}$ and $\tilde{\mathbb{A}}$ coincide. We can prove more.

Proposition 2.1. *If $\mathbb{A} = (A_1, \cdots, A_n)$ is a doubly commuting n-tuple of operators in $HU(p)$, then*

$$\sigma_{Te}(\mathbb{A}) = \sigma_e^r(\mathbb{A}) = \sigma_e^r\left(\tilde{\mathbb{A}}\right) = \sigma_{Te}\left(\tilde{\mathbb{A}}\right). \tag{2.1}$$

Proof. For the first equality of (2.1), let ρ be a faithful representation of the C^*-algebra generated by $\pi(A_1), \cdots, \pi(A_n)$ and $\pi(I)$ on a Hillbert space $\mathcal{H}_\rho$. By [**4**, Theorem 7], $\sigma_T(\rho(\pi(\mathbb{A}))) = \sigma^r(\rho(\pi(\mathbb{A})))$. Since by the spectral permanence of σ_T and σ^r (see [**6**, p. 38]

$$\sigma_T(\rho(\pi(\mathbb{A}))) = \sigma_T(\pi(\mathbb{A})) = \sigma_{Te}(\mathbb{A})$$

and

$$\sigma^r(\rho(\pi(\mathbb{A}))) = \sigma^r(\pi(\mathbb{A})) = \sigma_e^r(\mathbb{A}),$$

we have that $\sigma_{Te}(\mathbb{A})$. The second equality of (2.1) was essentially shown in Theorem 3 in [**10**]. The third equality of (2.1) is already a well known fact ([**3**] or [**5**]) since $\tilde{\mathbb{A}}$ is a doubly commuting n-tuple of hyponormal operators.

When $\mathbb{A}$ is a commuting n-tuple of operators in $B(\mathcal{H})$, recall ([**5**],[**6**],[**11**]) that if $\lambda \in \sigma_T(\mathbb{A}) \backslash \sigma_{Te}(\mathbb{A})$, then the *index* of $\mathbb{A} - \lambda$, denoted ind $(\mathbb{A} - \lambda)$, is defined by the *Euler* *characteristic* of the Koszul complex of $\mathbb{A}-\lambda$ and the *Taylor-Weyl spectrum*, denoted by $\sigma_{Tw}(\mathbb{A})$, of $\mathbb{A}$ is defined by

$$\sigma_{Tw}(\mathbb{A}) = \sigma_{Te}(\mathbb{A}) \cup \left\{\lambda \in \mathbb{C}^n : \text{ ind } (\mathbb{A}-\lambda) \neq 0\right\}. \tag{2.2}$$

Also recall ([5, Corollary 7.3]) that if $\mathbb{A} = (A_1, \cdots, A_n)$ is a doubly commuting n-tuple of operators in $B(\mathcal{H})$, then for each $\lambda(\lambda_1, \cdots, \lambda_n) \in \sigma_T(\mathbb{A}) \backslash \sigma_{Te}(\mathbb{A})$,

$$\text{ind } (\mathbb{A}-\lambda) = \sum_k (-1)^{k+1} \sum_{f \in I_k} \dim \left(\bigcap_{i=1}^n \ker{}^f (A_i - \lambda_i) \right), \tag{2.3}$$

where $I_k = \{f : \{1, \cdots, n\} \to \{0, 1\} | f(i) = 0$ exactly k times$\}$ and ${}^f(A_i - \lambda_i)$ is meant to be $(A_i-\lambda_i)^*(A_i-\lambda_i)$ and $(A_i-\lambda_i)(A_i-\lambda_i)^*$ according to $f(i) = 0$ and 1, respectively.

THEOREM 2.2. *If $\mathbb{A} = (A_1, \cdots, A_n)$ is a doubly commuting n-tuple of operators in $HU(p)$, then for each $\lambda \in \sigma_T(\mathbb{A}) \backslash \sigma_{Te}(\mathbb{A})$, we have*

$$\text{ind } (\mathbb{A}-\lambda) = \text{ ind } \left(\tilde{\mathbb{A}}-\lambda\right) \tag{2.4}$$

and in particular

$$\sigma_{Tw}(\mathbb{A}) = \sigma_{Tw}\left(\tilde{\mathbb{A}}\right). \tag{2.5}$$

Proof. For (2.4), we shall show that for each $f \in I_k (k = 0, 1, 2, \cdots, n)$, we have

$$\dim \left(\bigcap_{i=1}^n \ker{}^f (A_i - \lambda_i) \right) = \dim \left(\bigcap_{i=1}^n \ker{}^f \left(\tilde{A}_i - \lambda_i\right) \right)$$

for each $\lambda \in \sigma_T(\mathbb{A}) \backslash \sigma_{Te}(\mathbb{A})$.

Fix k and put $I = \{1, \cdots, k\}$ and $J = \{k+1, \cdots, n\}$. Write $I' = \left\{t \in I : \lambda_t \neq 0\right\}$ and $J' = \left\{s \in J : \lambda_s \neq 0\right\}$. Now define

$$F = \prod_{t \in I'} X_t \cdot \prod_{s \in J'} Y_s,$$

where $X_t = |\hat{A}_t|^{1/2} |A_t|^{1/2}$ and $Y_s = |\hat{A}_s|^{1/2} V_s^* |A_s|^{1/2} U_s^*$. Then a straightforward calculation by using the argument of Duggal[10, Lemma 1] shows that

(*i*) $F = \prod_{t \in I'} X_t \cdot \prod_{s \in J'} Y_s = \prod_{s \in J'} Y_s \cdot \prod_{t \in I'} X_t$,

(*ii*) $\tilde{A}_t F = F A_t$ for each $t \in I$,

(*iii*) $\tilde{A}_s^* F = F A_s^*$ for each $s \in J$,

which implies that

$${}^f\tilde{A}_i \cdot F = F \cdot {}^f A_i \text{ and } {}^f\tilde{A}_i^* \cdot F = F \cdot {}^f A_i^* \text{ for } i = 1, \cdots, n.$$

Put

$$Z := \bigcap_{i=1}^{n} \ker{}^f(A_i - \lambda_i) \text{ and } \tilde{Z} := \bigcap_{i=1}^{n} \ker{}^f\left(\tilde{A}_i - \lambda_i\right).$$

We now claim

$$\dim Z \leq \dim \tilde{Z}.$$

To see this suppose $x(\neq 0) \in Z$. Then for each $i \in (I\backslash I') \cup (J\backslash J')$

$$^fA_i x = 0 \Rightarrow |^fA_i|^{1/2} x = 0 \Rightarrow \widehat{^fA_i} x = 0 \Rightarrow \widetilde{^fA_i} x = 0. \tag{2.6}$$

Put $y = Fx$. Then $y \neq 0$. Thus we have that for each $i \in I' \cup J'$,

$$^f\left(\tilde{A}_i - \lambda_i\right) \cdot Fx = F \cdot {}^f(A_i - \lambda_i)x. \tag{2.7}$$

But since $^f\tilde{A}_i$ commutes with F, it follows that

$$^f\tilde{A}_i \cdot Fx = F \cdot {}^f\tilde{A}_i x = 0 \text{ for each } i \in (I\backslash I') \cup (J\backslash J'). \tag{2.8}$$

By (2.7) and (2.8) we have that $Fx \in \tilde{Z}$. On the other hand, for a set of linearly independent vectors $\{x_1, \cdots, x_n\}$ in Z, assume

$$0 = \sum_{i=1}^{n} \alpha_i F x_i = F\left(\sum_{i=1}^{n} \alpha_i x_i\right) \text{ for any } \alpha_i \in \mathbb{C}.$$

Then by what we have asserted above,

$$\sum_{i=1}^{n} a_i F x_i = F\left(\sum_{i=1}^{n} \alpha_i x_i\right) = 0 \Rightarrow \sum_{i=1}^{n} \alpha_i x_i = 0 \Rightarrow \alpha_i = 0 \text{ for all } i = 1, \cdots, n.$$

Hence $\{Fx_1, \cdots, Fx_n\}$ is a set of linearly independent vectors in $\tilde{Z}$. Since F is a linear map from a finite dimensional subspace Z to a finite dimensional subspace $\tilde{Z}$, we have

$$\dim\left(\bigcap_{i=1}^{n} \ker{}^f(A_i - \lambda_i)\right) \leq \dim\left(\bigcap_{i=1}^{n} \ker{}^f\left(\tilde{A}_i - \lambda_i\right)\right). \tag{2.9}$$

For the reverse inequality of (2.9), use the map F^* instead of F and similar argument above. The second assertion (2.5) is an immediate result of (2.4).

If $\mathbb{A}$ is a commuting n-tuple of operators in $B(\mathcal{H})$, the *Taylor-Browder spectrum*, denoted $\sigma_{Tb}(\mathbb{A})$, of $\mathbb{A}$ is defined by (cf. [7], [**11**])

$$\sigma_{Tb}(\mathbb{A}) = \sigma_{Te}(\mathbb{A}) \cup \operatorname{acc}\sigma_T(\mathbb{A}),$$

where acc K is the set of the accumulation points of K. We have the following result.

THEOREM 2.3. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting n-tuple of operators in* $HU(p)$, *then*

$$\sigma_{Tb}(\mathbb{A}) = \sigma_T(\mathbb{A}) \backslash \pi_{00}(\mathbb{A}) = \sigma_{Tb}\left(\tilde{\mathbb{A}}\right),$$

where $\pi_{00}(\mathbb{A})$ *is the set of isolated eigenvalues of finite multiplicity.*

Proof. From Theorem 2.8 in **[8]** we have

$$\sigma_T(\mathbb{A}) = \sigma_e^r(\mathbb{A}) \cup \overline{\sigma_{pf}(\mathbb{A}^*)},$$

where $\sigma_{pf}(\cdot)$ is the set of joint eigenvalues of finite multiplicity. Thus since $\sigma_{pf}(\mathbb{A}) = (\sigma_{Te}(\mathbb{A}))^c \cap \sigma_p(\mathbb{A})$, it follows that

$$\begin{aligned}\sigma_T(\mathbb{A}) \backslash \pi_{00}(\mathbb{A}) &= \sigma_T(\mathbb{A}) \cap \left(\sigma_{pf}(\mathbb{A}) \cap \text{ iso } \sigma_T(\mathbb{A})\right)^c \\ &= \sigma_T(\mathbb{A}) \cup \left(\left(\sigma_{Te}(\mathbb{A}) \cup \left(\sigma_p(\mathbb{A})\right)^c\right) \cup \text{ acc } \sigma_T(\mathbb{A})\right) \\ &= \sigma_{Te}(\mathbb{A}) \cup \text{ acc } \sigma_T(\mathbb{A}) = \sigma_{Tb}(\mathbb{A}),\end{aligned}$$

which gives the first equality. The second equality follows from the observation that $\sigma_T(\mathbb{A}) = \sigma_T(\tilde{\mathbb{A}})$ and $\pi_{00}(\mathbb{A}) = \pi_{00}(\tilde{\mathbb{A}})$.

The joint (Chō-Takaguchi) Weyl spectrum $\omega(\mathbb{A})$, for a commuting n-tuple, $\mathbb{A}$ is defined by

$$\omega(\mathbb{A}) = \cap\{\sigma_T(\mathbb{A} + \mathbb{K}) : \mathbb{K} \text{ is an } n\text{-tuple of compact operators and}$$
$$\mathbb{A} + \mathbb{K} = (A_1 + K_1, \cdots, A_n + K_n) \text{ is a commuting } n\text{-tuple}\}.$$

Jeon and Lee **[11]** showed that in general

$$\sigma_{Tw}(\mathbb{A}) \subseteq \omega(\mathbb{A}) \text{ and } \sigma_{Tw}(\mathbb{A}) \subseteq \sigma_{Tb}(\mathbb{A})$$

and suggested a question: does it follow that $\omega(\mathbb{A}) \subseteq \sigma_{Tb}(\mathbb{A})$? We give a partial answer.

COROLLARY 2.4. *If* $\mathbb{A} = (A_1, \cdots, A_n)$ *is a doubly commuting n-tuple of operators in* $HU(p)$, *then*

$$\omega(\mathbb{A}) \subseteq \sigma_{Tb}(\mathbb{A}).$$

Proof. It immediately follows from Theorem 2.3 and **[10**, Theorem 2].

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