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SPORADIC GROUP GEOMETRIES AND THE ACTION OF INVOLUTIONS

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Abstract

This paper is an expository introduction to recent and current work on geometries associated with minimal parabolic subgroups and maximal 2-local subgroups of finite sporadic simple groups, based on lectures given by the author at the Canberra Group Actions Workshop, held at the Australian National University in June 1993.

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1. Introduction

The purpose of this paper is to survey certain ideas which have been used in recent work on geometries related to the sporadic (finite) simple groups. The common feature of such work is the exploitation of the way certain groups of order two act upon the points of the geometry. Geometries (see Section 2 below for definitions), in the sense used here, were introduced in Buekenhout [1, 2]. Such concepts were an outgrowth of the theory of buildings and a desire to have an enlarged framework into which the sporadic simple groups and their geometries would also fit. For an excellent account of the evolution of these ideas see Tits [19].

Earlier interest in sporadic group geometries concentrated primarily on geometries whose rank 2 residues were either generalized m-gons or the so-called 'circle geometries' (see [19]). Later additions to the family of interesting geometries were made in [9] and [10]. Among other interesting geometries we should also mention the Peterson geometries (see [7]). The collections in [9] and [10] were directly inspired by analogous ideas in buildings, and it is these geometries, the minimal parabolic geometries and maximal 2-local geometries, that we shall focus upon here.

In the following section we shall review definitions and notation about geometries sufficient for our purposes. Then we shall survey the success to date in using the '$\mathbb{Z}_2$-action' approach. In order to convey what these methods entail we give (in Sections 3 and 4) some typical arguments, which have been selected from the preprint [11].

As an aside we mention that geometries may be also studied by considering covers of the geometry; for an indication of what is involved here we refer the reader to [8] and [6].

This paper is based on lectures given at the Canberra Group Actions Workshop, June 1993—the author wishes to thank the organizers for their hospitality.

2. Geometries

DEFINITIONS AND NOTATION. We say that $(\Gamma, *)$ is a *geometry of rank n* if and only if

$$\Gamma = \Gamma_0 \dot{\cup} \Gamma_1 \dot{\cup} \cdots \dot{\cup} \Gamma_{n-1}$$

where each $\Gamma_i \neq \emptyset$ and $*$ is a symmetric incidence relation on Γ such that $x * y$ never holds when x and y are in the same Γ_i. As is customary we suppress $*$ and say that Γ is a geometry. We say that $x \in \Gamma$ is of *type i* if and only if $x \in \Gamma_i$. We establish the convention that objects of type 0 will be called *points* and those of type 1 *lines*.

A subset $\mathscr{F}$ of Γ is called a *flag* of Γ if $x * y$ for each $x,\ y \in \mathscr{F}$ with $x \neq y$ (so that distinct elements of $\mathscr{F}$ must be of different types). The *type* of a flag $\mathscr{F}$ is the set $\{ i \mid \mathscr{F}$ contains an object of type $i \}$.

We say that points $a,\ b \in \Gamma_0$ are *collinear* provided there exists $l \in \Gamma_1$ such that $a * l$ and $b * l$. A structure of frequent interest (and particularly so in this paper) is the *point-line collinearity graph* $\mathscr{G}$ of Γ. The vertices of this graph are the elements of Γ_0, with $a,\ b \in \Gamma_0$ adjacent in $\mathscr{G}$ if and only if they are collinear in Γ.

Any nonempty subset Δ of Γ can be regarded as a geometry via the restriction of $*$ to Δ, the separation of elements of Δ into types being given by the decomposition

$$\Delta = \dot{\bigcup_{i \in J}} (\Gamma_i \cap \Delta)$$

where $J = \{ i \mid \Gamma_i \cap \Delta \neq \emptyset \}$. In such situations we sometimes call Δ a *subgeometry* of Γ.

Residues of flags are an important source of subgeometries. If $\mathscr{F}$ is a flag in Γ then the *residue of* $\mathscr{F}$ (in Γ) is the set $\Gamma_{\mathscr{F}}$ defined by

$$\Gamma_{\mathscr{F}} := \{ y \in \Gamma \mid y * x \text{ for all } x \in \mathscr{F} \}.$$

Provided it is nonempty, $\Gamma_{\mathscr{F}}$ is a (sub)geometry. For the special case $\mathscr{F} = \{x\}$ we write Γ_x instead of $\Gamma_{\{x\}}$; note that if x is of type i then Γ_x contains no objects of type i.

Many interesting geometries are *residually connected.* This means that for every flag $\mathscr{F}$ of Γ (including the empty flag) for which the rank of $\Gamma_{\mathscr{F}}$ is at least 2, the graph whose vertices are the elements of $\Gamma_{\mathscr{F}}$ and whose edges are the pairs of incident elements of $\Gamma_{\mathscr{F}}$ is a connected graph. We are also interested in geometries with the following property: if a, b, c are objects of types i, j, k (respectively) with $i < j < k$, then $a * b$ and $b * c$ imply $a * c$. (Obviously this property presupposes some fixed ordering of the types.) Geometries satisfying this are called *string geometries.*

We now look at AutΓ, the group of automorphisms of Γ, where an automorphism is a type and incidence preserving permutation of Γ. That is, AutΓ is the set of all $g \in \mathrm{Symm}\Gamma$ such that for all $x, y \in \Gamma$, if $x \in \Gamma_i$ then $x^g \in \Gamma_i$, and if $x * y$ then $x^g * y^g$. A subgroup G of AutΓ is said to be *flag transitive* on Γ if, for every pair of flags $\mathscr{F}, \mathscr{F}'$ which are of the same type, there exists a $g \in G$ with $\mathscr{F}' = \mathscr{F}^g$. In particular, this condition implies that G must be transitive on each Γ_i.

For each $x \in \Gamma$ we define

$$Q(x) := \{ g \in G_x \mid y^g = y \text{ for all } y \in \Gamma_x \},$$

where G_x is the stabilizer of x in G. It is clear that $Q(x)$ is a normal subgroup of G_x, and $G_x/Q(x)$ is a subgroup of AutΓ_x.

We now give two examples of geometries with the kind of properties that interest us. These two play fundamental roles in Sections 3 and 4 below.

EXAMPLE 2.1. Let V be a three dimensional vector space over the field with two elements, and let Δ_0 and Δ_1 be, respectively, the sets of all 1-dimensional and 2-dimensional subspaces of V. Then $\Delta := \Delta_0 \dot\cup \Delta_1$ is a (rank 2) geometry where, for $x \in \Delta_0$ and $y \in \Delta_1$, we define $x * y$ whenever $x \subset y$. It is straightforward to check that $GL_3(2)$ is flag transitive on Δ. Furthermore, $|\Delta_x| = 3 = |\Delta_y|$ for all $x \in \Delta_0$ and $y \in \Delta_1$. Since any two distinct 1-subspaces of V are contained in a unique 2-subspace, the point-line collinearity graph of Δ is a complete graph on 7 vertices.

Example 2.1 looks rather innocent and uninteresting, but it makes frequent appearances as a residue in many geometries associated with the sporadic groups.

EXAMPLE 2.2. Our second example is in fact the triple cover of the generalized quadrangle associated with the group $\mathrm{Sp}_4(2) \cong S_6$. It is a rank 2 geometry, but it is convenient for us to suspend our convention on names of types, using 'line' and 'plane' rather than 'point' and 'line' for the two types. We describe each line as a 6-tuple $(a_1a_2|a_3a_4|a_5a_6)$ where the a_i are elements of the set $\{A, B, C\}$. Let $\widetilde{\Delta}_1$ be the set of all such 6-tuples appearing in the 'Lines' part of Figure 1. The planes of the geometry are similar 6-tuples, except that two of the entries of each plane 6-tuple are dots rather than elements of $\{A, B, C\}$. Let $\widetilde{\Delta}_2$ be the set of all such 6-tuples listed under 'Planes' in Figure 1. Let $\widetilde{\Delta} := \widetilde{\Delta}_1 \dot\cup \widetilde{\Delta}_2$, and for $l = (a_1a_2|a_3a_4|a_5a_6) \in \widetilde{\Delta}_1$

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and $\pi = (b_1 b_2 | b_3 b_4 | b_5 b_6) \in \widetilde{\Delta}_2$ we define $l * \pi$ if $b_i = a_i$ whenever b_i is not a dot. So for example, referring to the labelling in Figure 1, if $l = (1, 4, 5)$ and $\pi = 8$ then $\widetilde{\Delta}_l = \{1, 4, 5\}$ and $\widetilde{\Delta}_\pi = \{(3, 8, 9), (8, 37, 42), (8, 38, 41)\}$. Observe that the labelling has been chosen so that a line labelled (i, j, k) is incident with a plane labelled m if and only if $m \in \{i, j, k\}$.

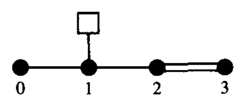
The coplanarity graph of $\widetilde{\Delta}$ has vertex set $\widetilde{\Delta}_1$, with $l_1, l_2 \in \widetilde{\Delta}_1$ being adjacent vertices if and only if $l_1 * \pi$ and $l_2 * \pi$ for some $\pi \in \widetilde{\Delta}_2$. Now fix $l = (1, 4, 5) \in \widetilde{\Delta}_1$. For each i from 1 to 4 the lines in the subset $\alpha_i(a, l) \subset \widetilde{\Delta}_1$ indicated in Figure 1 (ignore the 'a' for the moment) are precisely those vertices of distance i from l in the coplanarity graph. Observe that

$$|\alpha_1(a, l)| = 6, \quad |\alpha_2(a, l)| = 24, \quad |\alpha_3(a, l)| = 12 \quad \text{and} \quad |\alpha_4(a, l)| = 2.$$

The explicit description of Example 2.2 that we have given will be used heavily in Section 4 below.

In the next three theorems it is assumed that Γ is a residually connected string geometry and G a flag transitive subgroup of AutΓ. Our purpose in stating these theorems is to illustrate the kind of results that can be obtained by use of the $\mathbb{Z}_2$-action method, exploiting the way certain involutions in G act upon Γ and its subgeometries. The beginnings of this method lie in a paper of Segev [18], where the following theorem is established.

THEOREM 2.3. (Segev) *Suppose that Γ has diagram*



and that for each $a \in \Gamma_0$ and $X \in \Gamma_3$,

(i) *Γ_a is the dual of the maximal 2-local geometry of* M_{24},

(ii) *Γ_X is the geometry of points and lines and one class of maximal singular subspaces of the $\Omega_8^+(2)$-building.*

Then Γ is isomorphic to the maximal 2-local geometry of Co_1, *and* $G \cong Co_1$.

The precise meaning of the diagram need not concern us here; we note only that it provides information on the residues of flags. Thus the theme of Segev's paper is that information about the residues may enable the whole geometry Γ to be determined. Next we state two more recent results (along the same lines) which have also employed the $\mathbb{Z}_2$-action approach.

THEOREM 2.4. (Rowley and Walker [13]) *Assume Γ is a rank 3 geometry with type set $\{0, 1, 2\}$ such that for each $a \in \Gamma_0$ and $X \in \Gamma_3$,*

(i) *Γ_a is the geometry of duads and triduads of the Steiner system $S(22, 3, 6)$ and $G_a/Q(a) \cong$* $M_{22}:2$,

(ii) *Γ_X is the geometry of points and lines of a projective plane of order 2 (hence isomorphic to Example 2.1).*

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Lines

$\{l\}$	$(AA\mid AA\mid AA)$ (1,4,5)					
$\alpha_1(a,l)$	$(CB\mid AA\mid AA)$ (1,10,11)	$(BC\mid AA\mid AA)$ (1,12,13)	$(AA\mid CB\mid AA)$ (4,16,20)	$(AA\mid BC\mid AA)$ (4,17,21)	$(AA\mid AA\mid CB)$ (5,14,19)	$(AA\mid AA\mid BC)$ (5,15,18)
$\alpha_2(a,l)$	$(AC\mid AB\mid CB)$ (14,23,42)	$(AB\mid AC\mid CB)$ (14,45,27)	$(CA\mid BA\mid CB)$ (19,28,38)	$(BA\mid CA\mid CB)$ (19,34,31)	$(AB\mid CA\mid BC)$ (15,44,25)	$(AC\mid BA\mid BC)$ (15,22,41)
	$(CA\mid AB\mid BC)$ (18,29,37)	$(BA\mid AC\mid BC)$ (18,35,33)	$(AC\mid CB\mid AB)$ (16,23,40)	$(AB\mid CB\mid AC)$ (16,44,26)	$(CA\mid CB\mid BA)$ (20,29,36)	$(BA\mid CB\mid CA)$ (20,34,32)
	$(AB\mid BC\mid CA)$ (17,45,24)	$(AC\mid BC\mid BA)$ (17,43,22)	$(CA\mid BC\mid AB)$ (21,28,39)	$(BA\mid BC\mid AC)$ (21,35,30)	$(CB\mid AB\mid AC)$ (10,26,37)	$(CB\mid AC\mid AB)$ (10,27,39)
	$(CB\mid BA\mid CA)$ (11,38,24)	$(CB\mid CA\mid BA)$ (11,36,25)	$(BC\mid AB\mid CA)$ (12,32,42)	$(BC\mid AC\mid BA)$ (12,43,33)	$(BC\mid BA\mid AC)$ (13,30,41)	$(BC\mid CA\mid AB)$ (13,31,40)
$\alpha_3(a,l)$	$(BB\mid AC\mid BB)$ (6,27,33)	$(BB\mid CA\mid BB)$ (6,25,31)	$(AC\mid BB\mid BB)$ (2,22,23)	$(CA\mid BB\mid BB)$ (2,28,29)	$(BB\mid BB\mid CA)$ (7,24,32)	$(BB\mid BB\mid AC)$ (7,26,30)
	$(BA\mid CC\mid CC)$ (3,34,35)	$(AB\mid CC\mid CC)$ (3,44,45)	$(CC\mid AB\mid CC)$ (8,37,42)	$(CC\mid BA\mid CC)$ (8,38,41)	$(CC\mid CC\mid BA)$ (9,36,43)	$(CC\mid CC\mid AB)$ (9,39,40)
$\alpha_4(a,l)$	$(BB\mid BB\mid BB)$ (2,6,7)	$(CC\mid CC\mid CC)$ (3,8,9)				

Planes

$(..\mid AA\mid AA)$ 1	$(..\mid BB\mid BB)$ 2	$(..\mid CC\mid CC)$ 3	$(AA\mid ..\mid AA)$ 4	$(AA\mid AA\mid ..)$ 5	$(BB\mid ..\mid BB)$ 6
$(BB\mid BB\mid ..)$ 7	$(CC\mid ..\mid CC)$ 8	$(CC\mid CC\mid ..)$ 9	$(CB\mid A.\mid A.)$ 10	$(CB\mid .A\mid .A)$ 11	$(BC\mid A.\mid .A)$ 12
$(BC\mid .A\mid A.)$ 13	$(A.\mid A.\mid CB)$ 14	$(A.\mid .A\mid BC)$ 15	$(A.\mid CB\mid A.)$ 16	$(A.\mid BC\mid .A)$ 17	$(.A\mid A.\mid BC)$ 18
$(.A\mid .A\mid CB)$ 19	$(.A\mid CB\mid .A)$ 20	$(.A\mid BC\mid A.)$ 21	$(AC\mid B.\mid B.)$ 22	$(AC\mid .B\mid .B)$ 23	$(.B\mid .B\mid CA)$ 24
$(.B\mid CA\mid B.)$ 25	$(.B\mid .B\mid AC)$ 26	$(.B\mid AC\mid .B)$ 27	$(CA\mid B.\mid .B)$ 28	$(CA\mid .B\mid B.)$ 29	$(B.\mid B.\mid AC)$ 30
$(B.\mid CA\mid .B)$ 31	$(B.\mid .B\mid CA)$ 32	$(B.\mid AC\mid B.)$ 33	$(BA\mid C.\mid C.)$ 34	$(BA\mid .C\mid .C)$ 35	$(C.\mid C.\mid BA)$ 36
$(C.\mid AB\mid .C)$ 37	$(C.\mid BA\mid C.)$ 38	$(C.\mid .C\mid AB)$ 39	$(.C\mid C.\mid AB)$ 40	$(.C\mid BA\mid .C)$ 41	$(.C\mid AB\mid C.)$ 42
$(.C\mid .C\mid BA)$ 43	$(AB\mid C.\mid .C)$ 44	$(AB\mid .C\mid C.)$ 45			

FIGURE 1

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Assume further that the stabilizer G_{alX} is finite whenever $\{a, l, X\}$ is a maximal flag.

Then Γ is isomorphic to the Co_2 *minimal parabolic geometry, and* $G \cong \mathrm{Co}_2$.

THEOREM 2.5. (Rowley and Walker [16, 17]) *Assume Γ is a rank 3 geometry with type set $\{0, 1, 2\}$ such that for each $a \in \Gamma_0$ and $X \in \Gamma_3$,*

(i) *Γ_a is the geometry of duads and triduads of the Steiner system $S(22, 3, 6)$ and $G_a/Q(a) \cong \mathrm{M}_{22}$,*

(ii) *Γ_X is the geometry of non-zero isotropic vectors and isotropic 2-spaces in a 4-dimensional GF(4)-unitary space and $G_X/Q(X) \cong \mathrm{U}_4(2){:}2$.*

Assume that G_{alX} is finite whenever $\{a, l, X\}$ is a maximal flag.

Then Γ is isomorphic to the Fi_{22} *minimal parabolic geometry, and* $G \cong \mathrm{Fi}_{22}$.

(In the preceding two theorems, and below, our notation for group extensions is that used in the *Atlas* [3]: a colon indicates a split extension and a raised dot a nonsplit extension.)

A partial study of the point-line collinearity graph of the maximal 2-local geometry for Fi_{24}' was made in [20], while in [14, 15] a complete description is given of the point-line collinearity graph of the maximal 2-local geometry for J_4. This latter graph has 173,067,389 vertices and has a very intricate structure (partly because each line is incident with 5 points).

3. $\mathbb{Z}_2$-action

We illustrate the $\mathbb{Z}_2$-action method with arguments and results drawn from [11]. The following situation will be assumed to hold.

HYPOTHESIS 3.1. *Let $\Gamma = \Gamma_0 \dot\cup \Gamma_1 \dot\cup \Gamma_2$ be a residually connected rank 3 geometry and G a flag transitive subgroup of* $\mathrm{Aut}\Gamma$. *Assume that for all $a \in \Gamma_0$, $l \in \Gamma_1$ and $X \in \Gamma_2$ the following conditions are satisfied:*

(i) *Γ_a is isomorphic to the geometry $\widetilde{\Delta}$ described in Example 2.2;*

(ii) *Γ_X is isomorphic to the geometry Δ described in Example 2.1;*

(iii) *if $a * l$ and $l * X$ then $a * X$;*

(iv) *$Q(a)$ is elementary abelian of order 2^6 and $Q(X)$ is extraspecial of order 2^{1+6}, the quotient groups $G_a/Q(a)$ and $G_X/Q(X)$ being $3^{\cdot}\mathrm{S}_6$ and $GL_3(2)$ respectively;*

(v) *if $a * X$ then $\tau(X)$, the generator of $Z(G_X) = Z(Q(X))$, is contained in $Q(a)$.*

Hypothesis 3.1 looks like—and indeed is—the result of certain reductions, and it is the point at which the hard work starts. Later in this section we will place Hypothesis 3.1 in context. For now we remark that our perspective is that of deducing global information about Γ from the local data listed in Hypothesis 3.1. In the argument we present Hypothesis 3.1(iv) plays virtually no role, and is given just to fill out the picture. However, the element $\tau(X) \in Q(X)$ is the star of the show.

Objects of type 2 will be referred to as planes and (as usual) those of type 0 and 1 will be called, respectively, points and lines. Note that Hypothesis 3.1(i) explains the presence of the a in the notation $\alpha_i(a, l)$ that appears in Example 2.2.

We adopt the following useful notation: for $S \subseteq \Gamma$ and $i \in \{0, 1, 2\}$ let $\Gamma_i(S) := \{ y \in \Gamma_i \mid y * s \text{ for all } s \in S \}$. If $S = \{x\}$ then we write $\Gamma_i(x)$ for $\Gamma_i(\{x\})$.

If $l \in \Gamma_1$ and $X \in \Gamma_2$ with $l * X$, then, by Hypothesis 3.1(ii), l is incident with exactly three points in Γ_X. Hence, by Hypothesis 3.1(iii), l is incident with exactly three points in Γ. In view of the transitivity of G on Γ_1 it follows that each line in Γ is incident with exactly three points. Now let $a \in \Gamma_0$, $l \in \Gamma_1(a)$ and $X \in \Gamma_2(a)$ (so that l, $X \in \Gamma_a$). Without assuming that $l * X$, Hypothesis (3.1)(v) and $a * X$ imply that $\tau(X) \in Q(a)$; therefore $\tau(X)$ fixes l, as $l \in \Gamma_a$. Hence $\tau(X)$ leaves $\Gamma_0(l)$ invariant, and of course $\tau(X)$ fixes a. But what does $\tau(X)$ do to the two points in $\Gamma_0(l)\backslash\{a\}$?

LEMMA 3.2. ($\mathbb{Z}_2$-action) *Let $a \in \Gamma_0$, $l \in \Gamma_1(a)$ and $X \in \Gamma_2(a)$, and let $\tilde{l} \in \widetilde{\Delta}_1$, $\widetilde{X} \in \widetilde{\Delta}_2$ be the 6-tuples in Figure 1 that correspond to l, X. Then $\tau(X)$ interchanges $\Gamma_0(l)\backslash\{a\}$ if and only if $\widetilde{X}$ and $\tilde{l}$ agree in exactly one entry.*

We discuss briefly the lemmas corresponding to Lemma 3.2 for the geometries pertaining to the results mentioned at the end of Section 2. For Theorem 2.3 the lines and the objects of type 3 correspond respectively to sextets and octads in the Steiner system $S(24, 5, 8)$, and there $\tau(X)$ interchanges $\Gamma_0(l)\backslash\{a\}$ if and only if the octad X meets the tetrads of l in $(3, 1^5)$ (see [18, (4.19)]). In Theorems 2.4 and 2.5 the residue of a point is the same, but for the Co_2 case lines correspond to duads and planes (objects of type 2) to triduads, whereas for the Fi_{22} case it is the other way around. For Theorems 2.4 and 2.5 the criterion in the lemma analogous to Lemma 3.2 is that the duad and the hexad (which supports the triduad) intersect in exactly one element. In the J_4 maximal 2-local geometry lines and planes correspond, respectively, to trios and sextets of $S(24, 5, 8)$. Then we have that $x^{\tau(X)} \neq x$ for all $x \in \Gamma_0(l)\backslash\{a\}$ if and only if the trio l cuts the sextet X in $2222|311111|311111$ (meaning that the 3 octads of l intersect the tetrads of X in the sizes given).

LEMMA 3.3. *Let $a \in \Gamma_0$ and $l \in \Gamma_1(a)$. Then G_a is transitive on $\Gamma_1(a)$, and the G_{al}-orbits on $\Gamma_1(a)$ are $\{l\}$, $\alpha_1(a, l)$, $\alpha_2(a, l)$ $\alpha_3(a, l)$, $\alpha_4(a, l)$.*

Note that since $\alpha_i(a, l)$ consists of those lines in Γ_a whose distance from l in the coplanarity graph is i, it follows that if $g \in \mathrm{Aut}\Gamma$ then $\alpha_i(a, l)^g = \alpha_i(a^g, l^g)$.

Our next result gives a glimpse of what can be done with Lemma 3.2.

LEMMA 3.4. (Two points determine a line) *Let a and b be distinct collinear points of* Γ. *Then* $|\Gamma_1(\{a, b\})| = 1$.

PROOF. Suppose $l, m \in \Gamma_1(\{a, b\})$ with $l \neq m$, and argue for a contradiction. Working in Γ_a (note that $l, m \in \Gamma_a$) we may, without loss of generality since G_a is transitive on $\Gamma_1(a)$, take $l = (AA|AA|AA) = \text{line}\,(1, 4, 5)$.

If $m \in \alpha_1(a, l)$, then l and m are coplanar; that is, there exists $X \in \Gamma_2(a)$ such that $l * X$ and $m * X$. (Specifically, the required X will be plane 1, plane 4 or plane 5.) So $m, l \in \Gamma_X$ and, by Hypothesis 3.1(iii), $a, b \in \Gamma_X$ also. Then we have two distinct points both incident with two distinct lines in $\Gamma_X \cong \Delta$, which is impossible. Therefore $m \notin \alpha_1(a, l)$.

Since, by Lemma 3.3, $\alpha_2(a, l)$, $\alpha_3(a, l)$, and $\alpha_4(a, l)$ are G_{al}-orbits, it suffices to examine the following three cases:

(i) $m = (AC|AB|CB)$, line $(14, 23, 42)$,

(ii) $m = (BB|AC|BB)$, line $(6, 27, 33)$,

(iii) $m = (CC|CC|CC)$, line $(3, 8, 9)$.

In cases (i) and (ii) let $X = (..|AA|AA)$ (plane 1) and in case (iii) let $X = (CB|A.|A.)$ (plane 10). Using Lemma 3.2 we see that $\tau(X)$ fixes $\Gamma_0(l)\backslash\{a\}$ pointwise and interchanges $\Gamma_0(m)\backslash\{a\}$. But this gives a contradiction, since b is in both $\Gamma_0(l)\backslash\{a\}$ and $\Gamma_0(m)\backslash\{a\}$.

We now give some of the history behind Hypothesis 3.1. For the balance of this section we assume (i), (ii) and (iii) of Hypothesis 3.1, and assume in addition that if $\{a, l, X\}$ is a maximal flag of Γ then $G_a/Q(a) \cong 3^{\cdot}\mathrm{S}_6$, $G_X/Q(X) \cong \mathrm{GL}_3(2)$ and $G_l/Q(l) \cong \mathrm{GL}_2(2) \times GL_2(2)$.

THEOREM 3.5. (Rowley [12]) $|G| = 2^9$ *or* 2^{10}.

Actually more detailed information about the structure of G_{alX} is obtained in [12]; in particular, in the $|G| = 2^{10}$ case Hypothesis 3.1(iv),(v) hold. The proof of Theorem 3.5 is group theoretic in character. The motivation for looking at such Γ and G comes from there being examples in 'nature'—the $|G_{alX}| = 2^{10}$ case occurs in M_{24} and He.

THEOREM 3.6. (Heiss [4], Ivanov [5]) *If* $|G_{alX}| = 2^{10}$, *then* $G \cong \mathrm{M}_{24}$ *or* He, *and if* $|G_{alX}| = 2^9$, *then* $G \cong 3^7{}^{\cdot}\mathrm{Sp}_6(2)$.

Theorem 3.6 was obtained making essential use of a computer implementation of the Todd-Coxeter algorithm. Subsequently this result has been applied to other problems; so it would be desirable to have a computer-free proof of Theorem 3.6. Moreover, since the $|G_{alX}| = 2^{10}$ case appears (in connection with residues of certain flags) in other geometries, associated with such sporadic groups as $\cdot 1$, M and $\mathrm{F}i_{24}'$, insights gained from a computer-free proof of Theorem 3.6 could be valuable for other geometries.

4. $\mathbb{Z}_2$-action in action

Recall that our aim is to derive global information about Γ. Observe, for example, that it is by no means clear from Hypothesis 3.1 that Γ is a finite geometry. We concentrate our attention upon $\mathcal{G}$, the point-line collinearity graph of Γ. Our strategy is to fix a point a and 'work out' from a, investigating the ith disc of a,

$$\Delta_i(a) := \{ x \in \Gamma_0 \mid d(a, x) = i \}$$

for $i = 1, 2, 3, \dots$, where $d(\ ,\ )$ denotes the distance in $\mathcal{G}$. Since $\mathcal{G}$ is a connected graph, this approach may ultimately determine Γ.

Beginning with $\Delta_1(a)$ we have

LEMMA 4.1.

(i) $|\Delta_1(a)| = 90$,

(ii) *$\Delta_1(a)$ is a G_a-orbit,*

(iii) *if $b \in \Delta_1(a)$ and $\{l\} = \Gamma_1(\{a, b\})$, then $\overline{G}_{ab} = \overline{G}_{al}$ (where for $H \leq G_a$ we write $\overline{H}$ for $HQ(a)/Q(a)$).*

PROOF.

(i) We have seen that $|\Gamma_0(l)| = 3$ for all $l \in \Gamma_1$; hence each $l \in \Gamma_1(a)$ is incident with two points other than a. By Lemma 3.4 each $b \in \Delta_1(a)$ is incident with a unique $l \in \Gamma_1(a)$, and so it follows that $|\Delta_1(a)| = 2|\Gamma_1(a)|$. But $|\Gamma_1(a)| = 45$ by Hypothesis 3.1(i); so $|\Delta_1(a)| = 90$.

(ii) Let $l = (AA|AA|AA) = \text{line}\,(1, 4, 5) \in \Gamma_1(a)$ and $\{b, b'\} = \Gamma_0(l)\backslash\{a\}$. If c is an arbitary point in $\Delta_1(a)$ and $\{m\} = \Gamma_1(\{a, c\})$ then since G_a is transitive on $\Gamma_1(a)$ there exists $g \in G_a$ with $m^g = l$, and hence $c^g \in \Gamma_0(l)\backslash\{a\}$. So it suffices to prove that b and b' are in the same G_a-orbit. But choosing $X = (AC|B.|B.) = \text{plane } 22 \in \Gamma_2(a)$ and appealing to Lemma 3.2 gives that $\tau(X)$ interchanges $\{b, b'\}$, and since $\tau(X) \in G_a$ (because $\tau(X) \in Q(a)$ by Hypothesis 3.1(v)) we have (ii).

(iii) By Lemma 3.4, $G_{ab} \leq G_{al}$. Since there exists $\tau(X) \in Q(a)\backslash G_b$ (see part (ii)) we have $[G_{al} : G_{ab}] = 2 = [Q(a) : Q(a) \cap G_b]$. Hence (iii) holds.

From Lemma 4.1(iii) it follows that if $b \in \Delta_1(a)$ and $\{l\} = \Gamma_1(\{a, b\})$ then the orbits of G_{ab} on $\Gamma_1(a)$ are the same as the orbits of G_{al}, since $Q(a)$ acts trivially on Γ_a. Thus, by Lemma 3.3, the sets $\alpha_i(a, l)$ are G_{ab}-orbits.

It is convenient at this point to introduce a notational convention which has proved to be very useful in practice. Suppose that $a, b \in \Gamma_0$ are collinear, so that $\Gamma_1(\{a, b\}) = \{l\}$ for some $l \in \Gamma_1$ (by Lemma 3.4). Then we shall denote l by $a + b$ or by $b + a$, but we will use $a + b$ when looking at l as a line in Γ_a, and $b + a$ when looking at it as a line in Γ_b. We also define

$$T(a + b) = T(b + a) := \{ \tau(X) \mid X \in \Gamma_2 \text{ and } X * (a + b) \}.$$

Note that $|T(a+b)| = 3$ (since each line in $\widetilde{\Delta}$ is incident with three planes).

Note that if $\tau(X) \in T(a+b)$ then $a * X$ and $b * X$, by Hypothesis 3.1(iii). Hence, using Hypothesis 3.1(v), $\tau(X) \in Q(a) \cap Q(b)$. That is, $\tau(X)$ fixes all lines and planes incident with either a or b. It is this fact that underlies the importance of $T(a+b)$.

LEMMA 4.2. *Let $a, b, c \in \Gamma_0$ be distinct points with a and b collinear and b and c collinear (so that $d(a, b) = 1 = d(b, c)$) and $b + a \neq b + c$.*

(i) *If $b + c \in \alpha_1(b, b+a) \cup \alpha_4(b, b+a)$, then each $\tau(X) \in T(b+a)$ fixes $\Gamma_0(b+c)$ pointwise.*

(ii) *If $b + c \in \alpha_2(b, b+a) \cup \alpha_3(b, b+a)$, then one of the three elements $\tau(X) \in T(b+a)$ fixes $\Gamma_0(b+c)$ pointwise, while the other two interchange $\Gamma_0(b+c)\backslash\{b\}$.*

PROOF. Note first that all the ingredients of the situation are preserved by automorphisms of Γ. Thus, for example, if $g \in G$ then $b + c \in \alpha_i(b, b+a)$ if and only if $b^g + c^g \in \alpha_i(b^g, b^g + a^g)$, and $\tau(X) \in T(a+b)$ fixes $\Gamma_0(b+c)$ pointwise if and only if $g^{-1}\tau(X)g = \tau(X^g) \in T(a^g + b^g)$ fixes $\Gamma_0(b^g + c^g)$ pointwise. Now, working in Γ_b, we may apply an element of G_b and assume without loss of generality that $b + a = (AA|AA|AA)$. Then $T(b+a) = \{\tau(X_1), \tau(X_2), \tau(X_3)\}$, where

$$X_1 = (AA|AA|..), \quad X_2 = (AA|..|AA), \quad X_3 = (..|AA|AA).$$

Since the $\alpha_i(b, b+a)$ are G_{ba}-orbits, in each case we only need check one orbit representative as a candidate for $b + c$ (and we can choose it at random).

(i) If $b+c \in \alpha_1(b, b+a)$ take $b+c = (CB|AA|AA)$, and if $b+c \in \alpha_4(b, b+a)$ take $b+c = (CC|CC|CC)$. Then for each X_i the number of entries of $b+c$ which are the same as the corresponding entry of X_i is 0, 2 or 4, whence (i) holds by Lemma 3.2.

(ii) If $b + c \in \alpha_2(b, b+a)$ take $b + c = (AC|AB|CB)$. By Lemma 3.2, $\tau(X_1)$ fixes $\Gamma_0(b+c)$ pointwise and $\tau(X_2)$, $\tau(X_3)$ interchange $\Gamma_0(b+c)\backslash\{b\}$. If $b + c \in \alpha_3(b, b+a)$, take $b + c = (BB|AC|BB)$ and use Lemma 3.2 again.

We now investigate triangles in $\mathcal{G}$. These are crucial in many respects (see, in particular, Theorem 4.6).

LEMMA 4.3. *Let $\{a, b, c\}$ be a triangle in $\mathcal{G}$, and put $\Gamma_1(\{a, b\}) = \{m\}$, $\Gamma_1(\{b, c\}) = \{l\}$ and $\Gamma_1(\{a, c\}) = \{k\}$. Then either*

(i) *$a, b, c, m, l, k \in \Gamma_X$ for some $X \in \Gamma_2$, or*

(ii) *$m \in \alpha_4(a, k)$, $l \in \alpha_4(b, m)$ and $k \in \alpha_4(c, l)$.*

PROOF. We assume that (ii) does not hold and show that (i) does. If any two of m, l and k are equal, then this line is incident with all of a, b and c, and by Lemma 3.4 it is the unique line incident with any pair of these points. So $m = l = k$, and (i) holds with any choice of $X \in \Gamma_2(m)$. So we may suppose that k, l and m are all distinct.

In view of our supposition that (ii) does not hold, we may assume that $l \notin \alpha_4(b, m)$. Therefore $l \in \alpha_1(b, m) \cup \alpha_2(b, m) \cup \alpha_3(b, m)$. Suppose $l \in \alpha_2(b, m) \cup \alpha_3(b, m)$. Then appealing to Lemma 4.2(ii) yields the existence of an element $\tau(X) \in T(m)$ with $\tau(X)$ interchanging $\Gamma_0(l)\backslash b$. Thus $\Gamma_0(l) = \{b, c, c^{\tau(X)}\}$. Furthermore, $\tau(X)$ fixes k as well as l, since $\tau(X) \in Q(a) \cap Q(b)$, and since $c \in \Gamma_0(k)$ it follows that $c^{\tau(X)} \in \Gamma_0(k)$ also. Hence, using Lemma 3.4 and the fact that $c \neq c^{\tau(X)}$,

$$k = c + c^{\tau(X)} = l,$$

a contradiction. So $l \in \alpha_1(b, m)$, which implies that $l, m \in \Gamma_1(X)$ for some $X \in \Gamma_2(b)$. Hence a, b, $c \in \Gamma_X$ by the string condition Hypothesis 3.1(iii), and then, by Hypothesis 3.1(ii) and Lemma 3.4, it follows that $k \in \Gamma_X$ also. So (i) holds and the lemma is proved.

Triangles in $\mathscr{G}$ which satisfy (ii) of Lemma 4.3 will be called α_4-triangles.

In Lemma 4.1 we have learnt all there is to know about $\Delta_1(a)$; so now we focus upon $\Delta_2(a)$. Let $c \in \Delta_2(a)$ (that is, $d(a, c) = 2$). Then there exists $b \in \Delta_1(a) \cap \Delta_1(c)$, and so we have lines $b + a$ and $b + c$ (in Γ_b). Observe, using Lemma 4.1(iii) (with a and b interchanged), that $\overline{G}_{ab} = \overline{G}_{b\,b+a}$ (where $\overline{G}_{b\,b+a} = G_b/Q(b)$). Thus, because we will only be interested in G_a-orbits, we want to know for which i do we have $b + c \in \alpha_i(b, b + a)$. Now $i = 1$ is not possible since it would mean that $b + c$ and $b + a$ are both incident with some $X \in \Gamma_2$, which leads to $a, c \in \Gamma_X$, and then $d(a, c) = 1$ by Hypothesis 3.1(ii). We now take a closer look at the case $i = 2$, and begin by defining

$$\Delta_2^1(a) := \{c \in \Delta_2(a) \mid \text{there exists } b \in \Delta_1(a) \cap \Delta_1(c) \text{ with } b + c \in \alpha_2(b, b + a)\}.$$

LEMMA 4.4. *$\Delta_2^1(a)$ is a G_a-orbit.*

PROOF. Let c_1, $c_2 \in \Delta_2^1(a)$, and for each $i = 1, 2$ let $b_i \in \Delta_1(a) \cap \Delta_1(c_i)$ with $b_i + c_i \in \alpha_2(b_i, b_i + a)$. In view of Lemma 4.1(ii) there exists $g \in G_a$ with $b_2^g = b_1$, and so replacing c_2 by c_2^g we may suppose that $b_1 = b_2 = b$ (say). Then we have $b \in \Delta_1(a) \cap \Delta_1(c_1) \cap \Delta_1(c_2)$. By similar reasoning, since $\alpha_2(b, b + a)$ is a $\overline{G}_{ba}$-orbit we may further suppose that $b + c_1 = b + c_2$. From Lemma 4.2(ii) there exists $\tau(X) \in T(b + a)$ such that $\tau(X)$ interchanges $\Gamma_0(b + c_1)\backslash\{b\} = \{c_1, c_2\}$. Hence, as $\tau(X) \in G_a$, we have shown that c_1 and c_2 are in the same G_a-orbit. Since it is clear that $c \in \Delta_2^1(a)$ and $g \in G_a$ imply that $c^g \in \Delta_2^1(a)$, the lemma holds.

For $c \in \Delta_2^1(a)$ we put $\{a, c\}^\perp = \Delta_1(a) \cap \Delta_1(c)$.

THEOREM 4.5. *Let $c \in \Delta_2^1(a)$. Then $|\{a, c\}^\perp| = 3$.*

Apart from being useful for the analysis of other configurations in $\mathcal{G}$, Theorem 4.5 enables us to calculate $|\Delta_2^1(a)|$. We do this by counting in two ways the number of pairs (b, c) with $c \in \Delta_2(a)$ and $b \in \Delta_1(a) \cap \Delta_1(c)$ with $b + c \in \alpha_2(b, b + a)$. By Theorem 4.5 this number is $3|\Delta_2^1(a)|$, since c must be in $\Delta_2^1(a)$, and for each c there are 3 possibilities for b. On the other hand there are $|\Delta_1(a)|$ possibilities for b, for each of these there are $|\alpha_2(b, b + a)|$ possibilities for the line $b + c$, each giving 2 possibilities for c. Thus

$$3|\Delta_2^1(a)| = 2|\Delta_1(a)|\,|\alpha_2(b, b + a)|,$$

which yields $|\Delta_2^1(a)| = (2 \times 90 \times 24)/3 = 1440$.

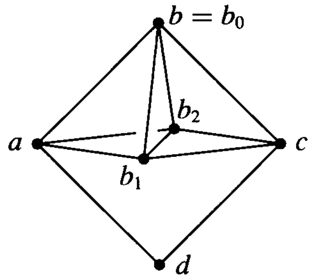
Moreover, knowing $\{a, c\}^\perp$ assists us in identifying G_{ac} (as a subgroup of G_a), which is itself important for investigating G_a-orbits of $\Delta_3(a)$.

PROOF OF THEOREM 4.5. (outline). Fix $c \in \Delta_2^1(a)$, and let $b \in \{a, c\}^\perp$ be such that $b + c \in \alpha_2(b, b + a)$. Inspection of $\widetilde{\Delta} \cong \Gamma_b$ reveals that there is a unique line $l \in \alpha_1(b, b + a)$ which is also adjacent to $b + c$ in the coplanarity graph. Then there exists $X \in \Gamma_2$ such that $l, b + a \in \Gamma_X$, and since $\Gamma_X \cong \Delta$ it follows that all the points in $\Gamma_0(l)$ are in $\Delta_1(a)$. Similarly they are in $\Delta_1(c)$ also. Thus

(4.5.1) $$(\Delta_1(b) \cup \{b\}) \cap \{a, c\}^\perp = \{b_0, b_1, b_2\} = \Gamma_0(l)$$

where $l \in \Gamma_1$ and $b_0 = b$.

We now assume the theorem is false and argue for a contradiction. Hence, (4.5.1) implies that there exists $d \in \{a, c\}^\perp$ with $d(b, d) = 2$. Thus we have the following situation.



Our next step is to rule out the possibility that $a + d \in \alpha_4(a, a + b)$. With this done, symmetry gives us

(4.5.2) $$\begin{aligned} a + d &\in \alpha_2(a, a + b) \cup \alpha_3(a, a + b) \quad \text{and} \\ c + d &\in \alpha_2(c, c + b) \cup \alpha_3(c, c + b). \end{aligned}$$

Put $T(a + b) = \{\tau_1, \tau_2, \tau_3\}$; by Lemma 4.2(ii) we may suppose our notation chosen so that τ_1 fixes c. We next prove that

(4.5.3) $$\tau_1 \text{ fixes } d.$$

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The key observation that leads to the desired contradiction is contained in

(4.5.4) *Let $\mu \in T(a+d)$. Then either μ interchanges $\Gamma_0(a+b_i)\backslash\{a\}$ for all $i = 0, 1, 2$, or else μ fixes $\Gamma_0(a+b_i)\backslash\{a\}$ pointwise for all $i = 0, 1, 2$.*

Now we bring the above information to bear on Γ_a. Letting $a+b = (1, 4, 5)$ we may, without loss of generality since $T(a+b)$ is a $G_{a\,a+b}$-conjugacy class, choose $\tau_1 = 5$. (For compactness of notation we will refer to planes and lines in Γ_a via the labelling in Figure 1.) Now since $a+b_0$, $a+b_1$ and $a+b_2$ are contained in a plane,

(4.5.5) *the three possibilities for $\{a+b_0, a+b_1, a+b_2\}$ are:*
(a) $\{(1, 4, 5),\ (1, 10, 11),\ (1, 12, 13)\}$,
(b) $\{(1, 4, 5),\ (4, 16, 20),\ (4, 17, 21)\}$ *and*
(c) $\{(1, 4, 5),\ (5, 14, 19),\ (5, 15, 18)\}$.

Combining (4.5.3) and Lemma 3.2, we can list the possibilities for $a+d$.

(4.5.6)
(i) *If $a+d \in \alpha_2(a, a+b)$, then $a+d \in$* $\{(14, 23, 42), (14, 45, 27), (19, 28, 38), (19, 34, 31), (15, 44, 25), (15, 22, 41), (18, 29, 37), (18, 35, 33)\}$.
(ii) *If $a+d \in \alpha_3(a, a+b)$, then $a+d \in$* $\{(7, 24, 32), (7, 26, 30), (9, 36, 43), (9, 39, 40)\}$.

And here is the contradiction.

(4.5.7) All possibilities in (4.5.5) and (4.5.6) contradict (4.5.4).

Suppose $a+d = (14, 23, 42)$ (the first possibility in (4.5.6)(i)). Then $T(a+d) = \{\tau(14), \tau(23), \tau(42)\}$. By Lemma 3.2 $\tau(14)$ fixes $\Gamma_0((1, 4, 5))\backslash\{a\}$ pointwise and interchanges both $\Gamma_0((1, 10, 11))\backslash\{a\}$ and $\Gamma_0((4, 16, 20))\backslash\{a\}$. Hence, by (4.5.4), (4.5.5)(a) and (b) cannot hold. However, using Lemma 3.2 yet again, $\tau(23)$ fixes $\Gamma_0((5, 14, 19))\backslash\{a\}$ pointwise and interchanges $\Gamma_0((1, 4, 5))\backslash\{a\}$, and so (4.5.4) also rules out (4.5.5)(c). Therefore we conclude that $a+d \neq (14, 23, 42)$.

Similar considerations eliminate all the possibilities for $a+d$ in (4.5.6), and complete the proof of Theorem 4.2.

Recall that there are two (very different) geometries which satisfy Hypothesis 3.1, yet no differences have come to light so far. Consider the following subset of $\Delta_2(a)$:

$$\Delta_2^2(a) := \{c \in \Delta_2(a) \mid \text{there exists } b \in \Delta_1(a) \cap \Delta_1(c) \text{ with } b+c \in \alpha_3(b, b+a)\}.$$

Let $c \in \Delta_2^2(a)$ and let $b \in \Delta_1(a) \cap \Delta_1(c)$ be such that $b+c \in \alpha_3(b, b+a)$. If there exists $b' \in \Delta_1(b) \cap \Delta_1(a) \cap \Delta_1(c)$, then we have triangles $\{a, b, b'\}$ and $\{c, b, b'\}$. Because $b+c \in \alpha_3(b, b+a)$, at least one of these must be an α_4-triangle. In particular, if $\mathscr{G}$ has no α_4-triangle, then we must have $\Delta_1(b) \cap \Delta_1(a) \cap \Delta_1(c) = \emptyset$. The presence or otherwise of α_4-triangles turns out to be critical; as an example we have

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THEOREM 4.6. (Rowley [11]) *If there exists an α_4-triangle in $\mathcal{G}$, then $G \cong M_{24}$ (and Γ is determined).*

Considerable progress has already been made in analysing $\mathcal{G}$ when it possesses no α_4-triangles.

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