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ON THE PROBABILITY DISTRIBUTION OF THE PRODUCT OF POWERS OF ELEMENTS IN COMPACT LIE GROUPS

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Abstract

In this paper, we study the probability distribution of the word map $w(x_1, x_2, \ldots, x_k) = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in a compact Lie group. We show that the probability distribution can be represented as an infinite series. Moreover, in the case of the Lie group SU(2), our computations give a nice convergent series for the probability distribution.

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1. Introduction

Erdős and Turán [3, 4] began the study of probabilistic questions in finite groups. Since then many papers investigated the number of solutions of equations and the probability distribution of word maps in finite groups. In the general setting, the problem can be formulated as follows. Let F_k be the free group of rank k and let $w \in F_k$ be a word. For a finite group G, the word w induces a word map $w_G : G^k \to G$. The problem is to study the probability distribution $P_{w,G}$ on G defined by $P_{w,G}(X) = |w_G^{-1}(X)|/|G|^k$ for $X \subset G$, where we use the counting measure on the finite group G. The probability distribution of the commutator word has been studied in [6, 15] for finite groups. The problem for other words on finite groups has been investigated in [2, 10, 13, 14].

Another direction is to study the same question for compact groups by using the Haar measure (see [5, 7, 8]). In these papers, the authors studied the probability distribution of the commutator word on compact groups by using the product Haar measure to define $|w_G^{-1}(X)|$.

In this paper, following Mulase and Penkava [12], we define $|w_G^{-1}(g)|$, $g \in G$, by the volume distribution

$$|w_G^{-1}(g)| = \int_{G^k} \delta(w_G(x_1, x_2, \ldots, x_k)g^{-1})\, dx_1 \cdots dx_k.$$

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Here, G is a compact Lie group, δ is the Dirac delta distribution on G and $dx_1 \cdots dx_k$ is the product Haar measure on G^k. The Dirac delta distribution has the expansion

$$\delta(x) = \frac{1}{|G|} \sum_{\lambda \in \widehat{G}} d_\lambda \chi_\lambda,$$

where $\widehat{G}$ is the set of all the irreducible representations of G. Also, d_λ and χ_λ are respectively the dimension and character of the representation λ. Therefore, the probability distribution $P_{w,G}$ is given by

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}} \sum_{\lambda \in \widehat{G}} \Big(\int_{G^k} d_\lambda \chi_\lambda(w(x_1, x_2, \ldots, x_k) g^{-1})\, dx_1\, dx_2 \cdots dx_k \Big), \tag{1.1}$$

provided that the sum on the right-hand side converges. (See [12] for further details on the volume distribution.)

The purpose of this paper is to give a formula for $P_{w,G}$ in the case $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, where $(n_1, n_2, \ldots, n_k)$ is a k-tuple of integers. Moreover, we show that in the case $G = \mathrm{SU}(2)$, the sum on the right-hand side of (1.1) converges and $P_{w,G}$ has a nice form. In the next section we will state and prove the main results.

2. Main results

We first note that it is enough to consider the case where all the exponents n_i are positive since changing x_i to x_i^{-1} does not affect $P_{w,G}$. Our first result is the following theorem.

Theorem 2.1. *Let G be a compact Lie group and $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$. The probability distribution of w on G is given by*

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}} \sum_{\lambda \in \widehat{G}} \frac{\chi_\lambda(g^{-1})}{d_\lambda^{k-1}} \int_G \chi_\lambda(x_1^{n_1})\, dx_1 \int_G \chi_\lambda(x_2^{n_2})\, dx_2 \cdots \int_G \chi_\lambda(x_k^{n_k})\, dx_k,$$

provided that the sum on the right-hand side converges.

To prove the formula, we need the following lemma.

Lemma 2.2. *For any $g \in G$ and $\lambda \in \widehat{G}$,*

$$\int_G \chi_\lambda(x^n g)\, dx = \frac{1}{d_\lambda} \int_G \chi_\lambda(x^n)\, dx \chi_\lambda(g).$$

Proof. Consider the action $\Phi : L^2(G) \to L^2(G)$ defined by $\Phi(f)(u) := \int_G f(x^n u)\, dx$. We can see that Φ commutes with the left regular representation of G on $L^2(G)$ given by

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$L_g(f)(u) = f(g^{-1}u)$. Indeed,

$$\begin{aligned}\Phi(L_g(f))(u) &= \int_G (L_g(f))(x^n u)\,dx = \int_G f(g^{-1}x^n u)\,dx \\ &= \int_G f((g^{-1}xg)^n g^{-1}u)\,d(g^{-1}xg) \\ &= \int_G f(x^n g^{-1}u)\,dx = \Phi(f)(g^{-1}u) = L_g(\Phi(f))(u).\end{aligned}$$

It is well known that the left regular representation $L^2(G)$ decomposes as the direct sum of all irreducible unitary representations. Let V_λ be such an irreducible summand corresponding to an irreducible representation ρ_λ with $\rho_\lambda(g) = L_g|_{V_\lambda}$. By Schur's lemma, Φ acts on V_λ by scalar multiplication by a constant C.

The left-hand side of the equality in the lemma can be written

$$\int_G \chi_\lambda(x^n g)\,dx = \mathrm{Tr}|_{V_\lambda}\Big(\int_G \rho_\lambda(x^n g)\,dx\Big) = \mathrm{Tr}|_{V_\lambda}(\Phi \circ L_g),$$

where the second equality follows from

$$\begin{aligned}\Big(\int_G \rho_\lambda(x^n g)\,dx(f)\Big)(u) &= \int_G f(g^{-1}x^{-n}u)\,dx = \int_G f(g^{-1}(x^{-1})^n u)\,d(x^{-1}) \\ &= \int_G f(g^{-1}x^n u)\,dx = (\Phi \circ L_g(f))(u) \quad \text{for all } f \in L^2(G).\end{aligned}$$

Since Φ acts on V_λ by scalar multiplication by a constant C,

$$\int_G \chi_\lambda(x^n g)\,dx = \mathrm{Tr}|_{V_\lambda}(\Phi \circ L_g) = C\chi_\lambda(g) \quad \text{for all } g \in G.$$

In particular, taking $g = 1$ gives $C = d_\lambda^{-1} \int_G \chi_\lambda(x^n)\,dx$ and the lemma is proved. □

Proof of Theorem 2.1. Using (1.1),

$$P_{w,G}(g) = \frac{1}{|G|^{k+1}} \sum_{\lambda \in \widehat{G}} \Big(\int_{G^k} d_\lambda \chi_\lambda(x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} g^{-1})\,dx_1\,dx_2 \cdots dx_k\Big).$$

By using Lemma 2.2 and taking integration over one variable at a time, we obtain the desired formula in the theorem. □

For a compact Lie group,

$$\int_G \chi_\lambda(x)\,dx = \begin{cases} |G| & \text{if } \lambda \text{ is the trivial representation,} \\ 0 & \text{otherwise,}\end{cases}$$

so we immediately deduce the following corollary.

Corollary 2.3. *Let G be a compact Lie group. Suppose that $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is a word where at least one of the exponents n_i is equal to* 1*. Then $P_{w,G}$ is the uniform distribution $P_{w,G}(g) = |G|^{-1}$ for all g.*

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The integral $\int_G \chi_\lambda(x^n)\,dx$ is a generalisation of the Frobenius–Schur indicator on finite groups and is hard to compute in general. Therefore, in general, it is hard to check whether or not the right-hand side of the formula in Theorem 2.1 converges. However, for the case of the group SU(2), we get an explicit formula which converges. The case of the word $w = x_1^2 x_2^2 \cdots x_k^2$ has been covered in [12]. For higher exponents, we get the following result.

THEOREM 2.4. *Consider $G = \mathrm{SU}(2)$ with the normalised Haar measure. Suppose that $w = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ is a word where all the n_i are greater than or equal to 2 and at least one is greater than 2. Then for $k \geq 4$ the probability distribution of w is given by*

$$P_{w,G}(g) = \sum_{j=0}^{\infty} \frac{\chi_{2j}(g^{-1})}{(2j+1)^{k-1}},$$

where χ_{2j} is the $(2j)$th irreducible character of SU(2).

PROOF. First, we recall some standard facts about representations of the group SU(2) from [1]. Note that $\widehat{G}$ can be identified with the set of nonnegative integers $\mathbb{Z}_{\geq}$. The jth irreducible representation has the dimension $d_j = j + 1$ and the character

$$\chi_j(e(\theta)) = \frac{\sin(j+1)\theta}{\sin\theta} \quad \text{where } e(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \text{ for } j \in \mathbb{Z}_{\geq}.$$

To prove the theorem, we need to compute $\int_G \chi_j(x^n)\,dx$. For $n = 2$, it is the Frobenius–Schur indicator $\int_G \chi_j(x^2)\,dx = (-1)^j$. For $n \geq 3$, we use the following lemma.

LEMMA 2.5. *Consider $G = \mathrm{SU}(2)$ with the normalised Haar measure. Then, for $j \in \mathbb{Z}_{\geq}$ and $n \geq 3$,*

$$\int_G \chi_j(x^n)\,dx = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ 1 & \text{if } j \text{ is even.} \end{cases}$$

PROOF. We recall that the Chebyshev polynomials of the first kind and the second kind are defined respectively by

$$T_j(\cos\theta) = \cos(j\theta) \quad \text{and} \quad U_j(\cos\theta) = \frac{\sin((j+1)\theta)}{\sin\theta}, \quad j \in \mathbb{Z}_{\geq}.$$

To compute the integrals of class functions on a Lie group, we use the following result from [1, page 86]:

$$\int_{\mathrm{SU}(2)} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(e(\theta))\sin^2\theta\,d\theta.$$

So,

$$\int_G \chi_j(x^n)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{\sin((j+1)n\theta)}{\sin(n\theta)}\sin^2\theta\,d\theta = \frac{2}{\pi}\int_0^{\pi} U_j(T_n(\cos\theta))\sin^2\theta\,d\theta.$$

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We need the following identities for Chebyshev polynomials from [11, Section 2.5, Exercise 3]:

$$\tfrac{1}{2}U_{2m+1} = T_1 + T_2 + \cdots + T_{2m+1} \quad \text{and} \quad \tfrac{1}{2}U_{2m} = \tfrac{1}{2}T_0 + T_2 + \cdots + T_{2m}.$$

These identities together with the fact that $T_n \circ T_m = T_{mn}$ allow us to write

$$U_{2m+1}(T_n(\cos\theta)) = 2T_n(\cos\theta) + 2T_{2n}(\cos\theta) + \cdots + 2T_{(2m+1)n}(\cos\theta)$$

and

$$U_{2m}(T_n(\cos\theta)) = T_0(\cos\theta) + 2T_{2n}(\cos\theta) + \cdots + 2T_{2mn}(\cos\theta).$$

Therefore,

$$U_{2m+1}(T_n(\cos\theta)) = 2\cos(n\theta) + 2\cos(2n\theta) + \cdots + 2\cos((2m+1)n\theta) \tag{2.1}$$

and

$$U_{2m}(T_n(\cos\theta)) = 1 + 2\cos(2n\theta) + \cdots + 2\cos(2mn\theta). \tag{2.2}$$

By [9, Section 3.631, page 397, (7) and (12)],

$$\int_0^{\pi} \cos(m\theta)\sin^2(\theta)\,d\theta = 0 \quad \text{for } m \geq 3. \tag{2.3}$$

In the case of j odd, it follows from (2.1) and (2.3) that the integrals of all the terms vanish and $\int_G \chi_j(x^n)\,dx = 0$. If j is even, it follows from (2.2) and (2.3) that $\int_G \chi_j(x^n)\,dx = (2/\pi)\int_0^{\pi} sin^2\theta\,d\theta = 1$, as required. □

Combining Theorem 2.1 and Lemma 2.5, we arrive at the formula

$$P_{w,G}(g) = \sum_{j=0}^{\infty} \frac{\chi_{2j}(g^{-1})}{(2j+1)^{k-1}}. \tag{2.4}$$

As noted above, the value of the character $\chi_{2j}(g^{-1})$ is of the form

$$U_{2j}(\cos\theta) = T_0 + 2T_2(\cos\theta) + \cdots + 2T_{2j}(\cos\theta).$$

So, we deduce that $|\chi_{2j}(g^{-1})| \leq 2j+1$ for all j. Therefore, the series on the right-hand side of (2.4) converges for $k \geq 4$ and the theorem is proved. □

It is interesting that for the group SU(2) the probability distribution $P_{w,G}$ does not depend on the exponents n_i appearing in w. It is natural to ask the following question.

Question 2.6. It is true that on the Lie group SU(n), for a fixed n, the probability distribution $P_{w,G}$ with $w = x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is finite when k is big enough and it does not depend on the n_i when the n_i are big enough?

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References

[1] T. Bröcker and T. T. Dieck, *Representations of Compact Lie Groups*, Graduate Texts in Mathematics, 98 (Springer, Berlin–Heidelberg, 2013).

[2] A. K. Das and R. K. Nath, 'On solutions of a class of equations in a finite group', *Comm. Algebra* **37**(11) (2009), 3904–3911.

[3] P. Erdős and P. Turán, 'On some problems of statistical group theory I', *Z. Wahrscheinlichkeitstheorie verw. Geb.* **4** (1965), 175–186.

[4] P. Erdős and P. Turán, 'On some problems of statistical group theory IV', *Acta Math. Acad. Sci. Hungar.* **19** (1968), 413–435.

[5] A. Erfanian and R. Rezaei, 'On the commutativity degree of compact groups', *Arch. Math. (Basel)* **93** (2009), 201–212.

[6] S. Garion and A. Shalev, 'Commutator maps, measure preservation, and T-systems', *Trans. Amer. Math. Soc.* **361**(9) (2009), 4631–4651.

[7] K. H. Hofmann and F. G. Russo, 'The probability that x and y commute in a compact group', *Math. Proc. Cambridge Philos. Soc.* **153**(3) (2012), 557–571.

[8] K. H. Hofmann and F. G. Russo, 'The probability that x^m and y^n commute in a compact group', *Bull. Aust. Math. Soc.* **87**(3) (2013), 503–513.

[9] A. Jeffrey and D. Zwillinger (eds.), *Tables of Integrals, Series, and Products*, 7th edn (Elsevier, Burlington, MA, 2007).

[10] M. Larsen and A. Shalev, 'On the distribution of values of certain word maps', *Trans. Amer. Math. Soc.* **368**(3) (2016), 1647–1661.

[11] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials* (Chapman and Hall/CRC, Boca Raton, FL, 2003).

[12] M. Mulase and M. Penkava, 'Volume of representation varieties', Preprint, 2002, arXiv:math/0212012.

[13] R. K. Nath and S. K. Prajapati, 'On the number of solutions of a generalized commutator equation in finite groups', *Acta Math. Hungar.* **156**(1) (2018), 18–37.

[14] O. Parzanchevski and G. Schul, 'On the Fourier expansion of word maps', *Bull. Lond. Math. Soc.* **46**(1) (2014), 91–102.

[15] M. R. Pournaki and R. Sobhani, 'Probability that the commutator of two group elements is equal to a given element', *J. Pure Appl. Algebra* **212** (2008), 727–734.

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