Bull. Aust. Math. Soc. (First published online 2024), page 1 of 11*
doi:10.1017/S0004972724000728
*Provisional—final page numbers to be inserted when paper edition is published

ON THE EVANS CHAIN COMPLEX

S. JOSEPH LIPPERT

(Received 27 June 2024; accepted 8 July 2024)

Abstract

We elaborate on the construction of the Evans chain complex for higher-rank graph C^*-algebras. Specifically, we introduce a block matrix presentation of the differential maps. These block matrices are then used to identify a wide family of higher-rank graph C^*-algebras with trivial K-theory. Additionally, in the specialised case where the higher-rank graph consists of one vertex, we are able to use the Künneth theorem to explicitly compute the homology groups of the Evans chain complex.

2020 *Mathematics subject classification*: primary 46L80; secondary 46L37.

Keywords and phrases: operator algebras, higher-rank graphs, K-theory, spectral sequences.

1. Introduction

In the field of C^*-algebra classification, K-theory computation has long been a captivating question. An algebra's K-theory groups will fully classify simple, separable, nuclear and purely infinite C^*-algebras satisfying the universal coefficient theorem (UCT) by the Kirchberg–Phillips classification theorem [8, 9]. Further, the Elliott invariant, which consists of the K-theory groups as well as information about the trace simplex, has shown incredible versatility in C^*-algebra classification [2, 4, 5].

On the other hand, higher-rank graph C^*-algebras have been an active field of research for the last quarter century. These algebras were introduced in [10], building on work in [12]. They have offered a number of creative solutions to long-standing problems in the field. They were used to show that every UCT Kirchberg algebra has nuclear dimension zero [14], they have been related to solutions of the Yang–Baxter equations [16, 18] and they have reformulated multipullback quantum odd sphere information in terms of universal properties [6].

With all of these promising results and many more on the horizon, it seems crucial to refine our understanding of higher-rank graph algebra K-theory. A landmark paper in this direction was Evans [3], in which the work of [7, 13, 15] was developed into a spectral sequence converging to the K-theory of a higher-rank graph C^*-algebra. The goal of this paper is to better understand the tools developed by Evans and to use them to make a number of explicit computations.

2. Preliminaries

2.1. Notation. We assume that $0 \in \mathbb{N}$. Further for $1 \leq j \leq k$, we use $\kappa_j : \mathbb{N} \to \mathbb{N}^k$ to denote the inclusion into the jth coordinate and $\iota_j : \mathbb{N}^j \to \mathbb{N}^k$ to denote the canonical inclusion onto the first j generators. For a small category, Λ, we denote the object set as Λ^0. Lastly, for a countable category, Λ, we define the group $\mathbb{Z}\Lambda^0 := \bigoplus_{v \in \Lambda^0} \mathbb{Z}$.

Definition 2.1 [10, Definition 1.1]. Let Λ be a countable category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. If (Λ, d) satisfies the *factorisation property*, that is, for every morphism $\lambda \in \Lambda$ and $n, m \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$, then (Λ, d) is a *k-graph* (or graph of rank k).

As mentioned in the Introduction, these objects can be used to build C^*-algebras. Specifically, this is done using the following relationships.

Definition 2.2 [10, Definition 1.5]. Let Λ be a source-free row-finite k-graph. Then $C^*(\Lambda)$ is the universal C^*-algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(ii) $s_{\lambda\mu} = s_\lambda s_\mu$ for all composable λ, μ;
(iii) $s^*_\lambda s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(iv) $s_v = \sum_{\lambda \in r^{-1}(v) \cap d^{-1}(n)} s_\lambda s^*_\lambda$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

Many facets of this algebra can be studied by means of the higher-rank graph from which it is built. The focus of this article is on the K-theory groups of $C^*(\Lambda)$. These groups have a deep relationship with the coordinate graphs of Λ, which we now define.

Definition 2.3 [10, Definition 1.9]. Let $f : \mathbb{N}^\ell \to \mathbb{N}^k$ be a monoid morphism. If (Λ, d) is a k-graph, then the *pullback* is the ℓ-graph $f^*(\Lambda)$ defined by

$$f^*(\Lambda) = \{(\lambda, n) : d(\lambda) = f(n)\}$$

with $d(\lambda, n) = n$, $s(\lambda, n) = s(\lambda)$ and $r(\lambda, n) = r(\lambda)$.

We call $\kappa_i^*(\Lambda)$ the ith *coordinate graph*, and we use $M_i(\Lambda)$ to denote the adjacency matrix of the ith coordinate graph (simplified to M_i when there is no ambiguity).

As alluded to in the Introduction, a useful tool for K-theory computation is the spectral sequence. We now introduce basic definitions, notation and results for this tool.

Definition 2.4. A spectral sequence is a set of abelian groups $E^r_{p,q}$, together with differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$, such that $E^{r+1}_{p,q} \cong \ker(d^r_{p,q}) / \operatorname{im}(d_{p+r,q-r+1})$

A helpful metaphor for these objects is a book where r indicates the page you are on. In general, the E^2 page would take the shape shown in Figure 1.

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

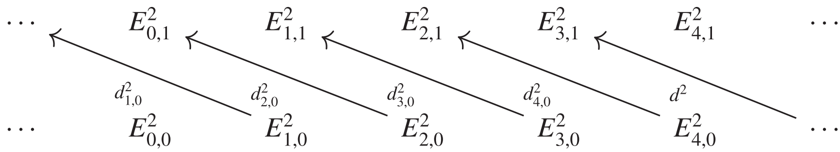

FIGURE 1. The E^2 page for a general spectral sequence.

DEFINITION 2.5. A spectral sequence E is said to *stabilise* if there exists some r_0 such that $E^r \cong E^{r_0}$ for all $r \geq r_0$. This stable page is denoted E^∞. A stable spectral sequence is said to *converge* to H_* (a family of abelian groups) if, for each n, there exists a filtration

$$0 = F_s(H_n) \subseteq \cdots \subseteq F_p(H_n) \subseteq \cdots \subseteq H_n$$

such that $E^\infty_{p,q} \cong F_p(H_{p+q})/F_{p-1}(H_{p+q})$.

THEOREM 2.6 (The Evans chain complex: [3, Theorem 3.15]). *For $p, k \in \mathbb{N}$ with $0 \leq p \leq k$, define the set*

$$\mathbb{N}_{p,k} := \{a = (a_1, \ldots, a_p) \in \{1, \ldots, k\}^p : a_i < a_{i+1} \text{ for all } i\}.$$

Equip these sets with the standard map such that, for $a \in \mathbb{N}_{p,k}$, the element $a^i \in \mathbb{N}_{p-1,k}$ is the tuple a with the ith coordinate removed. Additionally, $N_{0,k} := \{*\}$ with $a^1 = *$ for all $a \in N_{1,k}$.

For a row-finite, source-free k-graph, Λ, there exists a spectral sequence E, d converging to $K_*(C^*(\Lambda))$ with $E^\infty_{p,q} \cong E^{k+1}_{p,q}$ and

$$E^2_{p,q} \cong \begin{cases} H_p(\mathcal{D}) & 0 \leq p \leq k \wedge 2|q, \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathcal{D}$ is the chain complex

$$\mathcal{D}_p \cong \begin{cases} \bigoplus_{a \in N_{p,k}} \mathbb{Z}\Lambda^0 & 0 \leq p \leq k, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

with differentials

$$\bigoplus_{a \in N_{p,k}} m_a \mapsto \bigoplus_{b \in N_{p-1,k}} \sum_{a \in N_{p,k}} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a) \tag{2.2}$$

for $1 \leq p \leq k$ and $B_{a_i} := I - M^T_{a_i}$ (which we call the co-adjacency matrix).

3. Block matrix presentation

In this section, we investigate the indexing sets $\mathbb{N}_{p,k}$ more closely. Using this information, we partition the maps ∂ of $\mathcal{D}$ into recursive block matrices. This recursion is then leveraged to make explicit computations for k-graphs with an invertible co-adjacency matrix and monoid k-graphs.

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

Proposition 3.1. *For a fixed k, there exists a partition of $N_p = N_p^+ \sqcup N_p^-$ with $N_p^+ := \{a \in N_p : a_p = k\}$ (the k index has been supressed for ease of reading).*

(i) *For $a \in N_p^-$ and $1 \leq i \leq p$, we have $a^i \in N_{p-1}^-$.*
(ii) *Define $\psi : N_p^+ \to N_{p-1}^-$ by $a \mapsto a^p$. Then the function ψ is bijective.*
(iii) *There is a natural bijective inclusion $\varphi : N_{p,k-1} \to N_{p,k}^-$.*

Proof. (i) Let $a \in N_p^-$. Since $a_1 < \cdots < a_p$, we conclude that $a_i < k$ for all i. Thus, it is not possible for a^i to have a k in its last position.

(ii) The map $a \mapsto a^p$ is injective since $a, b \in N_p^+$ ensures that they agree in the last position and $a^p = b^p$ ensures that they agree everywhere else. Surjectivity is also immediate since $b \in N_{p-1}^-$ implies that $b_i < k$ for all i. Thus, $(b_1, \ldots, b_{p-1}, k)$ will map to b.

(iii) After observing that this map is well defined, bijectivity can be established quickly from the definition of $N_{p,k}^-$. □

We now use this information to reindex the image of (2.2).

$$\bigoplus_{b \in N_{p-1}^+} \sum_{a \in N_p} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a) = \bigoplus_{b \in N_{p-1}^+} \sum_{a \in N_p^+} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a) \tag{3.1}$$

$$\begin{aligned} \bigoplus_{b \in N_{p-1}^-} \sum_{a \in N_p} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a) &= \bigoplus_{b \in N_{p-1}^-} (-1)^{p+1} B_k(m_{\psi^{-1}(b)}) \\ &\quad \bigoplus_{b \in N_{p-1}^-} \Big(\sum_{a \in N_p^-} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a) \Big). \end{aligned} \tag{3.2}$$

Equalities (3.1) and (3.2) come from Proposition 3.1(i) and (ii), respectively.

Notice that our image has split neatly into three pieces. These three pieces become the blocks of our matrix. Before writing that down explicitly, we notice a recursive relationship between Λ and the pullback, $\iota_{k-1}^*(\Lambda)$.

Proposition 3.2. *Suppose Λ is a source-free, row-finite k-graph. Define the subgraphs $\Lambda_j := \iota_j^*(\Lambda)$ and call the Evans chain complexes of these subgraphs $\mathcal{D}^j$ with differentials ∂_p^j. The differentials of the Evans chain complex on Λ_{k-1} naturally map $\partial_p^{k-1} : \bigoplus_{N_{p,k}^-} \mathbb{Z}\Lambda^0 \to \bigoplus_{N_{p-1,k}^-} \mathbb{Z}\Lambda^0$ and $\partial_{p-1}^{k-1} : \bigoplus_{N_{p,k}^+} \mathbb{Z}\Lambda^0 \to \bigoplus_{N_{p-1,k}^+} \mathbb{Z}\Lambda^0$.*

Proof. Since the map of Proposition 3.1(iii) is bijective, we can simply reindex (2.2).

$$\bigoplus_{a \in N_{p,k-1}} m_a \mapsto \bigoplus_{b \in N_{p-1,k-1}} \sum_{a \in N_{p,k-1}} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a)$$

$$\bigoplus_{\varphi(a) \in N_{p,k}^-} m_a \mapsto \bigoplus_{\varphi(b) \in N_{p-1,k}^-} \sum_{\varphi(a) \in N_{p,k}^-} \sum_{i=1}^{p} (-1)^{i+1} \delta_{b,a^i} B_{a_i}(m_a).$$

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

	(2, 3, 4)	(1, 3, 4)	(1, 2, 4)	(1, 2, 3)
(3, 4)	B_2	B_1	0	0
(2, 4)	$-B_3$	0	B_1	0
(1, 4)	0	$-B_3$	B_2	0
(2, 3)	B_4	0	0	B_1
(1, 3)	0	B_4	0	$-B_2$
(1, 2)	0	0	B_4	B_3

	(1, 2, 3, 4)
(2, 3, 4)	B_1
(1, 3, 4)	$-B_2$
(1, 2, 4)	B_3
(1, 2, 3)	$-B_4$

FIGURE 2. The differentials ∂_3^4 and ∂_4^4.

The mapping of $\bigoplus_{N_{p,k}^+} \mathbb{Z}\Lambda^0$ by ∂_{p-1}^{k-1} follows directly from ψ preserving the positioning of coordinates when mapping $N_{p,k}^+ \to N_{p-1,k}^-$. □

Using the above proposition, we can write the image of ∂_p^k as

$$\partial_{p-1}^{k-1}\Big(\bigoplus_{a\in N_{p-1}^+} m_a\Big) + \Big(\bigoplus_{b\in N_{p-1}^-} (-1)^{p+1} B_k(m_{\psi^{-1}(b)})\Big) + \partial_p^{k-1}\Big(\bigoplus_{a\in N_{p,k}^-} m_a\Big).$$

Our main theorem organises all of this information into the well-studied perspective of block matrices.

THEOREM 3.3. *For a source-free row-finite k-graph, Λ, let $\mathcal{D}^j$ denote the Evans chain complex (2.1) for Λ_j. The differentials of $\mathcal{D}^i$ have the form*

$$\partial_p^j = \begin{bmatrix} \partial_{p-1}^{j-1} & 0 \\ (-1)^{p+1} B_j & \partial_p^{j-1} \end{bmatrix}$$

with ∂_p^ℓ that are not well defined (that is, for $\ell > p$) omitted. Here, B_i acts coordinate-wise on vectors.

PROOF. The proof is simply an application of the reindexing done in earlier propositions combined with the definition of block matrices. □

Before moving on, we look at some of the differentials for a 4-graph as an example to better visualise the pattern of these matrices. The differentials depicted in Figure 2 are ∂_3^4 and ∂_4^4, respectively. Each position is made zero if there is no way to delete an element of the column index and obtain the row index (δ_{b,a^i}). If an even position needs to be deleted, a negative is added ($(-1)^{i+1}$). Finally, whatever element was deleted determines which B_i is put in the slot (B_{a_i}).

THEOREM 3.4. *If Λ is a source-free row-finite k-graph such that B_i is bijective for some i with $1 \le i \le k$, then $K_*(C^*(\Lambda)) \cong 0$.*

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

PROOF. Since Λ is source free and row finite, there exists a spectral sequence E, d with E^2 page given by the homology groups of the Evans chain complex. Further, there exists a pullback $f^*(\Lambda)$ that is isomorphic to Λ and swaps B_i with B_k. Thus, without loss of generality, we suppose that B_k is bijective.

We demonstrate that $\mathrm{im}(\partial_{p+1}) = \ker(\partial_p)$ for $1 < p < k$. The special cases of $p = 1$ and $p = k$ are not particularly illuminating, but, for completeness, we include them in Lemma A.1.

Since it is already known that $\mathrm{im}(\partial_{p+1}) \subseteq \ker(\partial_p)$, we demonstrate the reverse inclusion. Since B_k is injective, it is relatively straightforward to check that

$$\ker\left(\begin{bmatrix} (-1)^{p-1}B_k & \partial_{p-1}^{k-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \partial_{p-1}^{k-1} & 0 \\ (-1)^{p+1}B_k & \partial_p^{k-1} \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -\partial_{p-1}^{k-1} & 0 \\ (-1)^{p+1}B_k & \partial_p^{k-1} \end{bmatrix}\right).$$

Since the adjacency matrices commute [10, Section 6], we see that the left-hand set simplifies to

$$\ker\left(\begin{bmatrix} 0 & 0 \\ (-1)^{p+1}B_k & \partial_p^{k-1} \end{bmatrix}\right).$$

This is exactly the set $\{(\alpha, \beta) : (-1)^{p+1}B_k(\alpha) = -\partial_p^{k-1}(\beta)\}$. Since B_k^{-1} exists, we can instead write $((-1)^{p+2}B_k^{-1}\partial_p^{k-1}(\beta), \beta)$.

Lastly, we note that $B_k^{-1}(\beta) \in \bigoplus_{N_p^-}(\mathbb{Z}\Lambda^0) = \bigoplus_{N_{p+1}^+}(\mathbb{Z}\Lambda^0)$. In particular,

$$\begin{bmatrix} \partial_p^{k-1} & 0 \\ (-1)^{p+2}B_k & \partial_{p+1}^{k-1} \end{bmatrix}\begin{bmatrix} B_k^{-1}(\beta) \\ 0 \end{bmatrix} = \begin{bmatrix} (-1)^{p+2}B_k^{-1}\partial_p^{k-1}(\beta) \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

We conclude that $\ker(\partial_p) = \mathrm{im}(\partial_{p+1})$. This implies that $H_p(\mathcal{D}) \cong 0$.

Since 0 has no nontrivial quotients, we conclude that $E_{p,q}^\infty \cong 0$ for all p, q in the Kasparov–Schochet spectral sequence. Then, looking at the definition of convergence, we conclude that the only family H_* that such a spectral sequence could converge to is the trivial family $H_* \cong 0$. Since $E \implies K_*(C^*(\Lambda))$, we deduce that $K_*(C^*(\Lambda)) \cong 0$. □

Our final result concerns tensor chain complexes. We begin by defining tensor complexes and explicitly stating their differential maps. We will then leverage the similarities between these maps and the Evans differentials.

DEFINITION 3.5. Given two chain complexes of R-modules $\mathcal{A}$ and C, we construct the chain complex $\mathcal{A} \otimes C$ by

$$(\mathcal{A} \otimes C)_n = \bigoplus_{i+j=n} \mathcal{A}_i \otimes_R C_j.$$

The differential is then defined on elementary tensors $a_i \otimes c_j \in \mathcal{A}_i \otimes C_j$ by

$$\partial^{\otimes}(a_i \otimes c_j) = \partial_i(a_i) \otimes_R c_j \oplus (-1)^i a_i \otimes_R \partial_j(c_j).$$

THEOREM 3.6. *For a nontrivial monoid k-graph, λ (that is, $|\Lambda^0| = 1$ and $B_i > 0$ for some i with $1 \leq i \leq k$), let $\mathcal{D}$ denote the Evans chain complex. If C^j denotes the chain complex $C^j : 0 \to \mathbb{Z} \xrightarrow{B_j} \mathbb{Z} \to 0$, then*

$$\mathcal{D} \cong \bigotimes_{j=1}^{k} C^j.$$

PROOF. Since $k < \infty$, it is enough to show that, for $\Lambda_{k-1} := \iota_{k-1}^*(\Lambda)$ with $\mathcal{D}^{k-1}$ the Evans chain complex for Λ_{k-1}, we have $\mathcal{D}^k \cong \mathcal{D}^{k-1} \otimes C$ with $C : 0 \to \mathbb{Z} \xrightarrow{B_k} \mathbb{Z} \to 0$.

First, we investigate the modules of the complex $\mathcal{D}^{k-1} \otimes C$. From Definition 3.5,

$$(\mathcal{A} \otimes C)_p = \mathcal{D}_p^{k-1} \otimes_{\mathbb{Z}} \mathbb{Z} \oplus \mathcal{D}_{p-1}^{k-1} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathcal{D}_p^{k-1} \oplus \mathcal{D}_{p-1}^{k-1} \cong \mathcal{D}_p^k.$$

The first isomorphism is the natural tensor isomorphism given by multiplication. The second is the reindexing bijections $\varphi : N_{p,k-1} \to N_{p,k}^-$ and $\psi^{-1} : N_{p-1,k}^- \to N_{p,k}^+$.

It remains to show that these isomorphisms, Ψ_p, are chain maps, that is, that they commute with the differentials. To avoid ambiguity, we denote the tensor differential by $\partial^{\otimes}$ and the Evans differential by ∂. Again, we will treat the cases $p = 1$ and $p = k$ separately because those differentials have a somewhat unique shape (see Lemma A.2).

Let $1 < p < k$. The map $\partial_p^k \Psi_p$ acts on elements $\alpha \otimes x \oplus \beta \otimes y$ with $\alpha \in \mathcal{D}_{p-1}^{k-1}$ and $\beta \in \mathcal{D}_p^{k-1}$ in the following way.

$$\partial_p^k \Psi_p(\alpha \otimes x \oplus \beta \otimes y) = \partial_p^k \begin{bmatrix} x \cdot \alpha \\ y \cdot \beta \end{bmatrix} = \begin{bmatrix} \partial_{p-1}^{k-1} & 0 \\ (-1)^{p+1} B_k & \partial_p^{k-1} \end{bmatrix} \begin{bmatrix} x \cdot \alpha \\ y \cdot \beta \end{bmatrix}$$
$$= \begin{bmatrix} x \cdot \partial_{p-1}^{k-1} \alpha \\ x(-1)^{p+1} B_k \alpha + y \partial_p^{k-1} \beta \end{bmatrix}.$$

Before moving on to the alternative ordering, we recall that $\partial_0^C = 0$ and $\partial_1^C = B_k$. This is used to simplify $\partial^{\otimes}$ in the working.

$$\Psi_{p-1} \partial_p^{\otimes}(\alpha \otimes x \oplus \beta \otimes y) = \Psi_{p-1}(\partial_{p-1}^{k-1}(\alpha) \otimes x \oplus (-1)^{p-1} \alpha \otimes B_k x + \partial_p^{k-1}(\beta) \otimes y)$$
$$= \begin{bmatrix} x \cdot \partial_{p-1}^{k-1}(\alpha) \\ (-1)^{p-1} B_k x \cdot \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ y \cdot \partial_p^{k-1}(\beta) \end{bmatrix}.$$

Finally, since the parity of $p - 1$ and $p + 1$ align, we conclude that these maps are equal. □

The benefit of phrasing the Evans complex in terms of a tensor complex is that it allows us to invoke the Künneth theorem

THEOREM 3.7 (Künneth theorem; see [17, Theorem 3.6.3]). *For a chain complex, C, with C_p and $\partial_C(C_p)$ flat for all p,*[1] *and an arbitrary complex $\mathcal{A}$, there exists a short exact sequence*

$$\bigoplus_{i+j=p} H_i(C)\otimes_R H_j(\mathcal{A}) \to H_p(C\otimes_R \mathcal{A}) \to \bigoplus_{i+j=p-1} \operatorname{Tor}_1^R(H_i(C), H_j(\mathcal{A}))$$

which splits unnaturally.

From this, we can state the E^2 page of the Kasparov–Schochet spectral sequence even more explicitly.

THEOREM 3.8. *For a nontrivial monoid k-graph, λ, define $g := \gcd(B_1, \ldots, B_k)$. The E^2 page of the Kasparov–Schochet spectral sequence has the form*

$$E^2_{p,q} = \begin{cases} \mathbb{Z}_g^{\binom{k-1}{p}} & 2 \mid q, \\ 0 & \textit{otherwise.} \end{cases}$$

PROOF. The goal is to use Theorem 3.6 which yields $\mathcal{D} \cong \bigotimes^k C^j$. Then, the Künneth Theorem with some classical finite group theory gives the necessary homology groups. As before, without loss of generality, we suppose that $B_1 > 0$.

We proceed by induction on k. Consider the base case $H_p(C^1)$. Since $B_1 \neq 0$, we observe that $E_{1,2q} = H_1(C^1) = \ker(B_1) = 0$ and $E_{0,2q} = H_0(C^1) = \mathbb{Z}_{B_1}$, as desired. Fix some $j \geq 1$ and define $g_j := \gcd(B_1, \ldots, B_j)$. Then suppose that

$$H_p(C^1 \otimes \cdots \otimes C^j) \cong (\mathbb{Z}_{g_j})^{\binom{j-1}{p}}.$$

Consider $(C^1 \otimes \cdots \otimes C^j) \otimes C^{j+1}$. Since each module in our complex is free, which implies that it is projective (see [17, Section 2]), we can use the Künneth Theorem and the inductive hypothesis to obtain

$$\begin{aligned} H_p((C^1 \otimes \cdots \otimes C^j) \otimes C^{j+1}) &\cong (\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}} \otimes H_1(C^{j+1}) \oplus (\mathbb{Z}_{g_j})^{\binom{j-1}{p}} \otimes H_0(C^{j+1}) \\ &\quad \oplus \operatorname{Tor}_1^{\mathbb{Z}}((\mathbb{Z}_{g_j})^{\binom{j-1}{p-2}}, H_1(C^{j+1})) \oplus \operatorname{Tor}_1^{\mathbb{Z}}((\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}}, H_0(C^{j+1})). \end{aligned}$$

There are two cases determined by B_{j+1}. First, suppose that $B_{j+1} = 0$ and thus $g_{j+1} = g_j$. In this case, $H_*(C^{j+1}) \cong \mathbb{Z}$. This would make the torsion groups 0 and

$$H_p((C^1 \otimes \cdots \otimes C^j) \otimes C^{j+1}) \cong (\mathbb{Z}_{g_j})^{\binom{j-1}{p}} \oplus (\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}} \cong (\mathbb{Z}_{g_{j+1}})^{\binom{j}{p}}.$$

Next, suppose that $B_{j+1} \neq 0$. This would mean that $H_1(C^{j+1}) = 0$ and $H_0(C^{j+1}) = \mathbb{Z}_{B_{j+1}}$. In particular, this means that the terms containing $H_1(C^{j+1})$ vanish leaving

$$(\mathbb{Z}_{g_j})^{\binom{j-1}{p}} \otimes \mathbb{Z}_{B_{j+1}} \oplus \operatorname{Tor}_1^{\mathbb{Z}}((\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}}, \mathbb{Z}_{B_{j+1}}).$$

[1] Some authors substitute C projective to simplify the initial conditions, but this is not equivalent.

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

It is a well-known fact (see [11, Section 6] that

$$\mathrm{Tor}_1^{\mathbb{Z}}((\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}}, \mathbb{Z}_{B_{j+1}}) \cong (\mathbb{Z}_{g_j})^{\binom{j-1}{p-1}} \otimes \mathbb{Z}_{B_{j+1}} \cong (\mathbb{Z}_{g_j} \otimes \mathbb{Z}_{B_{j+1}})^{\binom{j-1}{p-1}} \cong (\mathbb{Z}_{g_{j+1}})^{\binom{j-1}{p-1}},$$

which gives the desired form.

We conclude that $H_p(\mathcal{D}) \cong \mathbb{Z}_g^{\binom{k-1}{p}}$, which allows us to fill in the E^2 page of the Kasparov–Schochet spectral sequence with the desired result. □

REMARK 3.9. In the case where $\Lambda \cong \mathbb{N}^k$ (that is, trivial), $C^*(\Lambda) \cong C(\mathbb{T}^k)$ and thus $K_1(C^*(\Lambda)) \cong \mathbb{Z}^{2^{k-1}} \cong K_0(C^*(\Lambda))$. Because the K-theory is known, we omit the case from Theorem 3.8.

The above result for monoid k-graphs also sheds some light on Theorem 3.4. In particular, we find that an invertible co-adjacency matrix is not necessary for trivial K-theory groups.

COROLLARY 3.10. *Let Λ be a monoid k-graph with* $\gcd(B_1, \ldots, B_k) = 1$. *Then,* $K_*(C^*(\Lambda)) = 0$.

Corollary 3.10 has a very close relationship with the results of [1]. However, it is important to note that the results there required additional hypotheses on Λ.

We conclude this article by investigating monoid 3-graphs more generally. Specifically, this will demonstrate that knowing E^∞ does not always give perfect knowledge of $K_*(C^*(\Lambda))$.

COROLLARY 3.11. *Let Λ be a monoid 3-graph with $|\Lambda^{e_i}| > 1$ for some i with $1 \le i \le 3$ and define* $g = \gcd(B_1, B_2, B_3)$. *Then* $K_1(C^*(\Lambda)) = \mathbb{Z}_g^2$ *and there exists a short exact sequence*

$$\mathbb{Z}_g \to K_0(C^*(\Lambda)) \to \mathbb{Z}_g.$$

PROOF. Consider the map $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$. Observe that the parity of q means that $\mathrm{im}(d^2_{p,q}) = 0$ and $\ker(d_{p,q}) = E^2_{p,q}$. We conclude that $E^2 \cong E^3$; this is indeed true for any k-graph. Theorem 3.8 gives $E^3_{p,2q} \cong E^2_{p,2q} \cong \mathbb{Z}_g^{\binom{2}{p}}$. In particular, this means that $d^3_{p,q} : E^3_{p,q} \to E_{p-3,q}$ must be the zero map. We conclude, as before, that $E^3 \cong E^4$, and thus the sequence has stabilised.

First, we examine the K_1 group. We look at the family $\ldots, E_{-1,2}, E_{1,0}, E_{3,-2}, \ldots$. By Theorem 3.8, $E_{3,-2} \cong 0$ so all but $E_{1,0}$ are isomorphic to 0. Moreover, there exists a filtration of $K_1(C^*(\Lambda))$, $0 \le \cdots \le F_{-1} \le F_0 \le F_1 \le F_2 \le \cdots \le K_1(C^*(\Lambda))$ such that $F_p/F_{p-1} \cong E_{p,1-p}$. The zeros in all positions apart from $E_{1,0}$ allow us to conclude that $F_1 \cong K_1(C^*(\Lambda))$ and $F_0 \cong 0$. That is, $K_1(C^*(\Lambda)) \cong F_1/F_0 \cong E_{1,0} \cong \mathbb{Z}_g^2$.

To determine K_0, we must look at the family with even total degree $(2 \mid (p+q))$. This family is $\ldots, 0, 0, \mathbb{Z}_g, 0, \mathbb{Z}_g, 0, 0, \ldots$.

We now consider the filtration $0 \cong F_{-1} \le F_0 \le F_1 \le F_2 \cong K_0(C^*(\Lambda))$. Since $2 \nmid 1$, Theorem 2.1, ensures that $E_{1,-1} = 0$. So $F_1 \cong F_0$, which allows us to refine to

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press

the filtration $0 \leq F_0 \leq F_2$. This ensures that $F_2/F_0 \cong \mathbb{Z}_g$ providing the short exact sequence

$$0 \to \mathbb{Z}_g \to K_0(C^*(\Lambda)) \to \mathbb{Z}_g \to 0. \qquad \square$$

Appendix

LEMMA A.1. *Let Λ be a k-graph with B_k invertible. Then $H_0(\mathcal{D}) \cong H_k(\mathcal{D}) \cong 0$.*

PROOF. The differentials ∂_1 and ∂_k of the Evans chain complex always take the form

$$\partial_1 = \begin{bmatrix} B_k & B_{k-1} & \cdots & B_1 \end{bmatrix}, \quad \partial_k = \begin{bmatrix} B_1 \\ -B_2 \\ \vdots \\ (-1)^{p+1} B_k \end{bmatrix}.$$

Since $H_0(\mathcal{D}) = \operatorname{coker}(\partial_1)$ and $H_k(\mathcal{D}) = \ker(\partial_k)$, we need only demonstrate surjectivity and injectivity, respectively.

Since B_k is onto, $\partial_1 \begin{bmatrix} 0 \\ \vdots \\ \alpha \end{bmatrix}$ is onto $\mathbb{Z}\Lambda^0$. Lastly, $\partial_k(\alpha) = \partial_k(\beta)$ implies that $B_k(\alpha) = B_k(\beta)$ and thus, by injectivity of B_k, $\alpha = \beta$. $\square$

LEMMA A.2. *Under the hypotheses of Theorem 3.6, we have equalities $\partial_k^k \Psi_k = \Psi_{k-1} \partial^{\otimes}$ and $\partial_1^k \Psi_1 = \Psi_0 \partial^{\otimes}$.*

PROOF. Consider $\partial_k^k \Psi_k$ and notice that $N_{k,k}^- = \emptyset$. Since $(\mathcal{D}^{k-1} \otimes C)_k = \mathcal{D}_{k-1}^{k-1} \otimes C_1$, we let $\alpha \otimes x \in \mathcal{D}_{k-1}^{k-1} \otimes C_1$. Then

$$\partial_k^k \Psi_k(\alpha \otimes x) = \begin{bmatrix} \partial_{k-1}^{k-1} \\ (-1)^{k+1} B_k \end{bmatrix} [x \cdot \alpha] = \begin{bmatrix} x \partial_{k-1}^{k-1} \alpha \\ x(-1)^{k+1} B_k \alpha \end{bmatrix}$$

$$\Psi_{k-1} \partial^{\otimes}(\alpha \otimes x) = \Psi_{k-1}(\partial_{k-1}^{k-1}(\alpha) \otimes x \oplus (-1)^{k-1} B_1 x) = \begin{bmatrix} x \partial_{k-1}^{k-1} \alpha \\ x(-1)^{k-1} B_k \alpha \end{bmatrix}.$$

For the $p = 1$ case, we return to general elements $\alpha \otimes x \oplus \beta \otimes y \in \mathcal{D}_0^{k-1} \otimes \mathbb{Z} \oplus \mathcal{D}_1^{k-1} \otimes \mathbb{Z}$. Additionally, we utilise $\partial_0^{k-1} = 0$. Then

$$\partial_1^k \Psi_1(\alpha \otimes x \oplus \beta \otimes y) = \begin{bmatrix} (-1)^{k+1} B_k & \partial_1^{k-1} \end{bmatrix} \begin{bmatrix} x \cdot \alpha \\ y \cdot \beta \end{bmatrix} = \begin{bmatrix} x B_k \alpha + y \partial_1^{k-1} \beta \end{bmatrix}$$

$$\Psi_0 \partial^{\otimes}(\alpha \otimes x \oplus \beta \otimes y) = \Psi_0((-1)^0 \otimes B_k x + \partial_1^{k-1} \beta \otimes y) = \begin{bmatrix} x B_k \alpha + y \partial_1^{k-1} \beta \end{bmatrix}. \qquad \square$$

References

[1] S. Barlak, T. Omland and N. Stammeier, 'On the K-theory of C^*-algebras arising from integral dynamics', *Ergodic Theory Dynam. Systems* **38**(3) (2018), 832–862.

[2] G. A. Elliott, G. Gong, H. Lin and Z. Niu, 'The classification of simple separable KK-contractible C*-algebras with finite nuclear dimension', *J. Geom. Phys.* **158** (2020), Article no. 103861, 51 pages.

[3] D. G. Evans, ‘On the K-theory of higher rank graph C^*-algebras,’ *New York J. Math.* **14** (2008), 1–31.
[4] G. Gong, H. Lin and Z. Niu, ‘A classification of finite simple amenable Z-stable C^*-algebras, I: C^*-algebras with generalized tracial rank one’, *C. R. Math. Acad. Sci. Soc. R. Can.* **42**(3) (2020), 63–450.
[5] G. Gong, H. Lin and Z. Niu, ‘A classification of finite simple amenable Z-stable C^*-algebras, II: C^*-algebras with rational generalized tracial rank one’, *C. R. Math. Acad. Sci. Soc. R. Can.* **42**(4) (2020), 451–539.
[6] P. M. Hajac, R. Nest, D. Pask, A. Sims and B. Zieliński, ‘The K-theory of twisted multipullback quantum odd spheres and complex projective spaces’, *J. Noncommut. Geom.* **12**(3) (2018), 823–863.
[7] G. G. Kasparov, ‘Equivariant KK-theory and the Novikov conjecture’, *Invent. Math.* **91**(1) (1988), 147–201.
[8] E. Kirchberg, ‘Exact C^*-algebras, tensor products, and the classification of purely infinite algebras’, in: *Proceedings of the International Congress of Mathematicians (Zürich, 1994)*, Vols. 1, 2 (ed. S. D. Chatterji) (Birkhäuser, Basel, 1995), 943–954.
[9] E. Kirchberg and N. C. Phillips, ‘Embedding of exact C^*-algebras in the Cuntz algebra O_2’, *J. reine angew. Math.* **525** (2000), 17–53.
[10] A. Kumjian and D. Pask, ‘Higher rank graph C^*-algebras’, *New York J. Math.* **6** (2000), 1–20.
[11] W. S. Massey, *Singular Homology Theory*, Graduate Texts in Mathematics, 70 (Springer-Verlag, New York–Berlin, 1980).
[12] G. Robertson and T. Steger, ‘Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras’, *J. reine angew. Math.* **513** (1999), 115–144.
[13] G. Robertson and T. Steger, ‘Asymptotic K-theory for groups acting on $\widetilde{A}_2$ buildings’, *Canad. J. Math.* **53**(4) (2001), 809–833.
[14] E. Ruiz, A. Sims and A. P. W. Sørensen, ‘UCT-Kirchberg algebras have nuclear dimension one’, *Adv. Math.* **279** (2015), 1–28.
[15] C. Schochet, ‘Topological methods for C^*-algebras. I. Spectral sequences’, *Pacific J. Math.* **96**(1) (1981), 193–211.
[16] A. Vdovina, ‘Drinfeld–Manin solutions of the Yang–Baxter equation coming from cube complexes’, *Internat. J. Algebra Comput.* **31**(4) (2021), 775–788.
[17] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, 38 (Cambridge University Press, Cambridge, 1994).
[18] D. Yang. ‘The interplay between k-graphs and the Yang–Baxter equation’, *J. Algebra* **451** (2016), 494–525.

S. JOSEPH LIPPERT, Department of Mathematics and Statistics,
Sam Houston State University, Huntsville, Texas 77341, USA
e-mail: sjl054@shsu.edu

https://doi.org/10.1017/S0004972724000728 Published online by Cambridge University Press