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RESEARCH ARTICLE

Batch sojourn and delivery times in polling systems on a circle

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Abstract
In this paper, we analyze a polling system on a circle. Random batches of customers arrive at a circle, where each customer, independently, obtains a location that is uniformly distributed on the circle. A single server cyclically traverses the circle to serve all customers. Using mean value analysis, we derive the expected number of waiting customers within a given distance of the server. We exploit this to obtain closed-form expressions for both the mean batch sojourn time and the mean time to delivery.

1. Introduction

Polling systems are a class of queueing models in which servers attend to multiple queues. The most well-known variant models the case where a single server visits the queues in cyclical order. The analysis of polling systems has gained a lot of attention over the years, see, for instance, the literature reviews in [3, 16]. The strength of polling models lies in the wide range of applications they apply to, for example, in communication, traffic, and transportation systems [2, 9, 13]. In this paper, we focus on the application to warehouse logistics.

In warehouses, customer orders, consisting of several different items, need to be picked in a timely manner. One way of doing so is by deploying a milk-run system [19]. In this system, the picker continuously walks through the entire warehouse, picking any requested item she encounters. Whenever the picker finishes her route, she drops off the picks at the depot and starts anew. This system can be modeled as a polling system, where the server represents the picker and a customer represents a requested item; a batch of customers can thus be seen as an order. From the perspective of an order, it is important to know how long it takes for the order to be delivered at the drop-off point. As warehouses often store a large number of different items, it is natural to approximate the warehouse by allowing for *continuous* pick locations, that is, extending the polling model to polling on a circle.

Apart from warehouse logistics, continuous polling models have been applied to other domains, such as wireless communication networks. For instance, [10] uses these models to analyze ferry-assisted wireless local area networks (WLAN). In such systems, a ferry moves in a predetermined path and stops to communicate with nodes with either send or receive requests. This application highlights the versatility of continuous polling models in handling spatially distributed tasks across different fields. The extension of batch arrivals in our research is also relevant to the analysis conducted in [10]. In the context of ferry-assisted WLANs, one can envision scenarios where users need to send data packets to multiple recipients, randomly located along the ferry's path. This scenario aligns well with our model of batch arrivals on a circle, providing a robust framework for analyzing such network systems.

Polling on a circle, see, for example, [5, 11, 12], assumes that customers arrive according to a Poisson process and obtain locations according to a continuous distribution on the circle. The former two papers assume that these locations are uniform, while the latter considers an extension where the customers arrive according to a certain distribution with respect to the server. [7] extends the model to a so-called "snowblower model", which assumes a more general arrival process of jobs (excluding batch arrivals) and allows for a stochastic travel time over intervals. Using an analytical approach, the author derives some results on the amount of work in intervals; in particular, the expected amount of work in an interval is found. In later work, [8] proved that these snowblower models are limiting cases of discrete polling systems, where the number of queues tends to infinity. As a consequence, these continuous models can be useful for the approximation of large and complex polling models. Especially when analytical closed-form expressions can be found in the continuous model, these approximations may prove to be useful. For example, no method for finding the mean waiting time in a discrete polling model has been found that is faster than solving a system of N^2 linear equations, where N is the number of queues [22]. In the continuous model, on the other hand, an analytical expression can be found. On the other hand, obtaining more intricate performance measures than the mean is more challenging in polling on a circle.

A second extension discusses batch arrivals, under which groups of customers arrive simultaneously [14, 18]. Again, the customers are randomly placed in the different queues. In the current paper, we assume that these locations are uniformly distributed and independent of one another. Both [14] and [18] provide a mean value analysis for these discrete polling models with batch arrivals. [4] derives a conservation law for the weighted mean waiting time of customers in the system, making use of a stochastic decomposition. Other research on this topic includes that of [20] and [1]. The analysis of the batch sojourn time in polling systems is, as far as we know, restricted to discrete systems, see [18]. In the said paper, the authors provide a mean value analysis for the mean sojourn time of a batch, again requiring one to solve a system of N^2 linear equations.

In the current paper, we combine both extensions, viz. polling on a circle and batch arrivals, and present a mean value analysis for the batch sojourn time and time to delivery of a batch of customers. For this, we analyze the number of waiting customers in the system within a certain distance of the server. Using Little's law and an explicit expression for the expected waiting time, we derive an integral equation for the expected number of customers at distance x of the server. Our main contributions are (i) a novel technique for analyzing how, on average, waiting customers are spread on the circle and (ii) closed-form expressions for both the mean batch sojourn time and average time to delivery which lend themselves for optimization purposes. The techniques in this paper allow for the analysis of related systems, like systems with generally distributed arrival locations of customers or, in a sense equivalently, non-uniform server travel speed. The application of the approach to extensions of the model, however, is not straightforward and might not result in (exact) analytical expressions for the mean batch sojourn time and time to delivery.

The paper is built up as follows: Section 2 is devoted to the description of the model and corresponding definitions. We determine the steady-state mean number of waiting customers in Section 3. Section 4 is devoted to the determination of the mean of the total sojourn time of a batch of customers, and Section 5 discusses the mean time to delivery. Some numerical results are presented in Section 6.

2. Model description and preliminaries

Consider a circle with circumference 1, and assume that batches of customers arrive at this circle according to a Poisson process with intensity λ. Each arriving customer in the batch is assigned a location on this circle according to the uniform distribution, independent of all other customers. The size of a customer batch, K, is assumed to be strictly positive and to follow a known distribution with probabilities p_k and probability generating function $\tilde{K}(\cdot)$.

Service is provided by a single traveling server. We assume that the server travels at a fixed speed α^{-1} in clock-wise direction and serves any customer she encounters, taking a randomly distributed time

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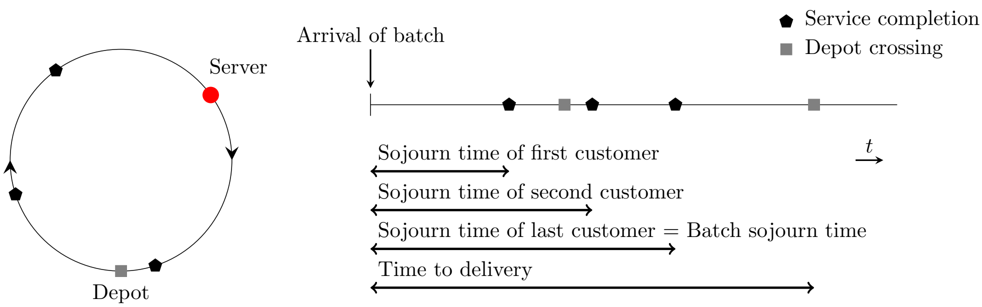

Figure 1. *Illustration of the batch sojourn time and time of delivery.*

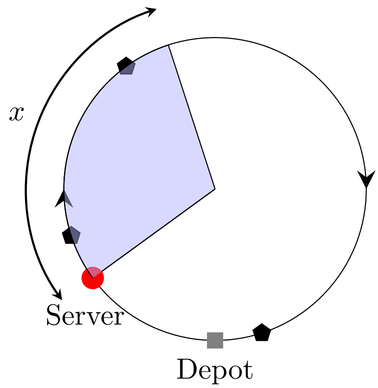

Figure 2. *Illustration of the polling model and corresponding range within distance x of the server. In this example,* $L(x) = 2$*.*

B. After finishing the service, the server continues traveling in the same direction. A depot is located at a fixed location on the circle, where the server delivers all served customers each time she passes this point. The time this takes is assumed to be negligible.

Extending the methodology of [12, Theorem 3.1] to batch arrivals shows that the system is stable when $\rho := \lambda\mathbb{E}[K]\mathbb{E}[B] < 1$, as is intuitively logical. Throughout this paper, we consider the steady-state behavior of the system and in particular focus on three performance statistics: (i) the steady-state number of waiting customers in the system; (ii) the long-run sojourn time of a job, S^B, that is, the time from the arrival of the batch of customers to the service completion of the last customer in that batch; and (iii) the long run time to delivery D of a batch, that is, the time from the arrival of the batch of customers to the delivery of the last customer in the batch at the depot (see Figure 1). We define the distance between a point on the circle and the server as the distance that the server has to travel to reach that point, that is, the clock-wise distance, see Figure 2. Throughout this paper, we rely on the symmetry of the model and often instead use that *arriving customers obtain locations at a uniform distance from the server*. We further focus on customers at a certain distance to the server, rather than at a location on the circle, and let $L(x)$ denote the steady-state number of waiting customers ahead of the server and within distance x of the server, see Figure 2; we often write the shorthand L for all waiting customers in the system (i.e. $L = L(1)$).

The long-run average number of waiting customers within distance x of the server is defined as $\mathbb{E}[L(x)]$. We further define the derivative of this function by f:

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{E}\big[L(x)\big]. \tag{2.1}$$

The function $f(x)$ describes how, on average, the waiting customers are spread on the circle, with respect to the server. Intuitively, one can interpret $f(x)\mathrm{d}x$ as the expected number of waiting customers with

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distances in a small interval $[x, x + dx]$ ahead of the server. We will later prove that $f(x) = a + b(1 - x)$ with $a, b > 0$. The existence of this function f is later discussed in the proof of Proposition 4.13.

3. The mean number of waiting customers

In this section, we derive an expression for the expectation of the steady-state number of waiting customers in the system. We use an approach based on mean value analysis. In two remarks, we briefly outline two other approaches to derive this result.

Lemma 3.1. *The expected steady-state number of waiting customers in the system is given by*

$$\mathbb{E}[L] = \frac{\lambda \mathbb{E}[K]}{2(1-\rho)}\left(\alpha + \rho\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} + \mathbb{E}[B]\frac{\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right). \tag{3.1}$$

Proof. According to Little's law ([21]), $\mathbb{E}[L] = \lambda\mathbb{E}[K]\mathbb{E}[W]$, where W is the steady-state waiting time of an arbitrary customer. Furthermore, $W = W_0 + W_1$, where W_0 is the travel time of the server during the waiting time and W_1 is the busy time of the server during the waiting time. Clearly, $\mathbb{E}[W_0] = \alpha/2$ because customers arrive uniformly on the circle. To determine $\mathbb{E}[W_1]$, we interchange locations of customers on the circle to preserve service in order of arrival. When a new batch of customers arrives, the arriving customers swap their locations with customers they would otherwise overtake, such that the order of service is equal to the order of arrival. Since all customers have the same service time distribution, interchanging locations of customers does not alter the number of customers waiting in the system, and hence, by Little's law, it also does not alter the mean waiting time. In case of service in order of arrival, we have

$$\mathbb{E}[W_1] = \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \mathbb{E}[L]\mathbb{E}[B] + \frac{\mathbb{E}[K(K-1)]}{2\mathbb{E}[K]}\mathbb{E}[B], \tag{3.2}$$

where the first term is the contribution of the customer in service, if any, the second term is the contribution of all the waiting customers at the moment that our arbitrary customer arrives, and the third term is the contribution of the customers who arrive in the same batch as our arbitrary customer and are served before the arbitrary customer; on average, this entails $\frac{\mathbb{E}[K(K-1)]}{2\mathbb{E}[K]}$ customers. Combined with $\mathbb{E}[L] = \lambda\mathbb{E}[K]\mathbb{E}[W_0 + W_1]$ and $\mathbb{E}[W_0] = \alpha/2$, the expression for (3.1) follows from (3.2). □

Remark 3.2. In discrete symmetric polling systems, an expression for the expected waiting time can be found directly from the pseudo-conservation law with batch arrivals, cf. Equation (3.21) of [4]. Due to the many-queue limit of the discrete polling system [8], the expected waiting time can be found by taking the limit of the number of queues to infinity, using the discrete system where customers arrive at the queues uniformly at random.

Remark 3.3. The lemma can also be proven by using the stochastic decomposition in [4, Theorem 2.1]: the expected amount of work in a queue is given by the sum of (i) the expected steady-state amount of work in a corresponding system without travel time and (ii) the expected amount of work at an epoch at which the server is traveling. As the current system is restricted to a single type of customers, the expected amount of work in a queue is $\mathbb{E}[B]$ times the mean queue length.

Term (i) follows from a known result for the $M/G/1$ queue with batch arrivals (see, e.g., [6], Section 5.10) and equals the last two components of (3.1). Term (ii) can be argued to equal the mean number of customers *at the end of a cycle* (one complete tour along the circle) of the server. A balancing argument can then be used to argue that the expected amount of work at a travel epoch equals the first component of (3.1).

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4. The mean sojourn time of a batch

In this section, we derive the expected sojourn time, $\mathbb{E}[S^B]$, of a batch of customers, that is, the time from the arrival of a batch until the service completion of its last customer. First, one should realize that the service of batches is pre-empted by travel periods, as the server has to travel between the customers in the same batch. Moreover, it might happen that other batches are (partially) served between the service of two customers in the same batch. Because of that, one cannot apply the stochastic decomposition nor the pseudo-conservation law in [4] to the number of batches in the system. Instead, we propose a direct analysis of the batch sojourn time. Using mean value analysis, we prove the following statement:

Theorem 4.1. *The expected batch sojourn time satisfies*

$$\begin{aligned}\mathbb{E}[S^B] = \frac{1}{1-\rho}\cdot\left[\alpha + \rho^2\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right]\mathbb{E}\left[\frac{K}{K+1}\right] \\ + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{1}{\lambda}\Big(\exp(\rho)-1\Big) \\ + \left(\rho\mathbb{E}[B] + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right)\int_{x=0}^{1}\exp(\rho x)\left[\tilde{K}(x)-1\right]\mathrm{d}x.\end{aligned} \tag{4.1}$$

Remark 4.2. Taking $\mathbb{E}[B] = \epsilon \to 0$ gives a total mean batch sojourn time of $\alpha\cdot\mathbb{E}[K/(K+1)]$. Note that this is exactly equal to the expected travel time of the server to the furthest customer.

Remark 4.3. The case $K \equiv 1$, that is, unit batch sizes, results in

$$\mathbb{E}[S^B] = \mathbb{E}[B] + \frac{\lambda\mathbb{E}[B^2]}{2(1-\rho)} + \frac{\alpha}{2(1-\rho)}.$$

This is as expected as the system now simplifies to the system discussed in [11]. Furthermore, as $\mathbb{E}[S^B] = \mathbb{E}[B] + \mathbb{E}[W] = \mathbb{E}[B] + \frac{1}{\lambda}\mathbb{E}[L]$ in this case, one can also verify this with (3.1).

Remark 4.4. Under light traffic, that is, $\lambda \to 0$, the last integral of (4.1) simplifies to $-\mathbb{E}[K/(K+1)]$. Further realizing that $[\exp(\lambda\mathbb{E}[K]\mathbb{E}[B]) - 1]/\lambda \to \mathbb{E}[K]\mathbb{E}[B]$ as $\lambda \to 0$ shows that

$$\mathbb{E}[S^B] \to \alpha\mathbb{E}\left[\frac{K}{K+1}\right] + \mathbb{E}[K]\mathbb{E}[B],$$

that is, in light traffic, the mean batch sojourn time reduces to the sum of the mean travel time to the furthest customer plus the mean service time of a batch.

We prove Theorem 4.1 by using two main ingredients. Firstly, in Proposition 4.5, we obtain an exact formula for the mean batch sojourn time conditional on the batch size K and the distance, X^B, from the server to the furthest customer in the batch. This expression still depends on $f(x)$ (see (2.1)). Secondly, we derive an expression for this function $f(x)$ (Proposition 4.6). This allows for the immediate derivation of the mean batch sojourn time.

Proposition 4.5. *The expected batch sojourn time satisfies*

$$\mathbb{E}[S^B] = \int_{x=0}^{1}\sum_{k=1}^{\infty} p_k k x^{k-1}\mathbb{E}[S^B | X^B = x, K = k]\mathrm{d}x, \tag{4.2}$$

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with

$$\begin{aligned}\mathbb{E}[S^B|X^B = x, K = k] = \mathbb{E}[B] + \frac{\alpha}{\rho}\Big[\exp(\rho x) - 1\Big] + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho x)\\ + \frac{(k-1)\mathbb{E}[B]}{x\rho}\Big[\exp(\rho x) - 1\Big] + \int_{y=0}^{x}\mathbb{E}[B]\exp(\rho(x-y))f(y)\mathrm{d}y.\end{aligned} \tag{4.3}$$

Proposition 4.6. *The function $f(\cdot)$, defined in (2.1), is given by*

$$f(x) = \rho\frac{\lambda\mathbb{E}[K]\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\alpha\lambda\mathbb{E}[K]}{1-\rho}(1-x) + \rho^2\frac{\lambda\mathbb{E}[K]\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]}(1-x) + \frac{\rho\mathbb{E}[K(K-1)]}{(1-\rho)\mathbb{E}[K]}(1-x). \tag{4.4}$$

The proofs of Propositions 4.5 and 4.6 are deferred to dedicated sections. We first prove Theorem 4.1, using both propositions.

Proof of Theorem 4.1. We start by focusing on the last term of (4.3) and apply partial integration, using that $f(x)$ has the form $f(x) = a + b(1-x)$:

$$\begin{aligned}\int_{y=0}^{x}\exp\big(\rho(x-y)\big)\big(a+b(1-y)\big)\mathrm{d}y &= \Big[-\frac{a+b(1-y)}{\rho}\exp\big(\rho(x-y)\big)\Big]_{y=0}^{x} - \int_{y=0}^{x}\frac{b}{\rho}\exp\big(\rho(x-y)\big)\mathrm{d}y\\ &= -\frac{a+b(1-x)}{\rho} + \frac{a+b}{\rho}\exp\big(\rho x\big) + \frac{b}{\rho^2}\Big[1-\exp\big(\rho x\big)\Big]\\ &= \frac{b}{\rho}x + \frac{1}{\rho}\Big[a - \frac{b(1-\rho)}{\rho}\Big]\Big[\exp\big(\rho x\big) - 1\Big].\end{aligned}$$

Consequently, using (4.4), we obtain

$$\begin{aligned}\int_{y=0}^{x}\mathbb{E}[B]\exp(\rho(x-y))f(y)\mathrm{d}y = \Big[\frac{\alpha}{1-\rho} + \frac{\rho^2\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{(1-\rho)\mathbb{E}[K]}\Big]x\\ + \frac{1}{\rho}\Big[-\alpha - \frac{\rho^2\mathbb{E}[B^2]}{2\mathbb{E}[B]} - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\Big]\Big[\exp(\rho x) - 1\Big].\end{aligned}$$

We substitute this in (4.3), grouping similar factors:

$$\begin{aligned}\mathbb{E}[S^B|X^B = x, K = k] = \mathbb{E}[B] + \frac{\alpha}{\rho}\Big[\exp(\rho x) - 1\Big] - \frac{\alpha}{\rho}\Big[\exp(\rho x) - 1\Big] + \frac{\alpha x}{1-\rho}\\ + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho x) - \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\Big[\exp(\rho x) - 1\Big] + \frac{\rho^2\mathbb{E}[B^2]x}{(1-\rho)\mathbb{E}[B]}\\ + \frac{(k-1)\mathbb{E}[B]}{x\rho}\Big[\exp(\rho x) - 1\Big]\\ - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\Big[\exp(\rho x) - 1\Big] + \frac{\rho\mathbb{E}[B]\mathbb{E}[K(K-1)]x}{(1-\rho)\mathbb{E}[K]}.\end{aligned}$$

Remark that many terms cancel, resulting in

$$\begin{aligned}\mathbb{E}[S^B|X^B = x, K = k] = \mathbb{E}[B] + \frac{\alpha x}{1-\rho} + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\rho^2\mathbb{E}[B^2]x}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]x}{\mathbb{E}[K](1-\rho)}\\ + \frac{(k-1)\mathbb{E}[B]}{\rho x}\Big[\exp(\rho x) - 1\Big] - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\Big[\exp(\rho x) - 1\Big].\end{aligned}$$

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We return to Equation (4.2) and decondition the expression above with respect to the batch size and location of the furthest customer. Rewriting everything in terms of the probability generating function of K, $\tilde{K}(x) := \sum_{k=1}^{\infty} p_k x^k$, gives

$$\begin{aligned}
\mathbb{E}[S^B] &= \mathbb{E}[B] + \int_{x=0}^{1} \sum_{k=1}^{\infty} k p_k x^{k-1} \cdot \Bigg[\frac{\alpha x}{1-\rho} + \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\rho^2 \mathbb{E}[B^2] x}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]x}{\mathbb{E}[K](1-\rho)} \\
&\qquad + \frac{(k-1)\mathbb{E}[B]}{\rho x}\big[\exp(\rho x) - 1\big] - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \\
&\qquad \times \big[\exp(\rho x) - 1\big]\Bigg] \mathrm{d}x \\
&= \mathbb{E}[B] + \int_{x=0}^{1} \left[\frac{\alpha}{1-\rho} + \frac{\rho^2 \mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)} \right] \tilde{K}'(x) x \mathrm{d}x \\
&\quad + \int_{x=0}^{1} \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} \tilde{K}'(x) \mathrm{d}x + \int_{x=0}^{1} \frac{\mathbb{E}[B]}{\rho} \big[\exp(\rho x) - 1\big] \tilde{K}''(x) \mathrm{d}x \\
&\quad - \int_{x=0}^{1} \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \big[\exp(\rho x) - 1\big] \tilde{K}'(x) \mathrm{d}x.
\end{aligned}$$

We now apply partial integration to the second, fourth, and last terms:

$$\begin{aligned}
\int_{x=0}^{1} \tilde{K}'(x) x \mathrm{d}x &= \Big[x\tilde{K}(x) \Big]_{x=0}^{1} - \int_{x=0}^{1} \tilde{K}(x) \mathrm{d}x = \mathbb{E}\Big[\frac{K}{K+1} \Big]; \\
\int_{x=0}^{1} \tilde{K}''(x) \big[\exp(\rho x) - 1\big] \mathrm{d}x &= \Big[\big[\exp(\rho x) - 1\big] \tilde{K}'(x) \Big]_{x=0}^{1} - \int_{x=0}^{1} \rho \exp(\rho x) \tilde{K}'(x) \mathrm{d}x \\
&= \big[\exp(\rho) - 1\big] \mathbb{E}[K] - \int_{x=0}^{1} \rho \exp(\rho x) \tilde{K}'(x) \mathrm{d}x; \\
\int_{x=0}^{1} \tilde{K}'(x) \exp(\rho x) \mathrm{d}x &= \exp(\rho) - \int_{x=0}^{1} \rho \exp(\rho x) \tilde{K}(x) \mathrm{d}x.
\end{aligned}$$

Applying these to the former expression of $\mathbb{E}[S^B]$ results in

$$\begin{aligned}
\mathbb{E}[S^B] &= \mathbb{E}[B] + \left[\frac{\alpha}{1-\rho} + \frac{\rho^2 \mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)} \right] \mathbb{E}\Big[\frac{K}{K+1} \Big] \\
&\quad + \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\mathbb{E}[B]}{\rho} \Big(\exp(\rho)\mathbb{E}[K] - \mathbb{E}[K] - \rho \exp(\rho) + \rho \int_{x=0}^{1} \rho \exp(\rho x) \tilde{K}(x) \mathrm{d}x \Big) \\
&\quad - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \Big(\exp(\rho) - 1 - \int_{x=0}^{1} \rho \exp(\rho x) \tilde{K}(x) \mathrm{d}x \Big).
\end{aligned}$$

Grouping terms with $\mathbb{E}[B]$ and $\mathbb{E}[K(K-1)]$ now completes the proof. □

It is left to prove the remaining Propositions 4.5 and 4.6. Section 4.1 is devoted to the proof of the former. By linking future arrivals to the state of the system at an arrival epoch, we are able to derive a first expression for the batch sojourn time. The proof of Proposition 4.6, see Section 4.2, is quite intricate. Based on an observation related to Little's law, one can derive an integral equation for f. It is then left to prove that the aforementioned formula for f is the unique solution to the said fixed point equation.

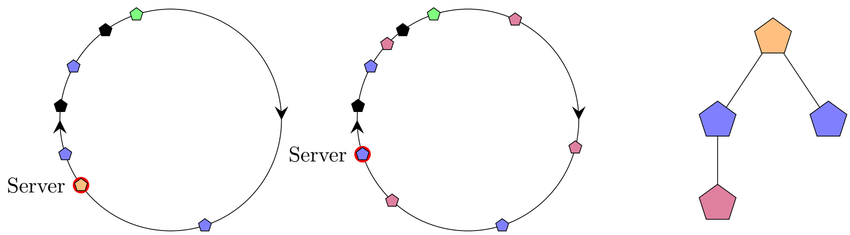

Figure 3. *Illustration of the extra waiting time of a tagged customer (green) that is generated by a service (of the orange customer) and the corresponding branching process. During the service of the orange customer, blue customers arrive, of which only the first two are considered. During the service of the first blue customer, the red customers arrive, of which only one will be served before the tagged customer.*

4.1. Proof of Proposition 4.5

In this section, we prove Proposition 4.5 by linking future arrivals to the current state of the system. Using this, one can derive an exact expression for the mean batch sojourn time.

The sojourn time of a batch of customers may include service times of customers who arrive at a later time point. That being the case, merely considering the state of the system at the arrival instant is not enough. We therefore link future arrivals in the system to the state of the system at the arrival moment. Inspired by [15], we construct a branching process that relates customers according to the following rules:

- A customer (a) is the offspring of a customer (b) when (a) arrives during the service of (b).
- A customer (a) is called an immigrant when she arrives during a travel period of the server.

From the perspective of a waiting customer, however, only customers who will be served before her are of interest. We therefore trim the branching process to only consider the customers who are served before the tagged customer, see Figure 3. As the offspring distribution of a customer now depends on the distance from this customer to the tagged customer, we take this distance as the type of a customer. Using this trimmed branching process, we introduce the following two definitions.

Definition 4.7. *S(x) denotes the service time of a customer in front and at distance x of the tagged customer plus the service times of all its descendants in the trimmed branching process. This is also referred to as the extra waiting time generated by a service at distance x.*

Definition 4.8. *T(x) denotes the added service times of all customers in the branching processes of immigrants, arriving during the travel time of the server over a distance x to the tagged customer. We call this the extra waiting time generated by traveling over a distance x.*

The expectations of these random variables can be found by conditioning with respect to the size of the offspring, resulting in an integral equation for the first moment of $S(x)$, which allows us to obtain $\mathbb{E}[S(x)]$ and $\mathbb{E}[T(x)]$:

Lemma 4.9. *The expected extra waiting time generated by a service at distance x of a customer,* $\mathbb{E}[S(x)]$*, is given by*

$$\mathbb{E}[S(x)] = \mathbb{E}[B]\exp(\rho x), \quad \forall\, 0 \le x \le 1. \tag{4.5}$$

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The mean total extra waiting time generated by traveling over a distance x to the customer, $\mathbb{E}[T(x)]$*, satisfies*

$$\mathbb{E}[T(x)] = \frac{\alpha}{\rho}\Big[\exp(\rho x) - 1\Big], \quad \forall\, 0 \leq x \leq 1. \tag{4.6}$$

Proof. The mean number of customer arrivals during a service of a customer equals $\lambda\mathbb{E}[K] \cdot \mathbb{E}[B] = \rho$ (because on average $\lambda\mathbb{E}[K]$ customers arrive each time unit). Each of these customers arrive uniformly on the circle. If a customer arrives at distance y of the tagged customer, that customer adds an extra amount $S(y)$ to the waiting time of our tagged customer. Hence,

$$\mathbb{E}[S(x)] = \mathbb{E}[B] + \rho \int_{y=0}^{x} \mathbb{E}[S(y)]\mathrm{d}y \Longrightarrow \mathbb{E}[S(x)] = \mathbb{E}[B]\exp(\rho x),$$

where the last step follows from the fact that $\mathbb{E}[S(0)] = \mathbb{E}[B]$.

We can now use the above expression to prove (4.6). The server takes a time $\alpha\mathrm{d}u$ to travel a small distance $\mathrm{d}u$. Therefore, in expectation, $\lambda\mathbb{E}[K]\alpha\mathrm{d}u$ customers arrive during the traveling of the said distance. Each of these arrivals adds a total waiting time of $S(y)$ when arriving between the tagged customer and the server at a distance y to the tagged customer:

$$\mathbb{E}[T(x)] = \alpha x + \int_{u=0}^{x} \lambda\mathbb{E}[K]\alpha \int_{y=0}^{u} \mathbb{E}[S(y)]\mathrm{d}y\mathrm{d}u = \alpha x + \alpha\lambda\mathbb{E}[K] \cdot \int_{u=0}^{x} \frac{1}{\lambda\mathbb{E}[K]}\Big[\exp(\rho u) - 1\Big]\mathrm{d}u$$
$$= \frac{\alpha}{\rho}\Big[\exp(\rho x) - 1\Big].$$

□

Remark 4.10. At the arrival instant of a customer, she might encounter a residual service time, B^R, first. The total expected extra waiting time generated by this residual service time, $\mathbb{E}[S^R(x)]$, satisfies relation (4.5), with $\mathbb{E}[B]$ replaced by $\mathbb{E}[B^R] = \mathbb{E}[B^2]/(2\mathbb{E}[B])$.

Using this construction, and the result above, we prove Proposition 4.5.

Proof of Proposition 4.5. The main idea behind this proof is that the waiting time of a customer at distance x of the server can be split into four elements: (i) the total extra waiting time generated by the traveling of the server over distance x: $T(x)$; (ii) the extra waiting time generated by the customer currently in service, if any: with probability ρ, there is a contribution $S^R(x)$, where $S^R(x)$ denotes the total extra waiting time generated by the *residual* service of a customer; (iii) the extra waiting time generated by all customers in the same batch $S(Y_1)+\ldots+S(Y_{k-1})$, where $Y_1, \ldots, Y_{k-1}$ denote the distance of these (unordered) customers to the tagged customer—remark that these are uniform on $[0, x]$; and (iv) the extra waiting time generated by all customers present at the time of arrival and in front of the tagged customer. A customer at distance y of the server has a distance $x - y$ to the tagged customer and hence generates a waiting time of $S(x - y)$. As the expected batch sojourn time is equal to the expected waiting time of the furthest customer in the batch plus the service time of this customer, we obtain

$$\mathbb{E}\big[S^B \big| X^B = x, K = k\big] = \mathbb{E}[B] + \mathbb{E}[T(x)] + \rho\mathbb{E}[S^R(x)] + (k-1)\int_{y=0}^{x} \frac{1}{x}\mathbb{E}[S(y)]\mathrm{d}y$$
$$+ \int_{y=0}^{x} \mathbb{E}\big[S(x-y)\big] f(y)\mathrm{d}y,$$

where we used the extra waiting time generated by a service, $S(x - y)$, which is independent of the current state of the system. □

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4.2. Proof of Proposition 4.6

In this section, we derive an explicit expression for the function f. The steps for this derivation can also be applied to extensions of this (continuous) polling model. Furthermore, the function f itself might be a valuable approximation of the average queue lengths in discrete polling systems, conditional on the server's location. The main ideas for the proof are itemized below:

- Step 1 Using a variant of Little's law, relate the number of customers in particular intervals to the waiting time of customers.
- Step 2 Using the result from step 1, construct an integral equation for f.
- Step 3 Find a solution to the integral equation and prove its uniqueness.

Lemma 4.11. *Let $W(x)$ be the waiting time of an arbitrary customer arriving within distance x ahead of the server. Then, we have*

$$\mathbb{E}[L(1)] - \mathbb{E}[L(1-x)] = \lambda\mathbb{E}[K]x\mathbb{E}\big[W(x)\big]. \tag{4.7}$$

Proof. This is a variant of Little's law. To argue (4.7), we use a money argument, along the lines of [17, page 51]. Suppose that each customer obtains \$1 for each time unit they spend at a distance of at least $1-x$ ahead of the server. One can pay each customer per time unit, which amounts to $\mathbb{E}[L(1)] - \mathbb{E}[L(1-x)]$ in expectation per unit of time (observe that $L(1) = L$).

Equivalently, one could pay each customer upon leaving the interval $[1-x, 1]$, which happens when the server moves closer to the customer such that this customer is now within distance $1-x$ of the server. Each customer leaving this interval should now receive a dollar for each time unit she spends in this interval: denoted as $\tau_{[1-x,1]}$. As the departure rate from this interval has to equal the arrival rate to this interval (due to the stability of the process), this implies that each time unit you expect to pay $\lambda\mathbb{E}[K]x$ customers.

Since both payout methods are equivalent, we thus have

$$\mathbb{E}[L] - \mathbb{E}[L(1-x)] = \lambda\mathbb{E}[K]x\mathbb{E}[\tau_{[1-x,1]}].$$

Now consider a customer arriving at distance $1-x+y$ and an imaginary customer arriving at y. Then the total time the first customer spends in the interval $[1-x, 1]$ is equal to the total time it takes the server to reach point y and hence the waiting time of the imaginary customer. Therefore, due to the symmetry of the model, we have $\mathbb{E}[\tau_{[1-x,1]}] = \mathbb{E}\big[W(x)\big]$. □

Remark 4.12. This observation is similar to that in [22, Eq. (6)], regarding discrete polling models. The equivalence becomes especially apparent when one no longer focuses on customers at distance x of the server but rather on the customers at a certain fixed location.

Lemmas 4.9 and 4.11 form the main ingredients for the integral equation that we construct in this subsection. We write the expected waiting time $\mathbb{E}[W(x)]$ as an integral by conditioning over the arrival location of the tagged customer, and by then taking the derivative with respect to x on both sides of (4.7), we find

Proposition 4.13. *The function $f(x)$ satisfies the following integral equation:*

$$\begin{aligned} f(1-x) = \lambda\mathbb{E}[K]\bigg\{&\frac{\alpha}{\rho}\big[\exp(\rho x) - 1\big] + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho x) + \mathbb{E}[B]\frac{\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\Big[\exp(\rho x) - 1\Big] \\ &+ \int_{z=0}^{x}\mathbb{E}[B]\exp\big(\rho(x-z)\big)f(z)\mathrm{d}z\bigg\}, \quad 0 \le x < 1. \end{aligned} \tag{4.8}$$

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Proof. First, we focus on $\mathbb{E}[W(x)]$ and use that the waiting time consists of the same four parts as before: (i) the extra waiting time generated by the traveling of the server to the customer; (ii) the extra waiting time generated by the possible residual service time; (iii) the total extra waiting time generated by the customers in the same batch who arrive in front of the tagged customer. On average, a tagged customer arrives with a total of $\mathbb{E}[K(K-1)]/\mathbb{E}[K]$ siblings, independent of the arrival location of this tagged customer. A customer arriving a distance z of the server generates an extra expected waiting time of $\mathbb{E}[S(u-z)]$ if $z < u$ and 0 otherwise; (iv) the extra waiting time generated by customers already present at the moment of arrival. Conditioning on the exact location of the customer shows

$$\mathbb{E}\big[W(x)\big] = \frac{1}{x}\int_{u=0}^{x}\Bigg\{\underbrace{\mathbb{E}[T(u)]}_{\text{(i)}} + \underbrace{\rho\mathbb{E}[S^R(u)]}_{\text{(ii)}} + \underbrace{\frac{\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\int_{z=0}^{u}\mathbb{E}[S(u-z)]\mathrm{d}z}_{\text{(iii)}} + \underbrace{\int_{z=0}^{u}\mathbb{E}[S(u-z)]f(z)\mathrm{d}z}_{\text{(iv)}}\Bigg\}\mathrm{d}u, \tag{4.9}$$

and hence

$$x\mathbb{E}\big[W(x)\big] = \int_{u=0}^{x}\Bigg\{\frac{\alpha}{\rho}\big[\exp(\rho u) - 1\big] + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho u) + \mathbb{E}[B]\frac{\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\Big[\exp(\rho u) - 1\Big] + \int_{z=0}^{u}\mathbb{E}[B]\exp\big(\rho(u-z)\big)f(z)\mathrm{d}z\Bigg\}\mathrm{d}u.$$

By writing the left-hand side of (4.7) as an integral over $f(u)$, that is, $\mathbb{E}[L] - \mathbb{E}[L(1-x)] = \int_{u=1-x}^{1} f(u)\mathrm{d}u$, we obtain the following equality:

$$\int_{u=1-x}^{1} f(u)\mathrm{d}u = \lambda\mathbb{E}[K]\int_{u=0}^{x}\Bigg\{\frac{\alpha}{\rho}\big[\exp(\rho u) - 1\big] + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho u) + \mathbb{E}[B]\frac{\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\Big[\exp(\rho u) - 1\Big] + \int_{z=0}^{u}\mathbb{E}[B]\exp\big(\rho(u-z)\big)f(z)\mathrm{d}z\Bigg\}\mathrm{d}u.$$

Taking the derivative on both sides results in (4.8). □

Corollary 4.14. *Any solution f to the fixed point equation is linear, that is, satisfies* $f''(x) = 0$.

Proof. Remark that the right-hand side of (4.8) is differentiable, and therefore $f(1-x)$ is. This results in

$$-f'(1-x) = \rho\lambda\mathbb{E}[K]\Bigg\{\frac{\alpha}{\rho} + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \mathbb{E}[B]\frac{\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\exp(\rho x) + \int_{z=0}^{x}\mathbb{E}[B]\exp\big(\rho(x-z)\big)f(z)\mathrm{d}z + \frac{\mathbb{E}[B]}{\rho}f(x)\Bigg\}.$$

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Clearly, the right-hand side again is differentiable. In particular, under the assumption that the $n-1$-th derivative exists, the following holds:

$$(-1)^n f^{(n)}(1-x) = \rho^n \lambda \mathbb{E}[K] \left\{ \frac{\alpha}{\rho} + \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} + \mathbb{E}[B] \frac{\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \exp(\rho x) \right. \\ \left. + \int_{z=0}^{x} \mathbb{E}[B] \exp\big(\rho(x-z)\big) f(z) \mathrm{d}z \right\} + \sum_{i=0}^{n-1} \rho^{n-i} f^{(i)}(x).$$

It then immediately follows that the n-th derivative of f exists. Therefore, by induction, the function f is infinitely differentiable.

Now, remark that the first derivative can also be rewritten as follows:

$$-f'(1-x) = \rho \lambda \mathbb{E}[K] \left\{ \frac{\alpha}{\rho} \big[\exp(\rho x) - 1 \big] + \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} \exp(\rho x) + \mathbb{E}[B] \frac{\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \big[\exp(\rho x) - 1 \big] \right. \\ \left. + \int_{z=0}^{x} \mathbb{E}[B] \exp\big(\rho(x-z)\big) f(z) \mathrm{d}z \right\} + \alpha \lambda \mathbb{E}[K] + \lambda \mathbb{E}[B] \mathbb{E}[K(K-1)] + \rho f(x) \\ = \rho f(x) + \alpha \lambda \mathbb{E}[K] + \lambda \mathbb{E}[B] \mathbb{E}[K(K-1)] + \rho f(1-x).$$

Consequently, the second derivative satisfies

$$f''(1-x) = -\rho f'(1-x) + \rho f'(x) = \rho^2 f(1-x) + \rho^2 f(x) - \rho^2 f(x) - \rho^2 f(1-x) = 0.$$

□

Now, we know that the average number of waiting customers at distance x of the server is linear in x. This can also be argued intuitively. All customers at a distance x of the server need to have arrived since the last visit to the said point. During this time, the server has traversed a total distance of $1-x$. The symmetry of the model suggests that the time the server takes, on average, to traverse this distance is proportional to that distance.

With these results, we can now prove Proposition 4.6.

Proof of Proposition 4.6. Since the solution is known to be linear, we propose $f(1-x) = a + bx$ in (4.8). By partial integration, we have that

$$\int_{z=0}^{x} \rho \exp\big(\rho(x-z)\big) f(z) \mathrm{d}z = \Big[-\big\{a + b(1-z)\big\} \cdot \exp\big(\rho(x-z)\big) \Big]_{z=0}^{x} - b \int_{z=0}^{x} \exp\big(\rho(x-z)\big) \mathrm{d}z \\ = -b(1-x) - a + (a+b)\exp(\rho x) + \frac{b}{\rho}\big[1 - \exp(\rho x)\big] \\ = bx + \left(a - \frac{b(1-\rho)}{\rho}\right)\big[\exp(\rho x) - 1\big].$$

Solving for a, b results in a linear system of 2 equations and 2 unknowns as the bx cancels against the left-hand side in (4.8):

$$\frac{b(1-\rho)}{\rho} - 2a = \frac{\lambda \mathbb{E}[K]\alpha}{\rho} + \frac{\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}$$
$$\frac{b(1-\rho)}{\rho} - a = \frac{\lambda \mathbb{E}[K]\alpha}{\rho} + \frac{\lambda \mathbb{E}[K]\rho \mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}.$$

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Note that this is a system of independent linear equations, and hence at most one solution can be found. Further note that this system of equations is solved by

$$a = \frac{\lambda \mathbb{E}[K]\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]}, \quad b = \frac{1}{1-\rho}\left(\alpha\lambda\mathbb{E}[K] + \rho^2\frac{\lambda\mathbb{E}[K]\mathbb{E}[B^2]}{\mathbb{E}[B]} + \frac{\rho\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right),$$

as in (4.4). □

5. Time to delivery

The polling system in the current paper is inspired by the so-called milk-run systems. In these systems, a picker continuously walks through the entire warehouse, being updated in real time with all pick locations. Any requested item that the picker encounters is picked and afterward delivered at a depot (located at a fixed location). Customer orders, often, include more than a single item. From the perspective of an incoming order, it is therefore important to analyze the performance of the *batch* of items. In these systems, the time it takes the picker to deliver these items is especially of interest, rather than the batch sojourn time.

In the polling system, this performance measure is the time it takes the picker to get to the depot after having served all customers in a batch. We denote this time as D. The expected time to delivery follows from similar derivations as for the mean batch sojourn time. First, we propose an initial expression for the mean time to delivery, dependent on the function f. By substituting f into this expression, $\mathbb{E}[D]$ follows.

Theorem 5.1. *The expected time to delivery of a batch of customers is given by*

$$\begin{aligned}\mathbb{E}[D] = {} & \frac{1}{1-\rho}\cdot\left[\alpha + \rho^2\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right]\mathbb{E}\left[\frac{K}{K+1} + \frac{1}{2}\right] + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} \\ & + \frac{1}{\rho\lambda}\left[\exp(2\rho) - \rho\exp(\rho) - 1 - \rho\right] - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho^2\mathbb{E}[K]}\left[\exp(\rho) - 1 - \rho\right] \\ & + \frac{\rho\mathbb{E}[B^2]}{\mathbb{E}[B]}\int_{u=0}^{1}\left[\tilde{K}(u) - 1\right]\left[\exp(\rho u) - 1\right]\mathrm{d}u \\ & + \mathbb{E}[B]\int_{u=0}^{1}\left[\tilde{K}(u) - 1\right]\left[\exp\left(\rho(1+u)\right) - u\rho\exp(\rho u) - \exp(\rho u)\right]\mathrm{d}u \\ & + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\int_{u=0}^{1}\left[\tilde{K}(u) - 1\right]\left[\exp\left(\rho(1+u)\right) - u\rho\exp(\rho u) - 1\right]\mathrm{d}u.\end{aligned} \tag{5.1}$$

Remark 5.2. The zero service time case, $B \equiv \epsilon \to 0$, results in an expected time to delivery of $\mathbb{E}[D] = \alpha\mathbb{E}\left[\frac{K}{K+1} + \frac{1}{2}\right]$. This is equal to the expected travel time of the server to the furthest customer in the batch plus the expected travel time from this customer to the depot.

Remark 5.3. In the case of $K \equiv 1$, that is, $\tilde{K}(u) = u$, the expected time to delivery reduces to

$$\begin{aligned}\mathbb{E}[D] = {} & \frac{\alpha}{1-\rho} + \frac{\rho^2\mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)} + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{1}{\rho\lambda}\left[\exp(2\rho) - \rho\exp(\rho) - 1 - \rho\right] \\ & + \frac{\rho\mathbb{E}[B^2]}{\mathbb{E}[B]}\left[\frac{1}{\rho} + \frac{1}{\rho^2} - \frac{1}{\rho^2}\exp(\rho) + \frac{1}{2}\right] + \mathbb{E}[B]\left[\frac{1}{\rho^2} + \frac{2}{\rho}\exp(\rho) - \frac{1}{\rho^2}\exp(2\rho)\right] \\ = {} & \mathbb{E}[B] + \frac{\alpha}{1-\rho} + \frac{\rho\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{1}{\rho}\left(\mathbb{E}[B] - \frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\right)\left(\exp(\rho) - 1 - \rho\right).\end{aligned}$$

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This property was not analyzed in [11], yet it leads to an interesting comparison with the sojourn time, see Remark 4.3.

Remark 5.4. Under light traffic, that is, $\lambda \to 0$, the expected time to delivery simplifies as follows:

$$\mathbb{E}[D] = \left[\alpha + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\right]\mathbb{E}\left[\frac{K}{K+1} + \frac{1}{2}\right] + \mathbb{E}[B]\mathbb{E}[K]$$
$$- \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{2\mathbb{E}[K]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}\int_{u=0}^{1}\left[\tilde{K}(u) - 1\right]\mathrm{d}u$$
$$= \alpha\mathbb{E}\left[\frac{K}{K+1} + \frac{1}{2}\right] + \mathbb{E}[B]\mathbb{E}[K].$$

Remark that this is exactly equal to the expected travel time that is required to deliver the batch of orders plus the service time of the batch.

The proof of Theorem 5.1 is very similar to that of Theorem 4.1. One main difference is the fact that D might involve more than one full cycle of travel from the server. Therefore, the extra time to delivery that is generated by a customer might no longer be sufficiently defined by the formulas for $S(x)$ and $T(x)$ for $0 \leq x \leq 1$. Extending their definitions and using an analysis comparable to before give the desired result. For the details of this proof, we refer the reader to Appendix A.

One might also remark that D is equal to S^B plus the residual cycle time of the server. The expectation of this latter term, however, is hard to find as it includes customers who have arrived during S^B and who will be served in this residual cycle. The expression for $\mathbb{E}[D] - \mathbb{E}[S^B]$ is also quite involved and does not allow for a very straightforward interpretation.

6. Numerical results

The results in this paper allow for the performance analysis of continuous polling models and, in particular, can be used to obtain insights into the performance effect of, for instance, the distribution of the batch size. Additionally, the given expressions for $\mathbb{E}[S^B]$ and $\mathbb{E}[D]$ lend themselves nicely for optimization. In previous work, [18, 19] present a mean value analysis for the discrete polling system with batch arrivals. They derive the expected batch sojourn time and time to delivery by first solving a system of N^2 equations, in which N is the total number of queues. For symmetric systems, this is reduced to a system of N linear equations. For systems with large N, this becomes computationally expensive, and in particular, insights into the effect of, for example, the distribution of the batches can no longer be directly obtained. In this section, we approximate a discrete (symmetric) polling system by its continuous variant and investigate the rate of convergence, as $N \to \infty$, of the mean batch sojourn and time to delivery, as well as the effect of the batch size distribution.

We consider symmetric discrete polling systems, consisting of N queues and deterministic equal switch-over times, with a total switch-over time of 1 (i.e. $\alpha = 1$). Additionally, we assume that each customer in a batch independently is assigned to a queue with equal probability ($1/N$). The service time distribution and total batch size distribution are taken the same in all models. For the continuous variant, we take 1 to be the total switch-over time between queues.

First, we study the approximations for different values of ρ and batch sizes. Consider the model with deterministic batch sizes and exponential service times (with mean 1). We consider four values of the workload, $\rho = 0.2, 0.45, 0.7, 0.95$. Figure 4 shows the effect of the batch size on the performance statistics and the convergence from the discrete model to the continuous model.

Logically, larger batch sizes result in bigger mean batch sojourn times and times to delivery due to the higher service requirement of a batch. We also see that the continuous model already proves to be

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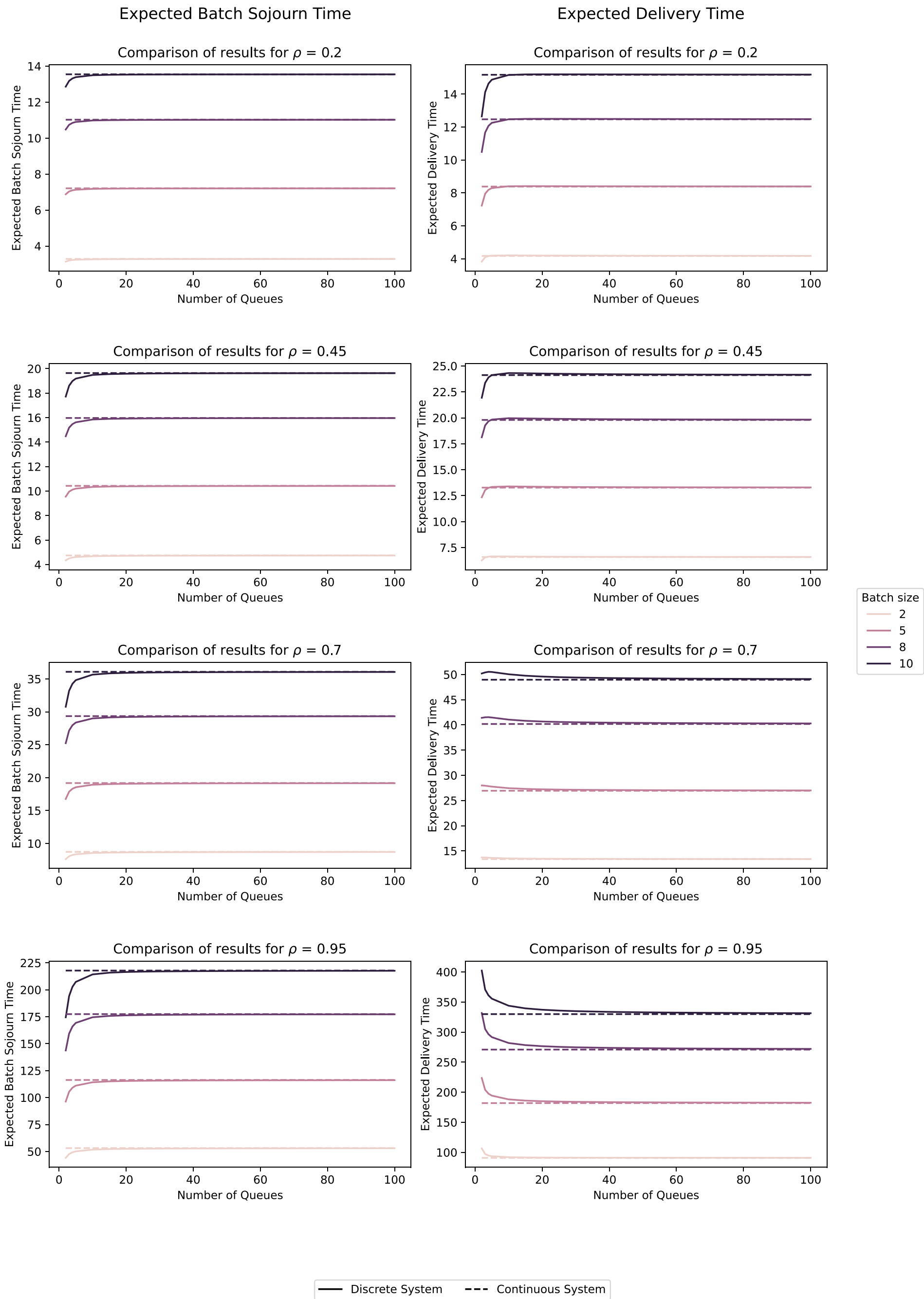

Figure 4. *The expected batch sojourn time and time to delivery for deterministic batch sizes and exponential service requirements with unit mean, comparing the discrete and continuous polling model.*

a good approximation for a relatively small number of queues: for $N = 10$, the difference is $< 3\%$, and for $N = 20$, the difference is $< 2\%$ in the mean batch sojourn times. The error in the expected time to delivery is $< 5\%$ for $N = 10$ and $< 3\%$ for $N = 20$. Additionally, the rate of convergence seems to be higher for lightly loaded systems. Also note that the continuous system consistently results in higher

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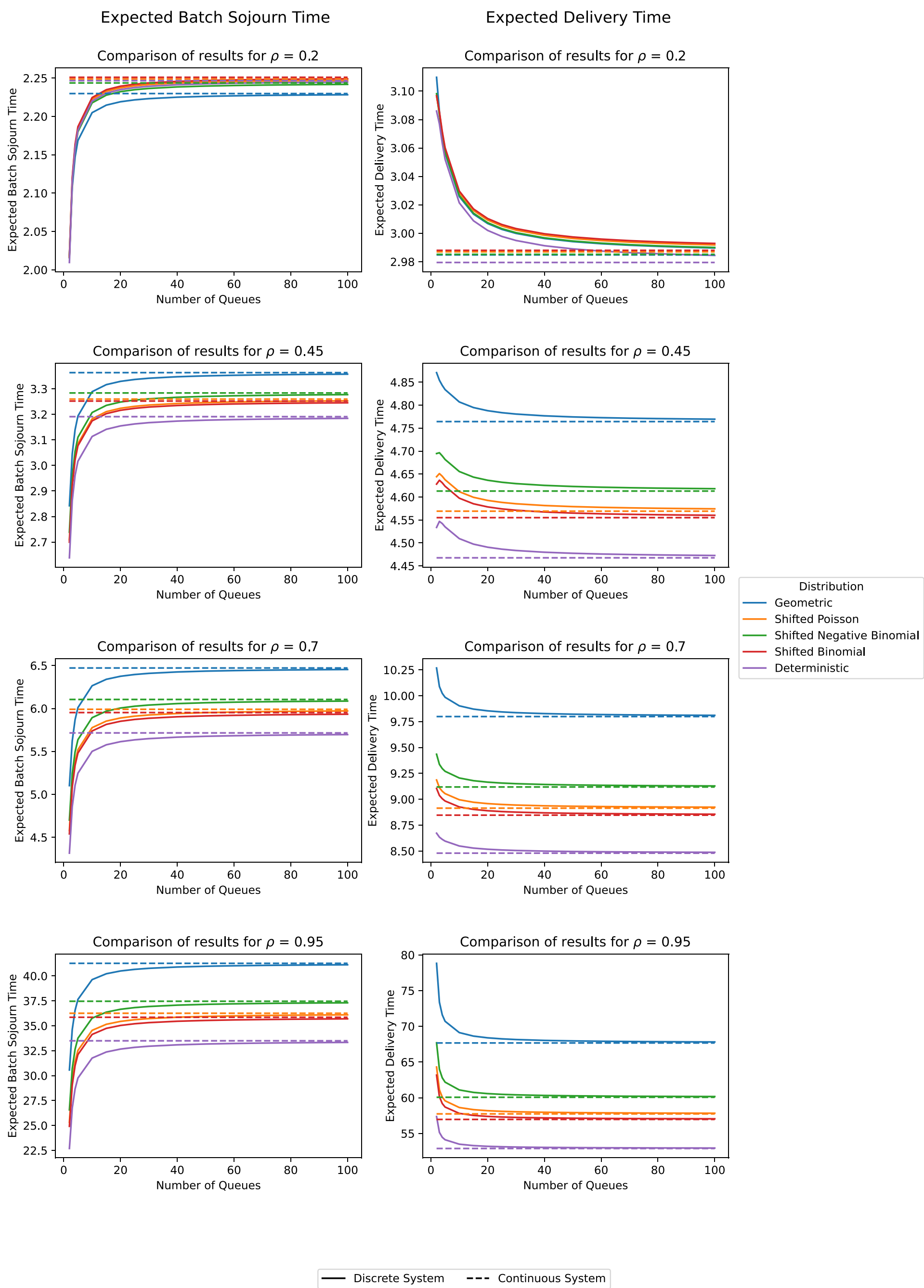

Figure 5. *Comparison of the expected batch sojourn time and time to delivery for different batch size distributions: geometric, Poisson, negative binomial with 5 successes, binomial with 15 trials, and deterministic. Services take* $1/5$ *time units, and the average batch size is 5.*

mean batch sojourn times, while it sometimes results in lower mean time to deliveries. An explanation for this lies within the length of the busy periods in the queues. During this busy period, the server remains in the same queue, without moving forward. In the continuous case, the server keeps moving toward the depot, albeit slowly.

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Secondly, we consider different batch size distributions, see Figure 5. We consider five batch size distributions, all with mean 5. We further assume that service of a customer takes a deterministic $1/5$ time unit. The results now show that the continuous model again provides a reasonable approximation for the discrete model but only for larger systems; here, the difference is $< 6\%$ in the mean batch sojourn time and $< 3\%$ in the mean time to delivery for $N = 10$. For $N = 20$, this reduces to $< 3\%$ for the mean batch sojourn time and $< 2\%$ for the mean time to delivery. Here, the error in the time to delivery is lower than that in the batch sojourn time, while it was the other way around in the previous example. This stems from the lower mean service time. Therefore, the main contribution to the mean batch sojourn time and mean time to delivery stems from the travel time. The approximated mean travel time to the furthest customer in a batch is worse than that of the mean travel distance to deliver a batch.

The effect of the batch size distribution on the mean sojourn time and time to delivery appears to be closely linked to the variance of this distribution; distributions with higher variance result in bigger mean sojourn times. This is in line with results on $M^K/G/1$ queues, where the same relation to the variance of the batch sizes holds. Only for systems with light loads, the mean batch sojourn time seems to be the lowest for geometric order sizes. This comes from the relatively low expected distance from the server to the furthest customer in this case (≈ 0.75 under geometric batch sizes and ≈ 0.83 under deterministic batch sizes).

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Competing interest. The authors declare none.

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Appendix A. Proof of Theorem 5.1

The approach to the derivation of the expected time to delivery is similar to that of the expected batch sojourn time. We first derive an initial expression for the expected value, which still depends on $f(x)$, the average spread of customers (cf. Proposition A.1). The proof of Theorem 5.1 then follows from substituting (4.4) into this expression.

Proposition A.1. *The expected time to delivery of a batch of customers satisfies*

$$\begin{aligned}\mathbb{E}[D] = &\int_{u=0}^{1} \sum_{k=1}^{\infty} p_k u^k \mathbb{E}[D|Y = u, X^B \le u, K = k]\mathrm{d}u \\ &+ \int_{u=0}^{1}\int_{x=u}^{1} \sum_{k=1}^{\infty} p_k k x^{k-1} \mathbb{E}[D|Y = u, X^B = x, K = k]\mathrm{d}x\mathrm{d}u,\end{aligned} \tag{A.1}$$

where:

$$\begin{aligned}\mathbb{E}[D|Y = u, X^B \le u, K = k] = &\frac{\alpha}{\rho}\big[\exp(\rho u) - 1\big] + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho u) + \frac{k\mathbb{E}[B]}{\rho u}\big[\exp(\rho u) - 1\big] \\ &+ \int_{z=0}^{u} f(z)\mathbb{E}[B]\exp\big(\rho(u-z)\big)\mathrm{d}z\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}&\mathbb{E}[D|Y = u, X^B = x, K = k] \\ &= \frac{\alpha}{\rho}\Big[\exp\big(\rho(1+u)\big) + \exp\big(\rho u\big) - 2 - \rho u \exp(\rho u)\Big] \\ &+ \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\big[\exp\big(\rho(1+u)\big) - \rho u\exp(\rho u)\big] + \mathbb{E}[B]\exp\big(\rho(1+u-x)\big) \\ &+ \frac{(k-1)\mathbb{E}[B]}{\rho x}\big[\exp\big(\rho(1+u)\big) - \exp\big(\rho(1+u-x)\big) + \exp(\rho u) - 1 - \rho u\exp(\rho u)\big] \\ &+ \int_{z=u}^{1} f(z)\mathbb{E}[B]\exp\big(\rho(1+u-z)\big)\mathrm{d}z \\ &+ \int_{z=0}^{u} f(z)\mathbb{E}[B]\Big[\exp\big(\rho(1+u-z)\big) - \rho(u-z)\exp\big(\rho(u-z)\big)\Big]\mathrm{d}z.\end{aligned} \tag{A.3}$$

We defer the proof of this statement to the end of this Appendix. We now turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. We first focus on the case that the furthest customer is positioned between the server and the depot, that is, the case in (A.2). For this, we use that f is of the form $f(x) = a + b(1 - x)$ and the following integral:

$$\int_{z=0}^{u} \rho \exp\big(\rho(u-z)\big)[a + b(1-z)]\mathrm{d}z = \Big(a - \frac{1-\rho}{\rho}b\Big)\exp(\rho u) - \Big(a - \frac{1-\rho}{\rho}b\Big) + bu.$$

Substituting this into (A.2) along with (4.4) then gives

$$\begin{aligned}\mathbb{E}[D|Y = u, X^B \le u, K = k] = {} & \frac{\alpha}{\rho}\big[\exp(\rho u) - 1\big] + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho u) + \frac{k\mathbb{E}[B]}{\rho u}\big[\exp(\rho u) - 1\big] \\ & - \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\exp(\rho u) + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\rho^2\mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)}u \\ & - \frac{\alpha}{\rho}\big[\exp(\rho u) - 1\big] + \frac{\alpha}{1-\rho}u \\ & - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\big[\exp(\rho u) - 1\big] + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)}u.\end{aligned}$$

Grouping and canceling the similar terms give

$$\begin{aligned}\mathbb{E}[D|Y = u, X^B \le u, K = k] = {} & \left[\frac{k}{\lambda\mathbb{E}[K]u} - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\right]\big[\exp(\rho u) - 1\big] \\ & + \rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \left[\frac{\rho^2\mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)} + \frac{\alpha}{1-\rho} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)}\right]u.\end{aligned} \tag{A.4}$$

Similarly, for the second case, see (A.3), we apply the following integration:

$$\begin{aligned}&\int_{z=0}^{1} \rho\exp\big(\rho(1+u-z)\big)[a + b(1-z)]\mathrm{d}z - \int_{z=0}^{u} \rho^2(u-z)\exp\big(\rho(u-z)\big)[a + b(1-z)]\mathrm{d}z \\ &\quad = \Big(a - \frac{1-\rho}{\rho}b\Big)\exp\big(\rho(1+u)\big) - \frac{1-\rho}{\rho}b\exp\big(\rho u\big) \\ &\qquad - \rho\Big(a - \frac{1-\rho}{\rho}b\Big)u\exp\big(\rho u\big) - \big(a - \frac{2-\rho}{\rho}b\big) + bu.\end{aligned}$$

Substituting this into (A.3) results in the following expression:

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$$
\begin{aligned}
&\mathbb{E}[D|Y=u, X^B=x, K=k] \\
&= \frac{\alpha}{\rho}\Big[\exp\big(\rho(1+u)\big)+\exp(\rho u)-2-u\rho\exp(\rho u)\Big] \\
&\quad+\rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}\Big[\exp\big(\rho(1+u)\big)-u\rho\exp(\rho u)\Big]+\mathbb{E}[B]\exp\big(\rho(1+u-x)\big) \\
&\quad+\frac{(k-1)\mathbb{E}[B]}{x\rho}\Big[\exp\big(\rho(1+u)\big)-\exp\big(\rho(1+u-x)\big)+\exp(\rho u)-1-\rho u\exp(\rho u)\Big] \\
&\quad+\left(\rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}-\rho\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\right)\exp\big(\rho(1+u)\big)-\rho\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\exp(\rho u)+\rho^2\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}u\exp(\rho u) \\
&\quad-\left(\rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}-\rho\frac{(2-\rho)\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]}\right)+\frac{\rho^2\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]}u \\
&\quad-\frac{\alpha}{\rho}\exp\big(\rho(1+u)\big)-\frac{\alpha}{\rho}\exp\big(\rho u\big)+\alpha u\exp\big(\rho u\big)+\frac{(2-\rho)\alpha}{\rho(1-\rho)}+\frac{\alpha}{1-\rho}u \\
&\quad-\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]\rho}\exp\big(\rho(1+u)\big)-\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]\rho}\exp\big(\rho u\big) \\
&\quad+\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}u\exp\big(\rho u\big) \\
&\quad+\frac{2-\rho}{\rho(1-\rho)}\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]}+\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)}u.
\end{aligned}
$$

Again, note that many terms cancel, such that we are left with

$$
\begin{aligned}
&\mathbb{E}[D|Y=u, X^B=x, K=k] \\
&= \frac{\alpha}{1-\rho}+\frac{\alpha}{1-\rho}u-\rho\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\exp(\rho u)+\frac{3-\rho}{1-\rho}\frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]}+\frac{\rho^2\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]}u \\
&\quad+\mathbb{E}[B]\exp\big(\rho(1+u-x)\big) \\
&\quad+\frac{(k-1)\mathbb{E}[B]}{x\rho}\Big[\exp\big(\rho(1+u)\big)-\exp\big(\rho(1+u-x)\big)-u\rho\exp(\rho u)-1+\exp(\rho u)\Big] \\
&\quad-\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K]\rho}\Big[\exp\big(\rho(1+u)\big)+\exp\big(\rho u\big)-u\rho\exp(\rho u)\Big] \\
&\quad+\frac{(2-\rho)\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho(1-\rho)\mathbb{E}[K]}+\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{(1-\rho)\mathbb{E}[K]}u.
\end{aligned}
$$

Reordering these terms such that the result is similar to that of the case $X^B \leq u$ gives

$$
\begin{aligned}
&\mathbb{E}[D|Y=u, X^B=x, K=k] \\
&= \left[\frac{k-1}{\lambda\mathbb{E}[K]x}-\frac{\mathbb{E}[K(K-1)]}{\lambda\mathbb{E}[K]^2}\right]\Big[\exp(\rho u)-1\Big]+\rho\frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]}+\Bigg[\frac{\rho^2\mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)}+\frac{\alpha}{1-\rho} \\
&\quad+\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)}\Bigg]u \\
&\quad+\frac{\alpha}{1-\rho}+\frac{\rho\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]}-\rho\frac{\mathbb{E}[B^2]}{\mathbb{E}[B]}\exp(\rho u)+\frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho(1-\rho)\mathbb{E}[K]} \\
&\quad-\frac{(k-1)\mathbb{E}[B]}{x\rho}\exp\big(\rho(1+u-x)\big)+\mathbb{E}[B]\exp\big(\rho(1+u-x)\big) \\
&\quad+\left[\frac{k-1}{\lambda\mathbb{E}[K]x}-\frac{\mathbb{E}[K(K-1)]}{\lambda\mathbb{E}[K]^2}\right]\cdot\Big[\exp\big(\rho(1+u)\big)-u\rho\exp(\rho u)\Big].
\end{aligned}
\tag{A.5}
$$

We now substitute (A.4) and (A.5) into (A.1) and remark that many elements in (A.5) are independent of x (the location of the furthest customer). Therefore, the conditional expectation simplifies as follows:

$$\begin{aligned}
\mathbb{E}[D] = &\int_{u=0}^{1} \left\{ \sum_{k=1}^{\infty} p_k u^k \left[\frac{k}{\lambda \mathbb{E}[K] u} - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \right] \left[\exp(\rho u) - 1 \right] \right\} \mathrm{d}u \\
&+ \int_{u=0}^{1} \int_{x=u}^{1} \left\{ \sum_{k=1}^{\infty} p_k k x^{k-1} \left[\frac{k-1}{\lambda \mathbb{E}[K] x} - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \right] \left[\exp(\rho u) - 1 \right] \right\} \mathrm{d}x \mathrm{d}u \\
&+ \int_{u=0}^{1} \left\{ \rho \frac{\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \left[\frac{\rho^2 \mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)} + \frac{\alpha}{1-\rho} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\mathbb{E}[K](1-\rho)} \right] u \right\} \mathrm{d}u \\
&+ \int_{u=0}^{1} \left\{ \sum_{k=1}^{\infty} p_k (1-u^k) \cdot \left[\frac{\alpha}{1-\rho} + \frac{\rho \mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho(1-\rho)\mathbb{E}[K]} \right] \right\} \mathrm{d}u \\
&- \int_{u=0}^{1} \left\{ \sum_{k=1}^{\infty} p_k (1-u^k) \cdot \frac{\rho \mathbb{E}[B^2]}{\mathbb{E}[B]} \exp(\rho u) \right\} \mathrm{d}u \\
&+ \int_{u=0}^{1} \int_{x=u}^{1} \left\{ \sum_{k=1}^{\infty} p_k k x^{k-1} \left[\mathbb{E}[B] - \frac{(k-1)\mathbb{E}[B]}{x\rho} \right] \exp\big(\rho(1+u-x)\big) \right\} \mathrm{d}x \mathrm{d}u \\
&+ \int_{u=0}^{1} \int_{x=u}^{1} \left\{ \sum_{k=1}^{\infty} p_k k x^{k-1} \left[\frac{k-1}{\lambda \mathbb{E}[K] x} - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \right] \cdot \left[\exp\big(\rho(1+u)\big) - u\rho \exp(\rho u) \right] \right\} \\
&\times \mathrm{d}x \mathrm{d}u.
\end{aligned}$$

Some of the integrals over x simplify as $\int_{x=u}^{1} k x^{k-1} \mathrm{d}x = 1 - u^k$; note that this also cancels a term u^k in the other integrals. We further rewrite the sums in terms of the probability generating function of K. We combine the first and second lines, work out the integral in the third line, and split the integrals in the last two lines to find

$$\begin{aligned}
\mathbb{E}[D] = &\frac{1}{\lambda} \int_{u=0}^{1} \left[\exp(\rho u) - 1 \right] \mathrm{d}u - \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \int_{u=0}^{1} \left[\exp(\rho u) - 1 \right] \mathrm{d}u \\
&+ \frac{\rho \mathbb{E}[B^2]}{2\mathbb{E}[B](1-\rho)} + \frac{\alpha}{2(1-\rho)} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{2\mathbb{E}[K](1-\rho)} \\
&+ \left[\frac{\alpha}{1-\rho} + \frac{\rho \mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho(1-\rho)\mathbb{E}[K]} \right] \int_{u=0}^{1} \left[1 - \tilde{K}(u) \right] \mathrm{d}u \\
&- \frac{\rho \mathbb{E}[B^2]}{\mathbb{E}[B]} \int_{u=0}^{1} \left[1 - \tilde{K}(u) \right] \exp(\rho u) \mathrm{d}u \\
&+ \mathbb{E}[B] \int_{u=0}^{1} \int_{x=u}^{1} \tilde{K}'(x) \exp\big(\rho(1+u-x)\big) \mathrm{d}x \mathrm{d}u \\
&- \frac{\mathbb{E}[B]}{\rho} \int_{u=0}^{1} \int_{x=u}^{1} \tilde{K}''(x) \exp\big(\rho(1+u-x)\big) \mathrm{d}x \mathrm{d}u \\
&+ \frac{1}{\lambda \mathbb{E}[K]} \int_{u=0}^{1} \left[\mathbb{E}[K] - \tilde{K}'(u) \right] \left[\exp\big(\rho(1+u)\big) - u\rho \exp(\rho u) \right] \mathrm{d}u \\
&- \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho \mathbb{E}[K]} \int_{u=0}^{1} \left[1 - \tilde{K}(u) \right] \left[\exp\big(\rho(1+u)\big) - u\rho \exp(\rho u) \right] \mathrm{d}u.
\end{aligned}$$

We now apply partial integration to the second integral with a $\exp(\rho(1+u-x))$ term; this then cancels a $\tilde{K}'(x)$ term. Additionally, we use partial integration to the penultimate integral. Specifically, we find

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$$
\begin{aligned}
\mathbb{E}[D] = {} & \frac{1}{\lambda}\int_{u=0}^{1}\big[\exp(\rho u)-1\big]\mathrm{d}u - \frac{\mathbb{E}[K(K-1)]}{\lambda\mathbb{E}[K]^2}\int_{u=0}^{1}\big[\exp(\rho u)-1\big]\mathrm{d}u \\
& + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B](1-\rho)} + \frac{\alpha}{2(1-\rho)} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{2\mathbb{E}[K](1-\rho)} \\
& + \left[\frac{\alpha}{1-\rho} + \frac{\rho\mathbb{E}[B^2]}{(1-\rho)\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho(1-\rho)\mathbb{E}[K]}\right]\int_{u=0}^{1}\big[1-\tilde{K}(u)\big]\mathrm{d}u \\
& - \frac{\rho\mathbb{E}[B^2]}{\mathbb{E}[B]}\int_{u=0}^{1}\big[1-\tilde{K}(u)\big]\exp(\rho u)\mathrm{d}u \\
& - \frac{\mathbb{E}[B]}{\rho}\int_{u=0}^{1}\Big\{\mathbb{E}[K]\exp(\rho u) - \tilde{K}'(u)\exp(\rho)\Big\}\mathrm{d}u \\
& + \frac{1}{\lambda}\int_{u=0}^{1}\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u)\big]\mathrm{d}u - \frac{1}{\lambda\mathbb{E}[K]}\big[\exp(2\rho) - \rho\exp(\rho)\big] \\
& + \mathbb{E}[B]\int_{u=0}^{1}\tilde{K}(u)\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u) - \exp(\rho u)\big]\mathrm{d}u \\
& + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\int_{u=0}^{1}\big[\tilde{K}(u)-1\big]\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u)\big]\mathrm{d}u.
\end{aligned}
$$

We now perform several operations: (i) we combine the first term of line 1 with the first term of line 6; (ii) we move the second and third line to the front and rewrite these to be similar to the first term for the expression of $\mathbb{E}[S^B]$ in (4.1); and (iii) we also evaluate the second integration term on the fifth line. This results in

$$
\begin{aligned}
\mathbb{E}[D] = {} & \frac{\rho^2\mathbb{E}[B^2]}{2\mathbb{E}[B](1-\rho)} + \frac{\rho\mathbb{E}[B^2]}{2\mathbb{E}[B]} + \frac{\alpha}{2(1-\rho)} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{2\mathbb{E}[K](1-\rho)} \\
& + \left[\frac{\alpha}{1-\rho} + \frac{\rho^2\mathbb{E}[B^2]}{\mathbb{E}[B](1-\rho)} + \frac{\rho\mathbb{E}[B^2]}{\mathbb{E}[B]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{(1-\rho)\mathbb{E}[K]} + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\right]\mathbb{E}\left[\frac{K}{K+1}\right] \\
& - \frac{\mathbb{E}[K(K-1)]}{\lambda\mathbb{E}[K]^2}\int_{u=0}^{1}\big[\exp(\rho u)-1\big]\mathrm{d}u - \frac{\rho\mathbb{E}[B^2]}{\mathbb{E}[B]}\int_{u=0}^{1}\big[1-\tilde{K}(u)\big]\exp(\rho u)\mathrm{d}u \\
& - \frac{1}{\lambda}\int_{u=0}^{1}\exp(\rho u)\mathrm{d}u + \frac{1}{\lambda\mathbb{E}[K]}\exp(\rho) \\
& + \frac{1}{\lambda}\cdot\int_{u=0}^{1}\big[\exp(\rho u) - 1 + \exp\big(\rho(1+u)\big) - u\rho\exp(\rho u)\big]\mathrm{d}u \\
& - \frac{1}{\lambda\mathbb{E}[K]}\big[\exp(2\rho) - \rho\exp(\rho)\big] + \mathbb{E}[B]\int_{u=0}^{1}\tilde{K}(u)\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u) \\
& - \exp(\rho u)\big]\mathrm{d}u \\
& + \frac{\mathbb{E}[B]\mathbb{E}[K(K-1)]}{\rho\mathbb{E}[K]}\int_{u=0}^{1}\big[\tilde{K}(u)-1\big]\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u)\big]\mathrm{d}u.
\end{aligned} \tag{A.6}
$$

The first two lines of (A.6), the second term of line 3, and the last line together result in the first, third, and last line of (5.1). For the remaining terms, we note that

$$
\frac{1}{\lambda\mathbb{E}[K]}\exp(\rho) - \frac{1}{\lambda\mathbb{E}[K]}\big[\exp(2\rho) - \rho\exp(\rho)\big] = -\mathbb{E}[B]\int_{u=0}^{1}\big[\exp\big(\rho(1+u)\big) - u\rho\exp(\rho u) - \exp(\rho u)\big]\mathrm{d}u.
$$

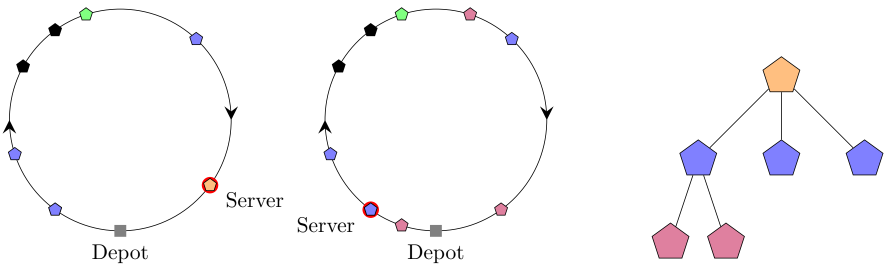

Figure A1. *Illustration of the time to delivery of a tagged customer (green) that is generated by a service (of the orange customer) and the corresponding (trimmed) branching process. During the service of the orange customer, blue customers arrive, of which only the first two are considered. During the service of the first blue customer, the red customers arrive, of which two will be served before the delivery of the green customer.*

Combined with the second term of line 6 in (A.6), this gives the fourth line of (5.1). The other terms in (5.1) follow from straightforward integrations. □

Proof of Proposition A.1

The proof of this statement is very similar to the proof in Section 4.1. Using the linking of future arrivals to the current state, one can derive an analytical expression for the expected time to delivery.

Consider the same branching construction as in Section 4.1, using the following rules:

- A customer (a) is the offspring of a customer (b) when (a) arrives during the service of (b).
- A customer (a) is called an immigrant when she arrives during a travel period of the server.

The main difference with the analysis of the batch sojourn time lies within the trimming of this branching process. Previously, we were interested in the waiting time of a tagged customer, therefore we trimmed the branching process such that only customers remained who were served before this tagged customer. For the time to delivery, however, one also needs to consider customers who are served after the service of but before the delivery of the tagged customer. We, therefore, now keep all customers in the branching process who will be served before the delivery of the tagged customer, see Figure A1. Note that the type of a customer now is the travel distance that the server has yet to travel before delivering the tagged customer. With this, we can define the effects of services and traveling on the time to delivery of a customer.

Definition A.2. *Let x be the distance that the server has yet to travel to deliver the tagged customer. We then define $S_D(x)$ as the service time of a customer plus the service times of all its descendants in the trimmed process. This is also referred to as the extra delivery time generated by a service.*

Definition A.3. *$T_D(x)$ denotes the added service times of all customers in the trimmed branching processes of immigrants, arriving during the travel of the server over a distance x. We call this the extra delivery time generated by the traveling of the server.*

It should be noted that the arguments of these definitions are no longer smaller than 1, as the server might travel more than a cycle before the tagged customer is delivered. This also implies that the trimmed branching process might include customers who are served after more than a cycle of travel from the server, see, for example, the last red customer in front of the depot in Figure A1. This slight nuance should also be considered in the analysis of these functions. We therefore consider two cases separately. Case 1: when the service of the customer happens in the same cycle as the delivery. In Figure A1, the service of the black customers occurs in the same cycle as the delivery of the tagged (green) customer. Case 2 is when the service of a customer happens in the cycle prior to the delivery. This occurs for the orange customer in Figure A1, as the tagged (green) customer is behind the depot and hence will be delivered in the next cycle. Remark that the time that is generated by a service is now determined by the distance the server has yet to travel to deliver the tagged customer, rather than the distance to the tagged customer.

Lemma A.4. *The expected values of $S(x), T(x)$ satisfy the following:*

$$\mathbb{E}[S_D(x)] = \begin{cases} \mathbb{E}[B]\exp(\rho x) & \text{if } 0 \le x \le 1; \\ \mathbb{E}[B]\Big[\exp(\rho x) - (x-1)\rho\exp\big(\rho(x-1)\big)\Big] & \text{if } 1 < x \le 2, \end{cases} \tag{A.7}$$

$$\mathbb{E}[T_D(x)] = \begin{cases} \frac{\alpha}{\rho}\big[\exp(\rho x) - 1\big] & \text{if } 0 \le x \le 1; \\ \frac{\alpha}{\rho}\Big[\exp(\rho x) + \exp\big(\rho(x-1)\big) - 2 - (x-1)\rho\exp\big(\rho(x-1)\big)\Big] & \text{if } 1 < x \le 2. \end{cases} \tag{A.8}$$

Proof. The analysis of case 1 is rather straightforward, as the server delivers the tagged customer within the same cycle, and hence we also know that the travel distance is less than 1. Therefore, following similar arguments as in Lemma 4.9, one immediately obtains $\mathbb{E}[S_D(x)] = \mathbb{E}[S(x)]$ and $\mathbb{E}[T_D(x)] = \mathbb{E}[T(x)]$ for $0 \le x \le 1$. For case 2, we find an integral equation which we solve; we do this by considering that the server still has to travel a distance of $1 + x$, $0 \le x \le 1$.

Remark that the customers arriving during the service of the customer themselves generated work. A customer arriving at distance u of the initial customer generates a work of $S_D(1 + x - u)$. Therefore,

$$\begin{aligned} \mathbb{E}[S_D(1+x)] &= \mathbb{E}[B] + \rho\int_{u=0}^{x}\mathbb{E}[S_D(1+x-u)]\mathrm{d}u + \rho\int_{u=x}^{1}\mathbb{E}[S_D(1+x-u)]\mathrm{d}u. \\ &= \mathbb{E}[B] + \rho\int_{u=0}^{x}\mathbb{E}[S_D(1+u)]\mathrm{d}u + \rho\mathbb{E}[B]\int_{u=x}^{1}\exp(\rho u)\mathrm{d}u \\ &= \mathbb{E}[B]\Big[1 + \exp(\rho) - \exp(\rho x)\Big] + \rho\int_{u=0}^{x}\mathbb{E}[S_D(1+u)]\mathrm{d}u, \end{aligned}$$

where we used that we know $\mathbb{E}[S_D(x)]$ for $x \le 1$, see Lemma 4.9. Let $h(x) := \mathbb{E}[S_D(1 + x)]$; taking the derivative of both sides then gives the following differential equation:

$$h'(x) = -\rho\mathbb{E}[B]\exp(\rho x) + \rho h(x).$$

It is readily verified that the expression in (A.7) solves this equation with initial condition $h(0) = \mathbb{E}[B]\exp(\rho)$.

For the work generated by travel, we focus on the work generated by the first x distance that is traveled, and the other traveled distance (the next cycle) is known to generate a total time of $\mathbb{E}[T_D(1)]$ on average. During this first distance, two types of customers may arrive again, ones who are served in this cycle and ones who are served in the next cycle:

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$$\begin{aligned}\mathbb{E}[T_D(1+x)] &= \mathbb{E}[T_D(1)] + \alpha x \\ &\quad + \int_{u=0}^{x} \lambda \mathbb{E}[K]\alpha \left\{ \int_{y=0}^{x-u} \mathbb{E}[S_D(1+x-u-y)]\mathrm{d}y + \int_{y=x-u}^{1} \mathbb{E}[S_D(1+x-u-y)]\mathrm{d}y \right\} \mathrm{d}u \\ &= \frac{\alpha}{\rho}\left[\exp(\rho)-1\right] + \alpha x + \alpha \int_{u=0}^{x}\int_{y=0}^{x-u} \rho \exp\left(\rho(1+y)\right)\mathrm{d}y\mathrm{d}u \\ &\quad - \alpha \int_{u=0}^{x}\int_{y=0}^{x-u} \rho^2 y \exp(\rho y)\mathrm{d}y\mathrm{d}u \\ &\quad + \alpha \int_{u=0}^{x}\int_{y=x-u}^{1} \rho \exp(\rho y)\mathrm{d}y\mathrm{d}u.\end{aligned}$$

We integrate the inner integrals to find

$$\begin{aligned}\mathbb{E}[T_D(1+x)] &= \frac{\alpha}{\rho}\left[\exp(\rho)-1\right] + \alpha x + \alpha \int_{u=0}^{x} \left\{ \exp\left(\rho(1+x-u)\right) - \exp(\rho) \right\} \mathrm{d}u \\ &\quad - \alpha \int_{u=0}^{x} \left\{ \rho(x-u)\exp\left(\rho(x-u)\right) - \int_{y=0}^{x-u} \rho \exp(\rho y)\mathrm{d}y \right\} \mathrm{d}u \\ &\quad + \alpha \int_{u=0}^{x} \left\{ \exp(\rho) - \exp\left(\rho(x-u)\right) \right\} \mathrm{d}u. \\ &= \frac{\alpha}{\rho}\left[\exp(\rho)-1\right] + \alpha x + \alpha \int_{u=0}^{x} \left[\exp\left(\rho(1+x-u)\right) - 1 \right] \mathrm{d}u \\ &\quad - \alpha \int_{u=0}^{x} \rho(x-u)\exp\left(\rho(x-u)\right)\mathrm{d}u.\end{aligned}$$

Another round of (partial) integration then results in (A.8). □

Remark A.5. As in Remark 4.10, we can extend the definition of S_D^R to the extra time to delivery generated by the residual service of a customer. The expected extra time generated by this residual service satisfies (A.7), with $\mathbb{E}[B]$ replaced by $\mathbb{E}[B^R]$.

With this, we can prove Proposition A.1

Proof of Proposition A.1. We condition over the distance from the server to the delivery point, u, which is uniformly distributed at the time of arrival. We further condition over the distance to the furthest customer in the arriving batch and the size of the batch. Note that with probability u^k a batch of size k will be completely picked within the current cycle, and hence we arrive at (A.1).

Proof of (A.2) Under the event of delivery in the current cycle, we can write the time to delivery as a sum of 4 elements: (i) the time to delivery generated by traveling over a distance of u: $T_D(u)$; (ii) the extra time to delivery generated by the possible residual service time at distance u of the delivery point, $S_D^R(u)$, which happens with probability ρ; (iii), the extra time to delivery generated by customers in the same batch, $S_D(Y_1) + \ldots + S_D(Y_k)$ where each customer is known to uniformly arrive within distance u of the server; and (iv) the extra time to delivery generated by all customers present on the circle at the time of arrival. A customer at distance z of the server generates a time $S_D(u-z)$. We thus obtain

$$\begin{aligned}\mathbb{E}[D|Y=u, X^B \le u, K=k] &= \mathbb{E}[T_D(u)] + \rho\mathbb{E}[S_D^R(u)] + k\int_{z=0}^{u} \frac{1}{u}\mathbb{E}[S_D(z)]\mathrm{d}z \\ &\quad + \int_{z=0}^{u} f(z)\mathbb{E}[S_D(u-z)]\mathrm{d}z.\end{aligned}$$

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Substituting the results of Lemma 4.9 gives the desired result.

Proof of (A.3) Under the event that the batch will not be delivered in the current cycle, we have a similar construction. However, we have to be careful that the server has to travel more than a cycle before the batch is delivered. Therefore, a customer might generate an extra time to delivery over a distance that is more than 1. The 4 elements now are (i) the extra time to delivery generated by traveling over a distance of $1+u$: $T_D(1+u)$; (ii) the extra time to delivery generated by the possible residual service time at distance u of the delivery point, $S_D^R(1+u)$, which happens with probability ρ; and (iii) the extra time to delivery generated by other customers in the same batch, $S_D(1+u-Y_1)+\ldots+S_D(1+u-Y_{k-1})$ where each customer is known to uniformly arrive within distance x of the server. Additionally, there is an extra time to delivery generated by the customer who arrives at distance x of the server: $S_D(1+u-x)$. Lastly we have (iv) the extra time to delivery generated by all customers present on the circle at the time of arrival. A customer at distance z of the server generates a time $S_D(1+u-z)$. Therefore,

$$\begin{aligned}\mathbb{E}[D|Y=u,X^B=x,K=k] &= \mathbb{E}[T_D(1+u)]+\rho\mathbb{E}[S_D^R(1+u)]+\mathbb{E}[S_D(1+u-x)]\\ &\quad+(k-1)\int_{z=0}^{x}\frac{1}{x}\mathbb{E}[S_D(1+u-z)]\mathrm{d}z+\int_{z=0}^{1}f(z)\mathbb{E}[S_D(1+u-z)]\mathrm{d}z.\end{aligned}$$

One can now split the integrals to differentiate between generated times over distances longer than 1 and shorter than 1:

$$\begin{aligned}\mathbb{E}[D|Y=u,X^B=x,K=k] &= \mathbb{E}[T_D(1+u)]+\rho\mathbb{E}[S_D^R(1+u)]+\mathbb{E}[S_D(1+u-x)]\\ &\quad+(k-1)\int_{z=0}^{u}\frac{1}{x}\mathbb{E}[S_D(1+u-z)]\mathrm{d}z+(k-1)\int_{z=u}^{x}\frac{1}{x}\mathbb{E}[S_D(1+u-z)]\mathrm{d}z\\ &\quad+\int_{z=0}^{u}f(z)\mathbb{E}[S_D(1+u-z)]\mathrm{d}z+\int_{z=u}^{1}f(z)\mathbb{E}[S_D(1+u-z)]\mathrm{d}z.\end{aligned}$$

Remark that the penultimate integral has $z<u$, and hence $1+u-z>1$; the last integral on the other hand considers the case $1+u-z\leq 1$. Using this distinction and Lemmas 4.9 and A.4 then completes the proof. □

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