Similitude and Inversion.

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The following paper contains little that can be regarded as new mathematical information. It aims only at showing, or rather at emphasising, the correspondence which exists between two geometrical theories which are related to each other in the same way as the arithmetical theories of multiplication and division. Such value, therefore, as it possesses is primarily pedagogical.

Attention should perhaps be drawn to the (unusual) use of the word "similar" in the sense of "similar and similarly situated," or "homothetic." The word homothesis (French geometers have adopted a curious form *homothétie*) is not naturalised in English; otherwise homothesis and homothetic might have been used instead of similitude and similar. The pair, similitude and homothetic, are somewhat incongruous.

The term antiparallel (said to have been first employed with a definite geometrical meaning by Antoine Arnauld in 1667) was not uncommon in English mathematical writings about a century ago. It is again, and deservedly, coming into use, and the following definition of it may be given:

Two straight lines intersecting the sides of an angle, or its vertically opposite angle, are antiparallel when the first straight line makes with one of the sides of the given angle an angle equal to that which the second straight line makes with the other side.

Figure 37′.

Thus, if $\angle$ OM′P′ = $\angle$ OPM, then M′P′ is antiparallel to MP with respect to $\angle$ O.

The following are some properties connected with antiparallels, the proof of which need not be given here.

(1) If M′P′ is antiparallel to MP with respect to $\angle$ O, then MM′ is antiparallel to PP′.

(2) The four points M, M′, P, P′ are concyclic.

(3) The rectangles OM·OM′, OP·OP′ are equal.

(4) The triangles OMP, OM′P′ are similar.

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SIMILITUDE.

§ 1. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered similar to each other, and the third may be taken as their centre of similitude. The ratio of the distances of the third point from the other two is called the ratio of similitude.

Thus, if O be chosen as centre of similitude, and P′ be considered similar to P, the ratio of similitude is OP : OP′.

When P and P′ are on the same side of O, the ratio of similitude is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the ratio of similitude is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ similar to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the similar point P′ will describe the similar curve.

§ 2. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the point similar to a given point P.

Figure 37.

Through O draw any straight line OM, and make $OM = r$, $OM' = r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r : r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ parallel to MP, and meeting OP in P′.

Then P′ is similar to P.

For $OP : OP' = OM : OM' = r : r'$.

§ 3. Given two points P, P′ similar to each other, and a ratio of similitude $r : r'$, to find the centre of similitude.

The centre of similitude is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may have the given ratio.

§ 4. Given two pairs of similar points P and P′, Q and Q′, to find the centre of similitude.

Case 1. When the four points are not collinear.

Inversion.

§ 1′. *Definition.*—If three collinear points O, P, P′ be given, any two of them may be considered inverse to each other, and the third may be taken as their centre of inversion. The rectangle under the distances of the third point from the other two is called the rectangle of inversion.

Thus, if O be chosen as centre of inversion, and P′ be considered inverse to P, the rectangle of inversion is OP·OP′.

When P and P′ are on the same side of O, the rectangle of inversion is positive, since OP and OP′ are drawn in the same direction; when P and P′ are on opposite sides of O, the rectangle of inversion is negative, since OP and OP′ are drawn in opposite directions.

When a given point P is variable, that is, when it occupies a series of consecutive positions, the point P′ inverse to it will also occupy a series of consecutive positions; in other words, when a given point P describes a certain curve, the inverse point P′ will describe the inverse curve.

§ 2′. Given a centre of inversion O, and a rectangle of inversion $r{\cdot}r'$ to find the point inverse to a given point P.

Figure 37′.

Through O draw any straight line OM, and make $OM=r$, $OM'=r'$. M and M′ will be on the same side of O or on opposite sides of O, according as $r{\cdot}r'$ is positive or negative. Join MP, OP, and through M′ draw M′P′ antiparallel to MP with respect to angle MOP, and meeting OP in P′.

Then P′ is inverse to P.

For $OP{\cdot}OP'=OM{\cdot}OM'=r.r'$.

§ 3′. Given two points P, P′ inverse to each other, and a rectangle of inversion $r{\cdot}r'$, to find the centre of inversion.

The centre of inversion is determined by joining PP′ and dividing it externally or internally at O so that the segments OP, OP′ may contain the given rectangle.

§ 4′. Given two pairs of inverse points P and P′, Q and Q′, to find the centre of inversion.

Case 1. When the four points are not collinear.

Join PP′, QQ′, and let them intersect at O. Then O is the centre of similitude.

This follows from § 1.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38.

Take any point M not collinear with the four points, and join PM, QM.

Through P′ draw a straight line parallel to PM; through Q′ draw a straight line parallel to QM; and let these straight lines intersect at M′.

MM′ will intersect PP′ at O, the centre of similitude.

For $$OP : OP' = OM : OM', \\ = OQ : OQ'.$$

SECOND METHOD.

Figure 39.

Take any point M not collinear with the four points, and join PM, QM.

Through Q′ describe a circle passing through P, M; through P′ describe a circle passing through Q, M; and let these circles intersect at M′.

MM′ will intersect PP′ at O, the centre of similitude.

For $$OP \cdot OQ' = OM\, OM'. \\ = OQ \cdot OP';$$

therefore $$OP : OP' = OQ : OQ'.$$

§ 5. It will be seen from the two preceding methods of solution that, when two pairs of similar points happen to be collinear, two pairs of inverse points also are obtained.

For the equality of the ratios $OP : OP'$, $OQ : OQ'$ necessitates the equality of the rectangles $OP \cdot OQ'$, $OQ \cdot OP'$.

§ 6. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the system of points similar to a given system A, B, C, …

Figure 41.

Join O to A, B, C, …; and on OA, OB, OC, … find A′, B′, C′, … such that

$$OA : OA' = OB : OB' = OC : OC' = \dots = r : r'.$$

Join PP′, QQ′, and let them intersect at O. Then O is the centre of inversion.

This follows from § 1′.

Case 2. When the four points are collinear.

FIRST METHOD.

Figure 38′.

Take any point M not collinear with the four points, and join PM, QM.

Through P′ describe a circle passing through P, M; through Q′ describe a circle passing through Q, M; and let these circles intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP \cdot OP' = OM \cdot OM', \quad = OQ \cdot OQ'.$$

SECOND METHOD.

Figure 39′.

Take any point M not collinear with the four points, and join PM, QM.

Through Q′ draw a straight line parallel to PM; through P′ draw a straight line parallel to QM; and let these straight lines intersect at M′.

MM′ will intersect PP′ at O, the centre of inversion.

For $$OP : OQ' = OM : OM', \quad = OQ : OP';$$

therefore $$OP \cdot OP' = OQ \cdot OQ'.$$

§ 5′. It will be seen from the two preceding methods of solution that, when two pairs of inverse points happen to be collinear, two pairs of similar points also are obtained.

For the equality of the rectangles $OP \cdot OP'$, $OQ \cdot OQ'$ necessitates the equality of the ratios $OP : OQ'$, $OQ : OP'$.

§ 6′. Given a centre of inversion O, and a rectangle of inversion $r \cdot r'$, to find the system of points inverse to a given system A, B, C, ...

Figure 41′.

Join O to A, B, C, ...; and on OA, OB, OC, ... find A′, B′, C′, ... such that

$$OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \ldots = r \cdot r'.$$

§ 7. If two systems of points be similar, the straight line joining any pair of points in the one is parallel to the straight line joining the corresponding pair in the other.

This follows from § 2.

§ 8. If two systems of points be similar, and three points of the first system be collinear, the three corresponding points of the second system will also be collinear.

Figure 40.

Let the system A, B, C be similar to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7, A′B′ is parallel to AB,

therefore $\angle OB'A' = \angle OBA.$

Similarly $\angle OB'C' = \angle OBC;$

therefore $\angle OB'A' + \angle OB'C' = \angle OBA + \angle OBC,$

$= 2$ right angles;

therefore A′, B′, C′ are collinear.

Hence the curve similar to a straight line is a straight line.

§ 9. If two systems of points be similar, with respect to a centre of similitude O and a ratio of similitude $r : r'$, then for every two points A, B and the points A′, B′ similar to them

$$A'B' : AB = r' : r.$$

Figure 41.

Since A′B′ is parallel to AB,

therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OA,$

$= r' : r.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p,$$

§ 10. If two systems of points be similar, then for every three points A, B, C and the points A′, B′, C′ similar to them

$$B'C' : C'A' : A'B' = BC : CA : AB.$$

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§ 7′. If two systems of points be inverse, the straight line joining any pair of points in the one is antiparallel to the straight line joining the corresponding pair in the other.

This follows from § 2′.

§ 8′. If two systems of points be inverse, and three points of the first system be collinear, the three corresponding points of the second system will not in general be collinear.

Figure 40′.

Let the system A, B, C be inverse to the system A′, B′, C′, and let A, B, C be collinear.

Since, by § 7′, A′B′ is antiparallel to AB,

therefore $\angle OA'B' = \angle OBA.$

Similarly $\angle OC'B' = \angle OBC$

therefore $\angle OA'B' + \angle OC'B' = \angle OBA + \angle OBC,$

$= 2$ right angles;

which is impossible, if A′, B′, C′ be collinear.

Hence the curve inverse to a straight line is not in general a straight line.

§ 9′. If two systems of points be inverse, with respect to a centre of inversion O and a rectangle of inversion $r \cdot r'$, then for every two points A, B and the points A′, B′ inverse to them

$$A'B' : AB = r \cdot r' : OA \cdot OB.$$

Figure 41′.

Since A′B′ is antiparallel to AB with respect to angle AOB, therefore triangles OA′B′, OAB are equiangular;

therefore $A'B' : AB = OA' : OB = OA \cdot OA' : OA \cdot OB,$

$= r \cdot r' : OA \cdot OB.$

If p', p be the perpendiculars from O on A′B′ and AB, it also follows from the equiangularity of the triangles OA′B′, OAB that

$$A'B' : AB = p' : p.$$

§ 10′. If two systems of points be inverse, then for every three points A, B, C and the points A′, B′, C′ inverse to them

$$B'C' : C'A' : A'B' = OA \cdot BC : OB \cdot CA : OC \cdot AB.$$

Figure 41.

For $$B'C' = BC.\frac{r'}{r}, \quad C'A' = CA.\frac{r'}{r};$$

therefore $$\frac{B'C'}{C'A'} = BC.\frac{r'}{r}/CA.\frac{r'}{r},$$
$$= BC/CA.$$

§ 11. If two systems of points be similar, then for every four points A, B, C, D and the points A′, B′, C′, D′ similar to them
$$B'C'\cdot A'D' : C'A'\cdot B'D' : A'B'\cdot C'D' =$$
$$BC\cdot AD : CA\cdot BD : AB\cdot CD.$$

Figure 41.

For $$B'C'\cdot A'D' = BC.\frac{r'}{r}.AD.\frac{r'}{r},$$

and $$C'A'\cdot B'D' = CA.\frac{r'}{r}.BD.\frac{r'}{r};$$

therefore $B'C'\cdot A'D' : C'A'\cdot B'D' = BC\cdot AD : CA\cdot BD.$

§ 12. Every straight line passing through a centre of similitude cuts two similar curves C, C′ at similar points.

This follows from § 1.

§ 13. Every straight line passing through a centre of similitude and touching a curve C will also touch the similar curve C′.

Figure 42.

Let a straight line through the centre of similitude O cut C at P and Q, then it will cut C′ at P′ and Q′ the points similar to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14. If on two similar curves C, C′ similar points P, P′ be taken, the tangents at P, P′ make equal angles with OPP′.

Figure 43.

On C take any point Q near to P, and on C′ find the point Q′ similar to Q. Draw the secants QPR, Q′P′R′.

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Figure 41′.

For $$B'C' = BC.\frac{rr'}{OB \cdot OC}, \quad C'A' = CA.\frac{r.r'}{OC \cdot OA};$$

therefore $$\frac{B'C'}{C'A'} = BC.\frac{rr'}{OB \cdot OC}/CA.\frac{rr'}{OC \cdot OA},$$
$$= OA \cdot BC/OB \cdot CA.$$

§ 11′. If two systems of points be inverse, then for every four points A, B, C, D and the points A′, B′, C′, D′ inverse to them
$$B'C' \cdot A'D' : C'A' \cdot B'D' : A'B' \cdot C'D' =$$
$$BC \cdot AD : CA \cdot BD : AB \cdot CD.$$

Figure 41′.

For $$B'C' \cdot A'D' = BC.\frac{rr'}{OB \cdot OC}.AD.\frac{rr'}{OA \cdot OD},$$

and $$C'A' \cdot B'D' = CA.\frac{rr'}{OC \cdot OA}.BD.\frac{rr'}{OB \cdot OD};$$

therefore $$B'C' \cdot A'D' : C'A' \cdot B'D' = BC \cdot AD : CA \cdot BD.$$

§ 12′. Every straight line passing through a centre of inversion cuts two inverse curves C, C′ at inverse points.

This follows from § 1′.

§ 13′. Every straight line passing through a centre of inversion and touching a curve C will also touch the inverse curve C′.

Figure 42′.

Let a straight line through the centre of inversion O cut C at P and Q, then it will cut C′ at P′ and Q′ the points inverse to P and Q. Now, when the points P and Q move up to each other and ultimately coincide, that is, when the straight line touches C, the points P′ and Q′ will move up to each other and ultimately coincide, that is, the straight line will touch C′.

§ 14′. If on two inverse curves C, C′ inverse points P, P′ be taken, the tangents at P, P′ make supplementary angles with OPP′.

Figure 43′.

On C take any point Q near to P, and on C′ find the point Q′ inverse to Q. Draw the secants QPR, Q′P′R′.

Then the secants QR, Q′R′ are parallel;

therefore $\angle OQP = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q′ moves to coincidence with P′, that is, the secant Q′R′ becomes the tangent P′T′.

Also when Q moves to coincidence with P, $\angle OQP$ becomes $\angle OPS$; and when Q′ moves to coincidence with P′, $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPS = \angle OP'S'$;

therefore $\angle OPT = \angle OP'T'$.

§ 15. If two curves intersect each other at any angle, the curves similar to them intersect each other at the same angle.

Figure 44.

Let the two curves C and D intersect each other at P; then C′ and D′ the curves similar to them will intersect each other at P′ the point similar to P.

Draw PT, PU tangents to C and D; and P′T′, P′U′ tangents to C′ and D′; and let O be the centre of similitude.

Then $\angle OPT = \angle OP'T'$

and $\angle OPU = \angle OP'U'$;

therefore $\angle OPT - \angle OPU = \angle OP'T' - \angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves similar to them will touch each other at the similar point P′.

§ 16. If two curves C′, C″ be both similar to a curve C with respect to the same centre of similitude O, then C′, C″ are similar to each other.

Let $r : r'$ and $r : r''$ be the two ratios of similitude.

Take any point P on C, and the similar points P′, P″ on C′ and C″.

Then
$$OP : OP' = r : r',$$
$$OP : OP'' = r : r'';$$

therefore
$$\frac{OP'}{OP}\Big/\frac{OP''}{OP} = \frac{r'}{r}\Big/\frac{r''}{r};$$

therefore
$$OP' : OP'' = \frac{r'}{r} : \frac{r''}{r}$$
$$= \text{a constant ratio.}$$

Hence C′ is similar to C″.

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Then the secants QR, Q′R′ are antiparallel;

therefore $\angle OPQ = \angle OQ'P'$.

Now, when Q moves to coincidence with P, that is, when the secant QR becomes the tangent PT, Q′ moves to coincidence with P′, that is, the secant Q′R′ becomes the tangent P′T′.

Also when Q moves to coincidence with P, $\angle OPQ$ becomes $\angle OPT$; and when Q′ moves to coincidence with P′, $\angle OQ'P'$ becomes $\angle OP'S'$;

therefore $\angle OPT = \angle OP'S'$;

therefore $\angle OPT =$ supplement of $\angle OP'T'$.

§ 15′. **If two curves intersect each other at any angle, the curves inverse to them intersect each other at the same angle.**

Figure 44′.

Let the two curves C and D intersect each other at P; then C′ and D′ the curves inverse to them will intersect each other at P′ the point inverse to P.

Draw PT, PU tangents to C and D; and P′T′, P′U′ tangents to C′ and D′; and let O be the centre of inversion.

Then $\angle OPT =$ supplement of $\angle OP'T'$

and $\angle OPU =$ supplement of $\angle OP'U'$;

therefore $\angle OPT - \angle OPU =$ supplement of $\angle OP'T'$ − supplement of $\angle OP'U'$;

therefore $\angle TPU = \angle T'P'U'$.

Hence, if two curves touch each other at any point P, the curves inverse to them will touch each other at the inverse point P′.

§ 16′. **If two curves C′, C″ be both inverse to a curve C with respect to the same centre of inversion O, then C′, C″ are similar to each other.**

Let $r{\cdot}r'$ and $r{\cdot}r''$ be the two rectangles of inversion.

Take any point P on C, and the inverse points P′, P″ on C′ and C″.

Then
$$OP{\cdot}OP' = r{\cdot}r',$$
$$OP{\cdot}OP'' = r{\cdot}r'';$$

therefore
$$OP{\cdot}OP'/OP{\cdot}OP'' = r.r'/r.r'';$$

therefore
$$OP' : OP'' = r{\cdot}r' : r{\cdot}r''$$
$$= \text{a constant ratio.}$$

Hence C′ is similar to C″.

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SIMILITUDE.

§ 17. Given a centre of similitude O and a ratio of similitude $r : r'$, to find the curve similar to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45.

Take any point P in PQ, and find in OP the point P′ such that $OP : OP' = r : r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r : r'$ is positive or negative.

Since the ratio OP : OP′ is fixed, as OP increases OP′ will also increase; and as OP diminishes OP′ will also diminish. In other words, as P moves farther and farther from O, P′ will also move farther and farther from O; as P moves nearer and nearer to O, P′ will also move nearer and nearer to O. And consequently when P is infinitely distant from O, P′ will also be infinitely distant from O; when P coincides with O, P′ will also coincide with O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from right to left or from left to right the same straight line.

Case 2. When PQ does not pass through O.

Figure 46.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP : OP' = r : r'$; through P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve similar to PQ.

Take any point Q in PQ, join OQ, and let it meet the perpendicular in Q′.

Since angles OPQ, OP′Q′ are right,

therefore PQ and P′Q′ are parallel;

therefore $$OQ : OQ' = OP : OP' = r : r'.$$

Hence Q′ is the point similar to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points similar to them situated in P′Q′; that is, the straight line P′Q′ is the curve similar to the given straight line PQ.

§ 18. If the straight line P′Q′ is similar to the straight line PQ with respect to a given centre of similitude O, the reciprocal relation also holds good, namely, that the straight line PQ is similar to the straight line P′Q′ with respect to the same centre of similitude.

§ 17′. Given a centre of inversion O and a rectangle of inversion $r.r'$, to find the curve inverse to a given straight line PQ.

Case 1. When PQ passes through O.

Figure 45′.

Take any point P in PQ, and find in OP the point P′ such that $OP \cdot OP' = r.r'$. P and P′ will be on the same side of O or on opposite sides of O according as $r.r'$ is positive or negative.

Since the rectangle OP·OP′ is fixed, as OP increases OP′ will diminish; and as OP diminishes OP′ will increase. In other words, as P moves farther and farther from O, P′ will move nearer and nearer to O; as P moves nearer and nearer to O, P′ will move farther and farther from O. And consequently when P is infinitely distant from O, P′ will coincide with O; when P coincides with O, P′ will be infinitely distant from O. Hence when P describes from right to left or from left to right the straight line PQ, P′ will also describe from left to right or from right to left the same straight line.

Case 2. When PQ does not pass through O.

Figure 46′.

Through O draw OP perpendicular to PQ, and find in OP the point P′ such that $OP \cdot OP' = r.r'$; on OP′ as diameter describe a circle.

This circle is the curve inverse to PQ.

Take any point Q in PQ, join OQ, and let it meet the circle in Q′; and join P′Q′.

Since angles OPQ, OQ′P′ are right,

therefore PQ and P′Q′ are antiparallel with respect to angle POQ;

therefore $$OQ \cdot OQ' = OP \cdot OP' = r.r'.$$

Hence Q′ is the point inverse to Q, and as Q was any point whatever in PQ, therefore all the points in PQ have the points inverse to them situated on the circle OP′Q′; that is, the circle OP′Q′ is the curve inverse to the given straight line PQ.

§ 18′. If the circle OP′Q′ is inverse to the straight line PQ with respect to a given centre of inversion O, the reciprocal relation also holds good, namely, that the straight line PQ is inverse to the circle OP′Q′ with respect to the same centre of inversion.

§ 19. Given a centre of similitude O, and a ratio of similitude $r : r'$, to find the curve similar to a circle.

Case 1. When the circle passes through O.

Figure 47.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P′ such that $OP : OP' = r : r'$; on OP′ as diameter describe the circle OP′Q′.

This circle is the curve similar to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the circle OP′Q′ at Q′. Join PQ, P′Q′.

Since angles OQP and OQ′P′ are right,

therefore PQ and P′Q′ are parallel;

therefore $$OQ : OQ' = OP : OP' = r : r'.$$

Hence Q′ is the point similar to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points similar to them situated in OP′Q′; that is, the circle OP′Q′ is the curve similar to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P′ similar to P. Join PC, and at P′ make angle OP′C′ equal to angle OPC, and let P′C′ meet OC at C′. With C′ as centre and C′P′ as radius describe a circle.

This circle will be similar to the circle C.

Since $$\angle OPC = \angle OP'C',$$

therefore PC and P′C′ are parallel;

therefore $$OP : OP' = CP : C'P'.$$

Hence $$C'P' = CP.\frac{OP'}{OP}$$

$$= \text{a constant,}$$

since CP is a constant length, and $OP : OP'$ is a constant ratio.

§ 19′. Given a centre of inversion O, and a rectangle of inversion $r{\cdot}r'$, to find the curve inverse to a circle.

Case 1. When the circle passes through O.

Figure 47′.

Let OPQ be the given circle.

Through O draw the diameter OP, and in OP find the point P′ such that $OP{\cdot}OP' = r{\cdot}r'$; at P′ draw P′Q′ perpendicular to OP′.

This perpendicular is the curve inverse to OPQ.

Take any point Q in OPQ, join OQ, and let it meet the perpendicular P′Q′ at Q′. Join PQ.

Since angles OQP and OP′Q′ are right,

therefore PQ and P′Q′ are antiparallel with respect to angle POQ;

therefore $$OQ{\cdot}OQ' = OP{\cdot}OP' = r{\cdot}r'.$$

Hence Q′ is the point inverse to Q, and as Q was any point whatever in OPQ, therefore all the points in OPQ have the points inverse to them situated in P′Q′; that is, the straight line P′Q′ is the curve inverse to the given circle OPQ.

Case 2. When the circle does not pass through O.

Figure 48′.

Let C be the centre of the given circle.

Take any point P on the circle C, join OP, and in OP find P′ inverse to P. Join PC, and at P′ make angle OP′C′ equal to the supplement of angle OPC, and let P′C′ meet OC at C′. With C′ as centre and C′P′ as radius describe a circle.

This circle will be inverse to the circle C.

Let OP meet the circle C again at Q, and join CQ.

Since supplement of $\angle OPC = \angle OP'C'$,

therefore $$\angle OQC = \angle OP'C';$$

therefore QC and P′C′ are parallel;

therefore $$OQ : OP' = CQ : C'P'.$$

Hence $$C'P' = CQ\,\frac{OP'}{OQ} = CQ.\frac{OP{\cdot}OP'}{OP{\cdot}OQ}$$
$$= \text{a constant},$$

since CQ is a constant length, and OP OP′, OP·OQ are constant rectangles.

Again $\quad OP : OP' = OC : OC'$;

therefore $$OC' = OC.\frac{OP'}{OP}$$
$$= \text{a constant.}$$

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20. If the circle C′ is similar to the circle C with respect to a given centre of similitude, the reciprocal relation also holds good, namely, that C is similar to C′ with respect to the same centre of similitude.

§ 21. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the centre of similitude.

Figure 48.

It will be seen from the construction and the reasoning in § 19, Case 2, that two circles have two centres of similitude, external or internal, according as their ratio of similitude is positive or negative.

Since $$OC : OC' = CP : C'P',$$
$$= c : c',$$

these centres of similitude are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22. Given two circles C, C′ (whose radii are c, c') similar to each other, to find the ratio of similitude.

Figure 48.

Find O the external or internal centre of similitude by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of similar points P and P′, Q and Q′; and join CP, C′P′.

Then $$c : c' = CP : C'P',$$
$$= OP : OP';$$

therefore the ratio of similitude of two circles similar to each other is the ratio of their radii.

Again $$OQ : OP' = OC : OC';$$

therefore $$OC' = OC.\frac{OP'}{OQ} = OC\cdot\frac{OP\cdot OP'}{OP\cdot OQ}$$
$$= \text{a constant};$$

therefore C′ is a fixed point.

Since C′ is a fixed point, and C′P′ is of constant length, therefore the locus of P′ is the circle C′.

§ 20′. If the circle C′ is inverse to the circle C with respect to a given centre of inversion, the reciprocal relation also holds good, namely, that C is inverse to C′ with respect to the same centre of inversion.

§ 21′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the centre of inversion.

Figure 48′.

It will be seen from the construction and the reasoning in § 19′, Case 2, that two circles have two centres of inversion, external or internal, according as their rectangle of inversion is positive or negative.

Since $$OC : OC' = CQ : C'P',$$
$$= c : c',$$

these centres of inversion are found by dividing CC′, the distance between the centres of the two circles, externally or internally, in the ratio of the radii.

§ 22′. Given two circles C, C′ (whose radii are c, c') inverse to each other, to find the rectangle of inversion.

Figure 48′.

Find O the external or internal centre of inversion by dividing CC′ externally or internally in the ratio $c : c'$; draw any straight line OPQ cutting the circles C, C′ in the pairs of inverse points P and P′, Q and Q′; and join CQ, C′P′.

Then $$c : c' = CQ : C'P',$$
$$= OQ : OP' = OP\cdot OQ : OP\cdot OP';$$

therefore the rectangle of inversion of two circles inverse to each other is a fourth proportional to c, c', and OP· OQ, the potency of O with respect to the circle C.

§ 23. A circle is similar to itself with respect to any centre of similitude O, when the ratio of similitude is unity.

For if P be any point on the circle,
since $$OP : OP' = 1,$$
therefore P′ is the same point as P.
Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes clockwise or counterclockwise the same circumference.

§ 24. If two circles be similar to each other their centres are similar points.

Figure 49.

Let C, C′ be two circles similar to each other, O their centre of similitude, and let OT touch the circle C at T. Then, by § 13, OT will touch the circle C′ at T′ the point similar to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point similar to it will, by § 12, be situated on OC. It will also, by § 7, be situated on a straight line through T′ parallel to TC. But since $\angle$ OTC and $\angle$ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is the point similar to C.

§ 25. The property of § 7 as applied to the circle may be enunciated thus:

If two circles be similar to each other the chord joining any two points of the one intersects the chord joining the two similar points of the other on the straight line at infinity.

The property of § 14 thus:

If two circles be similar to each other the tangents at two similar points of them intersect on the straight line at infinity.

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§ 23′. A circle is inverse to itself with respect to any centre of inversion O, when the rectangle of inversion is the potency of O with respect to the given circle.

For if P be any point on the circle,
since OP·OP′ = the potency of O with respect to the given circle, therefore P′ is the point where OP cuts the given circle a second time. Hence as P describes clockwise or counterclockwise the circumference of the given circle, P′ describes counterclockwise or clockwise the same circumference.

§ 24′. If two circles be inverse to each other their centres are not inverse points.

Figure 49′.

Let C, C′ be two circles inverse to each other, O their centre of inversion, and let OT touch the circle C at T. Then, by § 13′, OT will touch the circle C′ at T′ the point inverse to T. Join CT, C′T′.

Now, if the centre C be considered a point in the figure C, the point inverse to it will, by § 12, be situated on OC. It will also, by § 7′, be situated on a straight line through T′ antiparallel to TC. But since ∠ OTC and ∠ OT′C′ are right, therefore T′C′ is parallel to TC; therefore C′ is not the point inverse to C.

§ 25′. The property of § 7′ as applied to the circle may be enunciated thus:

If two circles be inverse to each other the chord joining any two points of the one intersects the chord joining the two inverse points of the other on the radical axis of the two circles.

The property of § 14′ thus:

If two circles be inverse to each other the tangents at two inverse points of them intersect on the radical axis of the two circles.