Note on a distance invariant and the calculation of Ruse's invariant

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(*Received* 13*th September*, 1940. *Read* 2*nd November*, 1940.)

1. Several papers on the subject of spatial distance in General Relativity appeared a few years ago, and a simple extension of this idea to any pair of points in any Riemannian space was given by me in a thesis[1]. A distance invariant was defined, and this was found to depend upon a certain two-point invariant which was first introduced by H. S. Ruse in a study of Laplace's Equation[2]. This invariant, now written ρ and defined in (3), has lately re-appeared[3], and it may now be of interest to publish the results found earlier. These include a geometrical interpretation of ρ, a simple method of calculation, and an expansion as a power series in the geodesic arc. The dependence of ρ upon the geodesic arc is also considered.

2. The *distance invariant*, Φ, for two points P, P' in a Riemannian space V_n is defined as follows:

If the geodesic PP' is not null, consider a thin cone of geodesics passing through P and near P'. Then Φ is proportional to the $(n-1)$-dimensional volume of cross-section of this cone at P', *i.e.* the section orthogonal to PP', and the constant of proportionality is chosen so that $\Phi \sim s^{n-1}$ when $s \to 0$, s being the geodesic arc measured from P. When the geodesic PP' is null, a cone of null geodesics is drawn to pass through P and near P'. Then Φ is proportional to the $(n-2)$-dimensional volume of cross-section at P' and is such that $\Phi \sim \sigma^{n-2}$ when $\sigma \to 0$, σ being a special parameter measured from P. In this case, any section at P' gives the same volume[4].

We shall now prove that

$$\Phi = \rho s^{n-1} \tag{1}$$

when PP' is not null, and that

$$\Phi = \rho \sigma^{n-2} \tag{2}$$

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[1] For the Senior Mathematical Scholarship, Oxford, 1934.

[2] *Proc. Edinburgh Math. Soc.* (2), 2 (1931), 137.

[3] Copson and Ruse, *Proc. Royal Soc. Edinburgh*, 60 (1940), 117.

[4] This theorem on null geodesics was given by me in *Quart. J. of Math.*, 4 (1933), 72, and has since been generalised by Prof. G. Temple, *Proc. Royal Soc.* A, 168 (1938), 122.

when PP' is null, where ρ is Ruse's invariant for the points P, P', defined in (3) below.

Let y^i be a set of normal coordinates with origin at P, the fundamental tensor then being a_{ij}. Then if the geodesic PP' is in the direction λ^i (a unit vector) at P, the coordinates of P' are $y^i = s\lambda^i$. It is well known that the element of volume of V_n at P' in the y-coordinate system is $\sqrt{\pm|a|}\,dy^1 \ldots . dy^n$, where $|a|$ denotes the determinant $|a_{ij}|$. If now we transform to a geodesic-polar system with P as origin, then, from $y^i = s\lambda^i$, it follows that the volume element is of the form $dV_n = \sqrt{\pm|a|}\,s^{n-1}\,ds\,d\omega$ where $d\omega$ is independent of s. Now if dV_{n-1} is the volume of cross-section of a thin cone with vertex at P, then $dV_n = ds\,.\,dV_{n-1}$, whence we see that along any cone, $dV_{n-1} \propto s^{n-1}\sqrt{\pm|a|}$. Since $\Phi \propto dV_{n-1}$, the constant being such that $\Phi \sim s^{n-1}$ when $s \to 0$, we finally have

$$\Phi = s^{n-1}\left(\frac{|a|_{P'}}{|a|_P}\right)^{\frac{1}{2}},$$

the suffixes denoting the points at which $|a|$ is evaluated. Passing now to a general coordinate system, with fundamental tensor g_{ij}, Ruse has shown how the coefficient of s^{n-1} above, which we call ρ, can be expressed in an invariant form. If x^i, x'^i are the coordinates of P and P', and if $\frac{1}{2}s^2$ is expressed in terms of these coordinates, then

$$\rho^2 = \frac{|a|_{P'}}{|a|_P} = \frac{|g|_P\,|g|_{P'}}{J^2}, \qquad J = \left|\frac{\partial^2\left(\frac{1}{2}s^2\right)}{\partial x^i\,\partial x'^j}\right|. \tag{3}$$

The relation (1) has now been established, and (2) can similarly be verified. Details of the latter proof will not be given here, but an equation equivalent to (2) was obtained by Etherington[1] in the case of 4-space with signature -2.

3. In order to apply the above formulae it is necessary either to find s in terms of the end points or to find a system of normal coordinates. These calculations are rarely easy, and we shall give another method for calculating ρ which has already been applied by the author to various spaces of General Relativity.

Let PP' be a non-null geodesic traced by the point $x^i(s)$, s being the arc measured from P, and let $\lambda^i_{p|}$, $p = 1, 2, \ldots, n-1$, be unit vectors displaced by parallel transport along PP', these vectors being

[1] *Phil. Mag.* (7), 15 (1933), 761.

orthogonal to each other and to the unit tangent vector $\lambda^i = dx^i/ds$ at each point of the curve. Now write

$$\gamma^p_q = e_p R_{hijk}\, \lambda^h_{p|}\, \lambda^k_{q|}\, \lambda^i\, \lambda^j, \qquad e_p = g_{ij}\, \lambda^i_{p|}\, \lambda^j_{p|} = \pm 1, \tag{4}$$

where R_{hijk} is the curvature tensor, so that γ^p_q are functions of s along PP'. Then it has been shown[1] that if $z^1, z^2, \ldots, z^{n-1}$ are small, the point $x^i + z^p\, \lambda^i_{p|}$ traces out a geodesic as s varies if the z's satisfy the equations

$$\ddot{z}^p = \gamma^p_q\, z^q$$

where dots denote differentiations with respect to s. For a geodesic which passes through P, $z^p = 0$ when $s = 0$, and the general solution subject to this restriction is of the form $z^p = \phi^p_q\, a^q$, where the a's are $n - 1$ arbitrary constants. Using matrix notation, $\gamma = (\gamma^p_q)$ and $\phi = (\phi^p_q)$ are square matrices of order $n - 1$, and ϕ satisfies the equations

$$\ddot{\phi} = \gamma\, \phi, \qquad (\phi)_0 = 0. \tag{5}$$

Consider now a thin cone of geodesics passing through P and lying near PP'. The small constants a^p can be regarded as coordinates in the cross-section, and from the definition of the z's it follows that the volume of cross-section is $\dfrac{\partial(z)}{\partial(a)}\, da^1 \ldots . da^{n-1} = |\,\phi\,|\, da^1 \ldots . da^{n-1}$, *i.e.* is proportional to $|\phi|$ since the limits of the cone are determined by constant values of the a's. Since each element of ϕ vanishes when $s = 0$, we have $|\,\phi\,| \sim |\,\dot{\phi}\,|_0\, s^{n-1}$ as $s \to 0$. Thus finally,

$$\Phi = \frac{|\,\phi\,|}{|\,\dot{\phi}\,|_0}, \qquad \rho = \frac{1}{s^{n-1}} \frac{|\,\phi\,|}{|\,\dot{\phi}\,|_0}, \tag{6}$$

where ϕ is any solution of (5).

As an example, let V_n be a space of constant curvature K. Then

$$R_{hijk} = K\, (g_{hj}\, g_{ik} - g_{hk}\, g_{ij}),$$

and from (4),

$$\gamma^p_q = -\, eK\delta^p_q, \qquad e = g_{ij}\, \lambda^i\, \lambda^j.$$

Thus (5) becomes $\ddot{\phi} + eK\phi = 0$, and this equation can at once be integrated to give

$$\phi = A \sin s\sqrt{eK},$$

[1] A. G. Walker, *Proc. Royal Soc. Edinburgh*, 52 (1932), 351.

with $(\phi)_0 = 0$, A being a constant non-singular matrix. Hence from (6)

$$\rho = \left(\frac{\sin s\sqrt{eK}}{s\sqrt{eK}}\right)^{n-1}. \tag{7}$$

As expected from the definition, it is seen that $\rho \to 1$ as $s \to 0$.

In flat space, $K = 0$ and we have $\rho = 1$ for any pair of points.

A formula similar to (6) holds when PP' is null, but in this case there are only $n - 2$ vectors $\lambda^i_{p|}$, and γ and ϕ are square matrices of order $n - 2$. The important case of a V_4 with signature $- 2$ is fully dealt with elsewhere by the author[1]. In any space of constant curvature, $e = 0$ in the above equations when PP' is null, so that $\gamma = 0$ and finally $\rho = 1$ as in flat space.

4. An advantage of the above method is that the required quantities concern only the geodesic PP', unlike methods requiring normal coordinates or the integrated form for $\frac{1}{2}s^2$. The use of vectors $\lambda^i_{p|}$ given by parallel transport along PP' leads to the remarkably simple form of equation (5), but these vectors can be avoided when the transport equations cannot be solved easily. For if $\psi^i_j = \sum_{p,q} e_q \phi^p_q \lambda^i_{p|} \lambda_{q|j}$, then the ψ's satisfy equations

$$\ddot{\psi}^i_j = \Gamma^i_k \psi^k_j, \qquad \psi^i_j \lambda^j = 0, \qquad \psi^i_j \lambda_i = 0, \qquad (\dot{\psi}^i_j)_0 = 0, \tag{9}$$

where $\ddot{\psi}^i_j$ denotes $\psi^i_{j,\,kl} \lambda^k \lambda^l$, and

$$\Gamma^i_k = R^i_{hjk} \lambda^h \lambda^j. \tag{10}$$

If now $l^i_{a|}$, $a = 1, 2, \ldots, n - 1$, are unit vectors orthogonal to each other and to λ^i, and if

$$\theta^a_b = e_a \psi^i_j l_{a|i} l^j_{b|}, \tag{11}$$

then $|\theta^a_b|$ is independent of the choice of $l^i_{a|}$ and is therefore equal to $|\phi^p_q|$. Thus we have:

If ψ^i_j is any solution of equations (9), *and if θ^a_b is given by* (11), *then*

$$\Phi = \frac{|\theta|}{|\dot{\theta}|_0}. \tag{12}$$

5. We shall now consider the expansion of ρ in powers of s, the point P and the direction PP' at P being fixed. For this purpose we shall find the following lemma useful:

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[1] *Quart. J. of Math.*, 4 (1933), 71, and *Monthly Notices R.A.S.*, 94 (1934), 159.

Lemma. If the elements of a square matrix F are functions of s so that F can be expanded in a series of powers of s with matrix coefficients, and if the first term is the unit matrix, then

$$\log |F| = \text{tr}\,(\log F) \qquad (13)$$

where tr A denotes the trace, *i.e.* the sum of the diagonal elements, of the matrix A. The log on the right is to be expanded in the ordinary way in powers of s, the order of the matrices in each product being unimportant since tr $(AB) = $ tr (BA).

To prove this lemma[1], we note that the rule for differentiating a determinant can be written $\frac{d}{ds}|F| = |F|\,\text{tr}\,(F^{-1}\dot{F})$ where $\dot{F} = \frac{dF}{ds}$, whence

$$\frac{d}{ds}\log |F| = \text{tr}\,(F^{-1}\dot{F}) = \text{tr}\left(\frac{d}{ds}\log F\right) = \frac{d}{ds}\,\text{tr}\,(\log F). \qquad (14)$$

Equation (13) is now given by integrating both sides of (14), since the term independent of s in F is assumed to be the unit matrix.

This method of obtaining the expansion of a determinant can be applied to any matrix provided the first term is non-singular. For if $f = f_0 + sf_1 + s^2 f_2 + \ldots$, then we can write $F = f_0^{-1} f$ and so obtain $\log |F|$, which is equal to $\log\left(|f|/|f_0|\right)$.

One method for expanding ρ is to return to normal coordinates and expand the fundamental tensor a_{ij} in powers of s. The above lemma can then be applied to (3) to give any desired number of terms in the expansion of ρ. The coefficients in the series for a_{ij} in powers of s are well known as the affine extensions of the fundamental tensor[2], and they can be expressed in terms of the curvature tensor and its derivatives. The final result can thus be expressed in invariant form applicable to any coordinate system. The great disadvantage of this method is that the extensions of the fundamental tensor require laborious calculations; it would be no easy matter to reach terms beyond s^4 in the expansion of ρ. Fortunately the method for calculating ρ given in this paper leads to a quite different method for finding the required expansion, this being so simple that any number of terms can be quickly calculated and expressed in invariant form.

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[1] This proof, which replaces a longer one of my own, is due to H. S. Ruse.

[2] See, for example, Veblen, "Invariants of Quadratic Differential Forms," *Cambridge Tract*, 24 (1927), 94.

6. We shall now expand ρ, or rather $\log \rho$, in powers of s by means of (5) and (6), and for this purpose we first require a recurrence relation for the matrix coefficients in the expansion of ϕ. From (5) we see that the rth derivative of ϕ can be expressed in the form

$$\frac{d^r \phi}{ds^r} = b_r \phi + c_r \dot{\phi},$$

where b_r, c_r are matrices, and on differentiating this equation, we find

$$b_{r+1} = \dot{b}_r + c_r \gamma, \qquad c_{r+1} = \dot{c}_r + b_r.$$

Eliminating the b's, we finally have

$$c_{r+2} = 2\, \dot{c}_{r+1} - \ddot{c}_r + c_r \gamma, \qquad c_0 = 0, \qquad c_1 = I, \tag{15}$$

which is a recurrence relation for the c's. Putting $s = 0$, it follows that $\left(\frac{d^r \phi}{ds^r}\right)_0 = c_r (\dot{\phi})_0$ where c_r is now evaluated at P, and we have

$$\phi\, (\dot{\phi})_0^{-1} = s\, I + \frac{s^2}{2!} c_2 + \frac{s^3}{3!} c_3 + \dots\dots$$

Hence from (13), since $\rho = | s^{-1} \phi\, (\dot{\phi})_0^{-1} |$ from (6),

$$\log \rho = \operatorname{tr} \log \left(I + \frac{s}{2!}\, c_2 + \frac{s^2}{3!} c_3 + \dots\dots\right). \tag{16}$$

This, together with (15), enables us to calculate quickly any number of terms in the expansion of $\log \rho$ and hence of ρ. For convenience we shall write

$$\log \rho = \frac{W_1}{2!}\, s + \frac{W_2}{3!}\, s^2 + \frac{W_3}{4!}\, s^3 + \dots\dots \tag{17}$$

Although γ involves $\lambda^i_{p|}$ it can be verified that these vectors disappear in the coefficients in (17) in consequence of the identities

$$\sum_p e_p\, \lambda^i_{p|}\, \lambda^j_{p|} = g^{ij} - e \lambda^i \lambda^j. \tag{18}$$

It can in fact be seen that the W's are unaltered when γ is replaced by Γ defined in (10), the derivatives being

$$\dot{\Gamma}^i_k = R^i_{hjk,\,l}\, \lambda^h \lambda^j \lambda^l, \qquad \ddot{\Gamma}^i_k = R^i_{hjk,\,lm}\, \lambda^h \lambda^j \lambda^l \lambda^m, \text{ etc.}$$

This substitution avoids continual application of (18).

From (15) the first few c's are $c_0 = 0, \quad c_1 = I, \quad c_2 = 0,$

$$c_3 = \gamma, \quad c_4 = 2\dot{\gamma}, \quad c_5 = 3\ddot{\gamma} + \gamma^2, \quad c_6 = 4\dddot{\gamma} + 4\dot{\gamma}\gamma + 2\gamma\dot{\gamma}, \text{ etc.}$$

Substituting in (16) and writing Γ in place of γ, we find

$$W_1 = 0, \qquad W_2 = \operatorname{tr} \Gamma, \qquad W_3 = 2 \operatorname{tr} \dot{\Gamma},$$

$$W_4 = \operatorname{tr} (3\ddot{\Gamma} - \tfrac{2}{3} \Gamma^2), \qquad W_5 = 4 \operatorname{tr} (\dddot{\Gamma} - \dot{\Gamma}\Gamma), \text{ etc.} \tag{19}$$

These coefficients can at once be expressed in terms of the curvature tensor R_{hijk} and the contracted tensor R_{ij}, and we finally have

$$W_2 = R_{ij}\, \lambda^i \lambda^j, \qquad W_3 = 2R_{ij,\,k}\, \lambda^i \lambda^j \lambda^k,$$
$$W_4 = (3R_{ij,\,kl} - \tfrac{2}{3} R^{h\,\cdot\,\cdot\,m}_{\cdot ij \cdot} R_{hklm}) \lambda^i\lambda^j\lambda^k\lambda^l, \text{ etc.} \tag{20}$$

These coefficients must all be evaluated at P, the vector λ^i giving the direction of the geodesic PP' at P.

7. In the above series we regard ρ, defined by two points P, P', as determined by space elements evaluated at P, the direction λ^i at P, and the geodesic arc s (or parameter σ) from P to P'. If now we keep P fixed and allow P' to vary, an obvious question presents itself: In what class of spaces is ρ dependent only upon s, *i.e.* independent of λ^i? Copson and Ruse have shown that such spaces are of considerable importance in the study of the generalised Laplace Equation; these spaces have been called *centrally harmonic* when the above requirement holds about one particular *base-point* P, and *completely harmonic* when it holds about every point of the space.

The conditions that a space shall be centrally harmonic about P have been given as an infinite set of equations to be satisfied by the fundamental and curvature tensors and their extensions at P. These conditions can be derived more simply from the expansion of the present paper, for from (17) we require that W_r, $r = 1, 2, \ldots.$ shall be independent of λ^i, this being possible since the λ's satisfy $g_{ij}\, \lambda^i \lambda^j = \text{constant}$. It is easily seen that W_r is of the form

$$W_r = W_{i_1 \ldots i_r}\, \lambda^{i_1} \ldots \lambda^{i_r}$$

where the coefficients on the right, assumed symmetric in the suffixes, are functions of the fundamental and curvature tensors and their derivatives. The required conditions are therefore

$$\begin{aligned} W_{i_1 \ldots i_r} &= 0, & r &= 1, 3, 5, \ldots. \\ W_{i_1 \ldots i_r} &= k_r \Sigma\, g_{i_1 i_2}\, g_{i_3 i_4} \ldots g_{i_{r-1} i_r}, & r &= 2, 4, 6, \ldots. \end{aligned} \tag{21}$$

where the k's are scalars, the sum being taken to give an expression symmetric in the suffixes. Any desired number of conditions can now be found and expressed directly in terms of the curvature tensor without involving the normal tensors.

When the above conditions are satisfied at P, then for any point P' it appears that ρ is a function of s, the arc PP', and is expressed as a power series in s^2. This is not strictly true except in space with positive definite metric, for the value of $g_{ij}\,\lambda^i\,\lambda^j$ changes sign when P' crosses the null cone with vertex at P. It follows from (21) that, strictly, *ρ is a function of $e.s^2$* where $e = g_{ij}\,\lambda^i\,\lambda^j = \pm 1$ or 0.

We already know that $\rho \to 1$ when $s \to 0$, and this value is reached again when $e = 0$, *i.e.* when PP' is null, for then $W_r = 0$, $r = 1, 2, \ldots.$ from (21), whence $\rho = 1$ from (17). Thus, *if a space is centrally harmonic about P and if the geodesic PP' is null, then $\rho = 1$.*

8. When a space is completely harmonic, conditions (21) are satisfied at every point. Since ρ is now a function of $e.s^2$ about each point of the space, then by alternatively keeping P fixed and moving P' and then keeping P' fixed and moving P, etc., it is easily seen that, in general, *the form of ρ as a function of $e.s^2$ must be the same for all base-points.* It follows that the scalars k_r in (21) are all constants, but we shall not assume this result in later calculations.

From (7) we see that any space of constant curvature is completely harmonic. This formula is an illustration of ρ as a function of $e.s^2$ throughout space.

Copson and Ruse have mentioned the possibility that spaces of constant curvature are the only spaces which are completely harmonic, and they have shown this to be the case for V_n when

(i) $n = 2$ or 3.

We shall now prove that this is the case also when

(ii) V_n is conformal to flat space;

(iii) $n = 4$, and the signature of V_4 is ± 2.

From (20) and (21) the condition given by $r = 2$ is

$$R_{ij} = k_2\, g_{ij}\,, \tag{22}$$

showing that V_n must be an Einstein space. When $r = 3$, it is easily seen from (20), (21) and (22) that k_2 must be constant for $n \geqq 2$. This latter result at once deals with the case (i) for $n = 2$ since k_2 is then the Gaussian curvature, and the result for $n = 3$ follows from the well known fact that an Einstein space V_3 has constant curvature. The case (ii) above follows from the fact that an Einstein space conformal to flat space has constant curvature.

Case (iii) above is of some importance since the 4-spaces of General Relativity have signature ± 2. The proof is not as simple as for the other cases and we require the next condition, given by $r = 4$.

Since $\operatorname{tr} \dot{\Gamma} = 0$, it follows that $\operatorname{tr} \ddot{\Gamma} = 0$, and the condition becomes

$$\Sigma\, g^{mm'} g^{nn'} R_{mijn} R_{m'kln'} = k_4 \Sigma\, g_{ij}\, g_{kl}\,, \tag{23}$$

where Σ denotes the sum obtained by rearranging i, j, k, l in all possible ways. These equations with (22) restrict the curvature tensor, and we shall prove that the space must have constant curvature when $n = 4$ and the signature is ± 2. It is more convenient to deal with quantities defined by

$$(ijkl) = \tfrac{1}{2}\,(R_{iklj} + R_{ilkj}) + \frac{k_2}{6}\,(g_{ik}\, g_{jl} + g_{il}\, g_{jk} - 2\, g_{ij}\, g_{kl}). \tag{24}$$

From the identities satisfied by the curvature components it is easily seen that these quantities satisfy

$$\begin{aligned} (ijkl) &= (jikl) = (ijlk) = (klij) \\ (ijkl) &+ (iklj) + (iljk) = 0. \end{aligned} \tag{25}$$

The algebra is now simplified by choosing a coordinate system so that, at any particular point P, $g_{ij} = 0$, $i \neq j$, and $g_{ii} = g^{ii} = e_i = \pm 1$. Then (22) and (23) become

$$\sum_{a=1}^{4} e_a\,(aaij) = 0, \tag{26}$$

$$\begin{aligned} \sum_{a,\,b}\; e_a\, e_b\, \{(abij)\,(abkl) + (abik)\,(ablj) + (abil)\,(abjk)\} \\ = k\,(g_{ij}\, g_{kl} + g_{ik}\, g_{lj} + g_{il}\, g_{jk}), \end{aligned} \tag{27}$$

where i, j, k, l take values 1, 2, 3, 4 and k is a scalar given by k_2 and k_4.

From (26) and (25) we get a number of equations of the form

$$e_1\, e_2\,(1122) = e_3\, e_4\,(3344), \tag{28.1}$$

$$e_2\,(1122) + e_3\,(1133) + e_4\,(1144) = 0, \tag{28.2}$$

$$e_1\,(1134) + e_2\,(2234) = 0, \tag{28.3}$$

etc., and from (27) and (25) we have, for $i = j = k = l$,

$$\begin{aligned} 2e_2\, e_3\,(1123)^2 + 2e_2\, e_4\,(1124)^2 + 2e_3\, e_4\,(1134)^2 \\ = k - (1122)^2 - (1133)^2 - (1144)^2, \end{aligned} \tag{28.4}$$

etc. Subtracting from (28.4) the similar equation found by interchanging 1 and 2, we find after using (28.1) and (28.3),

$$e_2\, e_3\,(1123)^2 - e_1\, e_4\,(2214)^2 = e_1\, e_3\,(2213)^2 - e_2\, e_4\,(1124)^2. \tag{28.5}$$

Interchanging 1 and 3, and also 2 and 4 in (28.5) and using (28.3), we get (28.5) again but with the signs changed on the right-hand side.

Thus each side of this equation must vanish, and we have a number of equations such as

$$(1123)^2 = e_1 e_2 e_3 e_4 (2214)^2.$$

When the signature of V_4 is ± 2, then $e_1 e_2 e_3 e_4 = -1$, and since all the above quantities are real, we have

$$(iijk) = 0, \qquad i, j, k \neq . \tag{28.6}$$

Equation (28.4) now becomes

$$k = (1122)^2 + (1133)^2 + (1144)^2. \tag{28.7}$$

Returning to (27) with $k = i$, $l = j$, $i \neq j$, we now find

$$(1234)^2 - \tfrac{1}{4}(1122)^2 - \tfrac{1}{2} e_3 e_4 (1133)(1144) = -k,$$

etc. Permuting 2, 3, 4 and adding,

$$(1234)^2 + (1342)^2 + (1423)^2 + 3k$$
$$= \tfrac{1}{4}\{e_2 (1122) + e_3 (1133) + e_4 (1144)\}^2 = 0$$

from (28.2). Hence, since k is a sum of squares from (28.7), we finally have, with (25), (28.1) and (28.6),

$$(ijkl) = 0, \qquad i, j; k, l = 1, 2, 3, 4. \tag{29}$$

This equation is in invariant form and therefore holds in all coordinate systems. It is also required to hold at all points of the space. It can easily be verified that (29) and (24) lead to

$$R_{iklj} = -\frac{k_2}{3}(g_{il} g_{jk} - g_{ij} g_{kl}),$$

and since this holds at all points, we have that the space must be of constant curvature.

9. Another special class of spaces, which we shall call *simply harmonic*, consists of those spaces which are completely harmonic and in which $\rho = 1$ for every pair of points. From (17), $\rho = 1$ only when $W_r = 0$ for all λ's, so that the conditions for a simply harmonic space are

$$W_{i_1 \ldots i_r} = 0, \qquad r = 1, 2, \ldots\ldots \tag{30}$$

These equations must be satisfied at all points of the space.

It has already been shown that flat space is simply harmonic, and it is probable that flat spaces are the only spaces which are simply harmonic. This is certainly the case under any set of conditions for which it is known that a space which is completely harmonic must have constant curvature. For a simply harmonic space is a special type of completely harmonic space, and (7) reduces to $\rho = 1$ only when $K = 0$, since e is not zero in all directions. We know therefore that a simply harmonic space V_n which satisfies (i), (ii) or (iii) of § 8 must be flat. In addition we can prove that: (iv) *If a real simply harmonic space has positive definite metric, then it is flat.*

Since $W_r = 0$ at all points of a simply harmonic space, we have from (19) that $\operatorname{tr} \Gamma = 0$ whence $\operatorname{tr} \dot{\Gamma} = 0$ and $\operatorname{tr} \ddot{\Gamma} = 0$. The next condition is therefore $\operatorname{tr} \Gamma^2 = 0$, *i.e.*

$$g^{ik} g^{jl} \Gamma_{ij} \Gamma_{kl} = 0, \qquad \Gamma_{ij} = R_{mijn} \lambda^m \lambda^n.$$

If the metric is positive definite we can choose a coordinate system so that $g_{ij} = g^{ij} = \delta^i_j$ at any one point P. At this point the above condition becomes $\sum_{i,j} (\Gamma_{ij})^2 = 0$, and since Γ_{ij} is real, it follows that $\Gamma_{ij} = 0$. This must hold for all λ's, whence $R_{mijn} + R_{nijm} = 0$, and it is easily verified that these equations lead to $R_{ijkl} = 0$. The curvature tensor thus vanishes at P and hence at every point of the space since P can take any position. The space is therefore flat, as stated.

T. Y. Thomas and E. W. Titt[1] have recently stated that the determinant of the fundamental tensor of a V_n is constant in every system of normal coordinates only when V_n is flat. From (3) we see that this is equivalent to saying that a space is simply harmonic only when it is flat, and although Thomas and Titt claim to have proved the theorem generally, they have in fact assumed that the metric is positive definite, *i.e.* they have proved no more than the theorem (iv) proved above. It is still possible, though I believe it improbable, that there are simply harmonic spaces with indefinite metrics which are not flat.

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[1] *Journal de Math.*, **18** (1939), 225.

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