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On Hankel Forms of Higher Weights: The Case of Hardy Spaces

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Abstract. In this paper we study bilinear Hankel forms of higher weights on Hardy spaces in several dimensions. (The Schatten class Hankel forms of higher weights on weighted Bergman spaces have already been studied by Janson and Peetre for one dimension and by Sundhäll for several dimensions). We get a full characterization of Schatten class Hankel forms in terms of conditions for the symbols to be in certain Besov spaces. Also, the Hankel forms are bounded and compact if and only if the symbols satisfy certain Carleson measure criteria and vanishing Carleson measure criteria, respectively.

1 Introduction and Main Results

The study of Hankel operators has played an important role in many areas in mathematics, such as approximation theory, the theory of Toeplitz operators, the Hilbert transform and singular integral operators [9].

The Hankel operators on Hardy and Bergman spaces can be viewed as bilinear forms; Janson–Peetre [6], and Peng–Zhang [10] discovered bilinear forms, generalizing these Hankel forms, the so-called Hankel forms of higher weights, which are defined below in (1.2) with nice invariance properties under the action of the Möbius group.

Schatten–von Neumann class Hankel forms of higher weights on Bergman spaces are characterized in [14, 15]. In the same way as for the case of Bergman spaces, Hankel forms of higher weights on a Hardy space are explicit characterizations of irreducible components in the tensor product of Hardy spaces under the Möbius group (see [10]).

Recall from [14, 15] the case of weighted Bergman spaces $L^2_a(d\iota_\nu)$ of holomorphic functions, square integrable with respect to the measure

$$d\iota_\nu(z) = c_\nu(1-|z|^2)^{\nu-(d+1)}\, dm(z), \tag{1.1}$$

where $\nu > d$, c_ν is a normalization constant and $dm(z)$ is the Lebesgue measure on the unit ball $\mathbb{B} = \{z \in \mathbb{C}^d : |z| < 1\}$. The bilinear Hankel forms of weight $s = 0, 1, 2, \ldots$ are given in [14] by

$$H^s_F(f_1, f_2) = \int_{\mathbb{B}} \langle \mathcal{T}_s(f_1, f_2), F\rangle_z\, (1-|z|^2)^{2\nu-(d+1)}\, dm(z). \tag{1.2}$$

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The transvectant $\mathcal{T}_s$ is given by

$$\mathcal{T}_s(f_1, f_2)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f_1(z) \odot \partial^{s-k} f_2(z)}{(\nu)_k (\nu)_{s-k}},$$

where

$$\partial^s f(z) = \sum_{j_1,\dots,j_s=0}^{d} \partial_{j_1} \cdots \partial_{j_s} f(z)\, dz_{j_1} \otimes \cdots \otimes dz_{j_s},$$

and $(\nu)_k = \nu(\nu+1)\cdots(\nu+k-1)$ is the Pochammer symbol. Also, the Möbius invariant inner product $\langle \cdot\,, \cdot \rangle_z$ is given in the following way; for $u, v \in \odot^s \left(\mathbb{C}^d\right)'$, where the tangent space at z is identified with $\mathbb{C}^d$,

$$\langle u, v \rangle_z = \langle \otimes^s B^t(z,z) u, v \rangle_{\otimes^s(\mathbb{C}^d)'}$$

where $B(z,z) = (1 - |z|^2)(I - \langle \cdot\,, z \rangle z)$ is the Bergman operator on $\mathbb{C}^d$ and $B^t(z,z)$ is the dual operator acting on the dual space of $\mathbb{C}^d$. The tensor-valued holomorphic function F is called the *symbol* corresponding to the Hankel form H_F^s.

Now let $\partial\mathbb{B}$ be the boundary of the unit ball $\mathbb{B}$ of $\mathbb{C}^d$. The irreducible components in the decomposition of tensor products of Hardy spaces $H^2(\partial\mathbb{B})$ in [10] can be given explicitly as *bilinear Hankel forms of weight s on the Hardy space* $H^2(\partial\mathbb{B})$ by

$$H_F^s(f_1, f_2) = \int_{\mathbb{B}} \langle \mathcal{T}_s(f_1, f_2), F \rangle_z (1 - |z|^2)^{d-1}\, dm(z), \tag{1.3}$$

where the transvectant $\mathcal{T}_s$ is here given by

$$\mathcal{T}_s(f_1, f_2)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f_1(z) \odot \partial^{s-k} f_2(z)}{(d)_k (d)_{s-k}},$$

where, in fact, this is the limiting case $\nu = d$ of (1.2).

The main results for Hankel forms H_F^s defined by (1.3) are given below in Theorems A and B.

Theorem A *H_F^s is (compact) bounded if and only if*

$$d\mu_F(z) = \|F\|_z^2 (1 - |z|^2)^{2d-1}\, dm(z)$$

is a (vanishing) Carleson measure on $H^2(\partial\mathbb{B})$, with equivalent norms.

Remark 1.1 Note that $\|F\|_z^2 = \langle F, F \rangle_z$.

Theorem B *H_F^s is of Schatten class $\mathcal{S}_p$, $2 \le p < \infty$, if and only if*

$$\|F\|_{pd,s,p} = \left(\int_{\mathbb{B}} \|F\|_z^p (1 - |z|^2)^{pd-d-1}\, dm(z) \right)^{1/p} < \infty,$$

with equivalent norms.

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Remark 1.2 If $s = 0$, we rewrite the Schatten class criterion as

$$\int_{\mathbb{B}} |(RF)(z)|^p (1 - |z|^2)^{(p-1)(d+1)}\, dm(z) < \infty, \tag{1.4}$$

where $R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}$ is the radial derivative. Theorem B is then extended to $1 \le p < \infty$, where (1.4) is equivalent to $\|F\|_{pd,0,p} < \infty$ for $1 < p < \infty$.

Remark 1.3 Janson and Peetre obtained Theorem A and Theorem B in the case $d = 1$ by using paracommutator arguments (see [6]). Our approach extends their results and provides a different proof of the case $d = 1$ they have treated.

1.1 Approach

In this paper we use different techniques to deal with the case of weight zero and the case of weight $s = 1, 2, \ldots$, and they are therefore treated separately in Section 3 and Section 4, respectively. In [14] the criteria for boundedness, compactness, and Schatten–von Neumann class for higher weights on weighted Bergman spaces are natural generalizations of the case of weight zero. For Hardy spaces, as Example 4.2 shows, the transvectant of various weights does not behave as in the case of Bergman spaces where the boundedness properties for the transvectant were necessary in order to generalize the weight zero case to arbitrary weights. This explains why we treat the weight zero and nonzero cases separately. The Hankel forms of weight zero on Hardy spaces can be rewritten, using the radial derivative, into the classical Hankel forms in [16] and then we use results from [17–19] to get the right conditions for the symbols. We have results for Carleson measures which together with invariance properties give criteria for the boundedness and compactness for Hankel forms of nonzero weights. The Schatten-class criteria are proved by using interpolation for analytic families of operators. For this purpose we need results about Hankel forms on Bergman–Sobolev-type spaces. The preliminaries in Section 2 give the prerequisites we need.

1.2 Notation

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two equivalent norms on a vector space X, then we write $\|x\|_1 \simeq \|x\|_2$, $x \in X$. Also, for two real-valued functions f and g on X we write $f \lesssim g$ if there is a constant $C > 0$, independent of the variables in question, such that $Cf(x) \le g(x)$.

2 Preliminaries

For $\alpha > -d$, let $\mathcal{A}^2_\alpha$ be the Bergman–Sobolev-type space of holomorphic functions $f : \mathbb{B} \to \mathbb{C}$ with the property that

$$\|f\|^2_\alpha = \sum_{m \in \mathbb{N}^d} |c(m)|^2 \frac{\Gamma(d+\alpha) m!}{\Gamma(d + |m| + \alpha)} < \infty,$$

where $f(z) = \sum_{m\in\mathbb{N}^d} c(m)z^m$ is the Taylor expansion of f. Then $\mathcal{A}^2_\alpha$ is a Hilbert space with the inner product

$$\langle f_1, f_2\rangle_\alpha = \sum_{m\in\mathbb{N}^d} c_1(m)\overline{c_2(m)}\frac{\Gamma(d+\alpha)m!}{\Gamma(d+|m|+\alpha)},$$

where $f_i(z) = \sum_{m\in\mathbb{N}^d} c_i(m)z^m$, $i = 1,2$. Now has a reproducing kernel, K^α_w for $w \in \mathbb{B}$, given by

$$K^\alpha_w(z) = \frac{1}{(1-\langle z,w\rangle)^{\alpha+d}}. \tag{2.1}$$

If $\alpha > 0$, then $\mathcal{A}^2_\alpha$ is the weighted Bergman space $L^2_a(d\iota_{\alpha+d})$, where $d\iota_{\alpha+d}$ is given by (1.1). Also, $\mathcal{A}^2_0$ is the Hardy space $H^2(\partial\mathbb{B})$.

2.1 Decomposition of $\mathcal{A}^2_\alpha \otimes \mathcal{A}^2_\beta$

Let G be the group of biholomorphic self-maps on $\mathbb{B}$. If $g \in G$ with $g(z) = 0$, then there is a linear fractional map φ_z on $\mathbb{B}$ and a unitary map $U \in \mathcal{U}(d)$ such that $g = U\varphi_z$. The fractional linear map φ_z is given by

$$\varphi_z(w) = \frac{z - P_z w - (1-|z|^2)^{1/2}Q_z w}{1-\langle w,z\rangle}, \tag{2.2}$$

where $P_z = \langle\,\cdot\,,z\rangle z/\|z\|^2$ and $Q_z = I - P_z$. The complex Jacobian is therefore given by $J_g = \det U \cdot J_{\varphi_z}$, where

$$J_{\varphi_z}(w) = (-1)^d\frac{(1-|z|^2)^{(d+1)/2}}{(1-\langle z,w\rangle)^{d+1}}.$$

The group G acts unitarily on $\mathcal{A}^2_\alpha$ via the following:

$$\pi_\nu(g)f(z) = f(g^{-1}(z))J_{g^{-1}}(z)^{\nu/(d+1)}, \tag{2.3}$$

where $\nu = \alpha + d$, and it gives an irreducible unitary (projective) representation of G. In addition, for $\beta > -d$, the group G acts on the Hilbert space tensor product $\mathcal{A}^2_\alpha \otimes \mathcal{A}^2_\beta$ by

$$\pi_{\nu_1}(g)\otimes\pi_{\nu_2}(g)(f_1(z), f_2(w)) = f_1(g^{-1}(z))f_2(g^{-1}(w))J_{g^{-1}}(z)^{\nu_1/(d+1)}J_{g^{-1}}(w)^{\nu_2/(d+1)},$$

where $\nu_1 = \alpha + d$, $\nu_2 = \beta + d$, and it gives a unitary (projective) representation of G. However this is not irreducible, and the irreducible decomposition is given in [10]. In particular, if $\alpha + \beta > -d - s$, then

$$\mathcal{A}^2_\alpha \otimes \mathcal{A}^2_\beta \simeq \sum_{s=0}^{\infty} \mathcal{H}^2_{\alpha+\beta+2d,s}, \tag{2.4}$$

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where $\mathcal{H}^2_{u,s}$, $u > d - s$ is the space of holomorphic functions $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$ with the property that $\int_{\mathbb{B}} \|F\|_z^2 (1 - |z|^2)^{u-d-1}\, dm(z) < \infty$, where we recall that $\|F\|_z^2 = \langle F, F\rangle_z = \langle \otimes^s B^t(z,z)F(z), F(z)\rangle_{\otimes^s(\mathbb{C}^d)'}$. The group G acts unitarily on $\mathcal{H}^2_{u,s}$ by

$$\pi_{u,s}(g^{-1})F(z) = \otimes^s dg(z)^t F(g(z))\, J_g(z)^{u/(d+1)}, \tag{2.5}$$

where $dg(z)\colon T_z(\mathbb{B}) \to T_{g(z)}(\mathbb{B})$ is the differential map and gives an irreducible unitary (projective) representation of G. Via the transvectant $\mathfrak{T}_s^{\alpha,\beta}$ defined on $\mathcal{A}^2_\alpha \otimes \mathcal{A}^2_\beta$ by

$$\mathfrak{T}_s^{\alpha,\beta}(f_1, f_2) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\partial^k f_1(z) \odot \partial^{s-k} f_2(z)}{(\alpha + d)_k (\beta + d)_{s-k}}, \tag{2.6}$$

the irreducible components in the decomposition (2.4) are realized in [10] as *Hankel forms of higher weights (order s)*:

$$H_F^{\alpha,\beta,s}(f_1, f_2) = \langle \mathfrak{T}_s^{\alpha,\beta}(f_1, f_2), F\rangle_{\alpha+\beta+2d,s,2}, \tag{2.7}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{u,s,2}$ is the $\mathcal{H}^2_{u,s}$-pairing, and $F \in \mathcal{H}^2_{\alpha+\beta+2d,s}$.

Remark 2.1 For $\alpha = \beta = 0$ in (2.7) we get the Hankel forms of weight s on Hardy spaces defined by (1.3).

The transvectant $\mathfrak{T}_s^{\alpha,\beta}\colon \mathcal{A}^2_\alpha \otimes \mathcal{A}^2_\beta \to \mathcal{H}^2_{\alpha+\beta+2d,s}$ is onto and has an intertwining property

$$\mathfrak{T}_s^{\alpha,\beta}\big(\pi_{\alpha+d}(g) f_1, \pi_{\beta+d}(g) f_2\big) = \pi_{\alpha+\beta+2d,s}(g)\mathfrak{T}_s^{\alpha,\beta}(f_1, f_2).$$

Hence,

$$H_F^{\alpha,\beta,s}(\pi_{\alpha+d}(g) f_1, \pi_{\beta+d}(g) f_2) = H^{\alpha,\beta,s}_{\pi_{\alpha+\beta+2d,s}(g^{-1})F}(f_1, f_2). \tag{2.8}$$

2.2 Spaces of Symbols and Schatten Class Hankel Forms

For $1 \le p < \infty$ and $u > d - \frac{ps}{2}$, let $\mathcal{H}^p_{u,s}$ be the space of all holomorphic functions $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$ such that

$$\|F\|^p_{u,s,p} = \int_{\mathbb{B}} \|F\|_z^p (1 - |z|^2)^{u-d-1}\, dm(z) < \infty.$$

Also, for $u \ge -\frac{s}{2}$, let $\mathcal{H}^\infty_{u,s}$ be the space of holomorphic functions $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$ such that $\|F\|_{u,s,\infty} = \sup_{z\in\mathbb{B}} \|F\|_z (1 - |z|^2)^u < \infty$. Then $\mathcal{H}^p_{u,s}$ for $1 \le p \le \infty$ are Banach spaces.

In [14, 15] there are several results about $\mathcal{H}^p_{p\nu,s}$ for $\nu > d$ and we can use these same arguments to generalize these results to $\mathcal{H}^p_{u,s}$. Hence, the results below will be stated without proofs. The reader is referred to [14, 15] for more details.

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Lemma 2.2 *Let $u + s > d$. Then the reproducing kernel of $\mathcal{H}^2_{u,s}$ is, up to a nonzero constant c, given by $K_{u,s}(w,z) = (1 - \langle w, z\rangle)^{-u} \otimes^s B^t(w,z)^{-1}$. Namely, for any $\mathbf{x} \in \odot^s(\mathbb{C}^d)'$ and any $F \in \mathcal{H}^2_{u,s}$,*

$$\begin{aligned}\langle F(z), \mathbf{x}\rangle_{\otimes^s(\mathbb{C}^d)'} &= c\langle F, K_{u,s}(\cdot, z)\mathbf{x}\rangle_{u,s,2}\\ &= c\int_{\mathbb{B}} \langle F, K_{u,s}(\,\cdot\,, z)\mathbf{x}\rangle_w (1 - |w|^2)^{u-d-1}\, dm(w).\end{aligned}$$

Lemma 2.3 *Let $1 < p < \infty$ and $1/p + 1/q = 1$. For $u > d - \frac{ps}{2}$ and $v > d - \frac{qs}{2}$ the following duality $(\mathcal{H}^p_{u,s})' = \mathcal{H}^q_{v,s}$ holds, with respect to the $\mathcal{H}^2_{(u/p)+(v/q),s}$-pairing. That is, for any bounded linear functional l on $\mathcal{H}^p_{u,s}$ there exists an element $G \in \mathcal{H}^q_{v,s}$ such that $l(F) = \langle F, G\rangle_{u/p+v/q,s,2}$ for all $F \in \mathcal{H}^p_{u,s}$, and $\|l\| \simeq \|G\|_{v,s,q}$.*

Lemma 2.4 *Let $u > -d - s$ and $v \geq -d - \frac{s}{2}$. If $2 < p < \infty$, then*

$$(\mathcal{H}^2_{u+2d,s}, \mathcal{H}^\infty_{v+d,s})_{[1-\frac{2}{p}]} = \mathcal{H}^p_{(p-2)v+u+pd,s}.$$

Lemma 2.5 *Let $\alpha, \beta > -d$ with $\alpha + \beta > -d - s$. Then there is a constant $C(\alpha, \beta, s, d) > 0$ such that*

$$\|H_F^{\alpha,\beta,s}\|_{\mathcal{S}_2(\mathcal{A}^2_\alpha, \mathcal{A}^2_\beta)} = C(\alpha, \beta, s, d)\|F\|_{\alpha+\beta+2d,s,2},$$

for all holomorphic $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$.

Remark 2.6 By computing the norms for $F = \otimes^s dz_1$, we can see that $C(\alpha, \beta, s, d)$ is continuous in α and β, since for some $C(d, s) > 0$ we have

$$C(\alpha, \beta, s, d)^2 = C(s, d)^2 \sum_{k=0}^{s} \binom{s}{k} \frac{1}{(\alpha + d)_k(\beta + d)_{s-k}}. \tag{2.9}$$

Lemma 2.7 *Let $\alpha, \beta > 0$. Then $H_F^{\alpha,\beta,s}$ is bounded on $\mathcal{A}^2_\alpha \times \mathcal{A}^2_\beta$ if and only if $F \in \mathcal{H}^\infty_{\frac{1}{2}(\alpha+\beta)+d,s}$, with equivalent norms.*

For $\alpha, \beta \geq 0$ define an operator $\tilde{\mathfrak{T}}_s^{\alpha,\beta}$ on $\mathcal{S}_\infty(\mathcal{A}^2_\alpha, \mathcal{A}^2_\beta)$ by

$$\tilde{\mathfrak{T}}_s^{\alpha,\beta}(A)(z) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} \frac{\left(\partial_w^k \odot \partial_\zeta^{s-k} \overline{A(K_w^\alpha, K_\zeta^\beta)}\right)(z,z)}{(\alpha + d)_k(\beta + d)_{s-k}},$$

where K_w^α is the reproducing kernel for $\mathcal{A}^2_\alpha$ given by (2.1).

Remark 2.8 If A has rank one, then $\tilde{\mathfrak{T}}_s^{\alpha,\beta}$ is the transvectant given by (2.6).

In the following next two results in this subsection we make use of Lemma 2.4. Namely, to get the results we need to interpolate the spaces $\mathcal{H}^2_{\alpha+\beta+2d,s}$ and $\mathcal{H}^\infty_{\frac{1}{2}(\alpha+\beta)+d,s}$, where $\alpha, \beta > 0$. In fact, by Lemma 2.4,

$$(\mathcal{H}^2_{\alpha+\beta+2d,s}, \mathcal{H}^\infty_{\frac{1}{2}(\alpha+\beta)+d,s})_{[1-\frac{2}{p}]} = \mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s}. \tag{2.10}$$

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Lemma 2.9 *Let $\alpha, \beta \geq 0$ and $2 \leq p \leq \infty$. Then $\tilde{\mathfrak{T}}_s^{\alpha,\beta}$ maps $\mathcal{S}_p(\mathcal{A}^2_\alpha, \mathcal{A}^2_\beta)$ into $\mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s}$ boundedly, and if $H_F^{\alpha,\beta,s} \in \mathcal{S}_p(\mathcal{A}^2_\alpha, \mathcal{A}^2_\beta)$, for $\alpha, \beta > 0$ and $2 \leq p < \infty$ or $\alpha = \beta = 0$ and $p = 2$, then $\tilde{\mathfrak{T}}_s^{\alpha,\beta}(H_F^{\alpha,\beta,s}) = F$.*

Using Lemma 2.5, Lemma 2.7 with (2.10) on one hand and Lemma 2.9 on the other hand, we get the following theorem.

Theorem 2.10 *Let $\alpha, \beta > 0$ and $2 \leq p \leq \infty$. Then $H_F^{\alpha,\beta,s}$ is in $\mathcal{S}_p(\mathcal{A}^2_\alpha, \mathcal{A}^2_\beta)$ if and only if $F \in \mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s}$, with equivalent norms.*

Remark 2.11 We want to extend this result to $\alpha, \beta > -1/p$, to include the Hardy case, and need therefore the theory for families of analytic operators. We use the approach by Bergh, Janson, *et al.* found in [8] and Theorem 2.12, given below.

Let X_0, X_1 be Banach spaces continuously imbedded into a Banach space X and respectively, Y_0, Y_1 and Y.

Theorem 2.12 *Let Γ be a bounded holomorphic function on the strip $0 < \Re(z) < 1$, continuous on $0 \leq \Re(z) \leq 1$ and taking values in the space of operators from $X_0 \cap X_1$ to $Y_0 + Y_1$. Suppose that*

(i) *for any $y \in \mathbb{R}$ the operator $\Gamma(iy)$ can be extended to a bounded operator from X_0 to Y_0 and $\sup_{y\in\mathbb{R}} \|\Gamma(iy)\|_{X_0 \to Y_0} = M_0 < \infty$;*
(ii) *for any $y \in \mathbb{R}$ the operator $\Gamma(1 + iy)$ can be extended to a bounded operator from X_1 to Y_1 and $\sup_{y\in\mathbb{R}} \|\Gamma(1 + iy)\|_{X_1 \to Y_1} = M_1 < \infty$.*

Then for any $\theta \in (0, 1)$ the operator $\Gamma(\theta)$ can be extended to a bounded operator from $X_{[\theta]} = (X_0, X_1)_{[\theta]}$ to $Y_{[\theta]} = (Y_0, Y_1)_{[\theta]}$ and $\|\Gamma(\theta)\|_{X_{[\theta]} \to Y_{[\theta]}} \leq M_0^{1-\theta} M_1^{\theta}$.

3 Hankel Forms of Weight Zero

To find the Schatten–von Neumann class Hankel forms of weight zero on Hardy spaces we shall rewrite H_F^0 in terms of the small Hankel operators studied in [16]. The problem then boils down to finding the relationship between the corresponding symbols.

The Hankel form H_G in [16] is given by

$$H_G(f_1, f_2) = \int_{\partial\mathbb{B}} f_1(w) f_2(w) \overline{G(w)}\, d\sigma(w), \tag{3.1}$$

where $d\sigma$ is the normalized Lebesgue measure on $\partial\mathbb{B}$. Denote by R the radial derivative, defined as

$$Rf(z) = \sum_{i=1}^{d} z_i \frac{\partial f}{\partial z_i}(z),$$

where $f\colon \mathbb{B} \to \mathbb{C}$ is holomorphic. If $R^d := (R + 2d - 1)(R + 2d - 2) \cdots (R + d)$, then for holomorphic functions f_1 and f_2 we have, by means of Taylor expansion,

$$\int_{\partial\mathbb{B}} f_1(w) \overline{f_2(w)}\, d\sigma(w) = c(d) \int_{\mathbb{B}} f_1(z) \overline{R^d f_2(z)} (1 - |z|^2)^{d-1}\, dm(z). \tag{3.2}$$

Lemma 3.1 *Let H_F^0 be given by* (1.3) *and H_G by* (3.1)*. Then $H_F^0 = H_G$ if and only if $R^d G(z) = c(d)F(z)$.*

Proof Since

$$H_F^0(f_1, f_2) = \int_{\mathbb{B}} f_1(z) f_2(z)\overline{F(z)}(1 - |z|^2)^{d-1}\, dm(z)$$

and

$$H_G(f_1, f_2) = \int_{\partial\mathbb{B}} f_1(w) f_2(w)\overline{G(w)}\, d\sigma(w),$$

then the result follows by applying (3.2) on $\tilde{f}_1 = f_1 f_2$ and $\tilde{f}_2 = G$. ■

3.1 Schatten-von Neumann Class $\mathcal{S}_p$ Hankel Forms

In this subsection we present sufficient and necessary conditions for Hankel forms of weight zero to be in Schatten–von Neumann class $\mathcal{S}_p$, $1 \le p < \infty$.

Theorem 3.2 *The Hankel form H_F^0 is of Schatten–von Neumann class $\mathcal{S}_p$, for $1 \le p < \infty$, if and only*

$$\int_{\mathbb{B}} |RF(z)|^p (1 - |z|^2)^{(p-1)(d+1)}\, dm(z) < \infty.$$

This theorem is a direct consequence of Lemma 3.1 and Theorem 1 in [16] (see also [5, Theorem C]).

Theorem 3.3 *Let $\alpha > -1$ and $1 \le p < \infty$. Then the Hankel form H_G, defined by* (3.1)*, is of Schatten–von Neumann class $\mathcal{S}_p$ if and only if*

$$\int_{\mathbb{B}} |R^{d+1} G(z)|^p (1 - |z|^2)^{(p-1)(d+1)}\, dm(z) < \infty.$$

3.2 Bounded and Compact Hankel Forms

In this subsection we present necessary and sufficient conditions for Hankel forms of weight zero to be bounded and compact (see Theorem 3.6). The definitions and results on Carleson measures and BMOA spaces used in this subsection can be found in [17, 18]. We remark that the one-dimensional case of Lemma 3.4 is already proved (see [19, Corollary 15]), but since we have not been able to find an explicit version of this result in several variables we prove this result.

Lemma 3.4 *Let $t > -1$ and $a \ge 0$. For any holomorphic function $g\colon \mathbb{B} \to \mathbb{C}$, $d\mu_1(z) = |g(z)|^2 (1 - |z|^2)^t\, dm(z)$ is a (vanishing) Carleson measure if and only if $d\mu_2(z) = |((R + a)g)(z)|^2 (1 - |z|^2)^{t+2}\, dm(z)$ is a (vanishing) Carleson measure.*

Proof We only prove equivalence for the Carleson measure case, the vanishing Carleson measure case then follows by using the same techniques. Also, we may assume that $a > 0$. Then

$$\|(R+a)f\|_{L^2\left((1-|z|^2)^{t+2}\,dm\right)} \simeq \|f\|_{L^2\left((1-|z|^2)^{t}\,dm\right)}, \tag{3.3}$$

for all holomorphic $f\colon \mathbb{B} \to \mathbb{C}$.

Assume first that $d\mu_1$ is a Carleson measure. Then there is a constant $C_1 > 0$ such that

$$\Big(\int_{\mathbb{B}} \big|\big((R+a)f\big)(z)\big|^2(1-|z|^2)^2\,d\mu_1(z)\Big)^{1/2} \leq C_1\|f\|_{H^2(\partial\mathbb{B})} \tag{3.4}$$

for all $f \in H^2(\partial\mathbb{B})$. Observing $f(R+a)g = (R+a)(fg) - ((R+a)f)g$ and applying (3.3) on $(R+a)(fg)$ and (3.4) on $((R+a)f)g$, it follows that $d\mu_2$ is a Carleson measure.

For the sufficiency, assuming $d\mu_2$ is a Carleson measure, we observe that there is a constant $s > 0$ such that

$$\sup_{w\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{d+2+s}}\,d\mu'(z) < +\infty,$$

where $d\mu'(z) = |g(z)|^2(1-|z|^2)^{t+2}\,dm(z)$. Then there is a constant $C_2 > 0$ such that

$$\Big(\int_{\mathbb{B}} \big|\big((R+a)f\big)\big|^2|g(z)|^2(1-|z|^2)^{t+2}\,dm(z)\Big)^{1/2} \leq C_2\|f\|_{H^2(\partial\mathbb{B})}, \tag{3.5}$$

for all $f \in H^2(\partial\mathbb{B})$. Now the result follows in the same way as for the necessity, using (3.5) instead of (3.4). ■

As a direct consequence of Lemma 3.4 we get a generalized version of Theorem 5.14 in [18].

Lemma 3.5 *Let k be a positive integer $a_1,\dots,a_k \geq 0$ and f holomorphic on $\mathbb{B}$. Then the following properties are equivalent:*

(i) $f \in$ (VMOA) BMOA.
(ii) $|((R+a_1)\cdots(R+a_k)f)(z)|^2(1-|z|^2)^{2k-1}\,dm(z)$ *is a (vanishing) Carleson measure.*

The classical Hankel form (small Hankel operator) H_G on the Hardy space $H^2(\partial\mathbb{B})$ as in [16] is bounded if and only if $G \in$ BMOA and H_G is compact if and only if $G \in$ VMOA (see [3, 4]). Then as a consequence of Lemma 3.5 and Lemma 3.1, we have the following theorem.

Theorem 3.6 *The Hankel form H_F^0 is (compact) bounded if and only if*

$$|F(z)|^2(1-|z|^2)^{2d-1}\,dm(z)$$

is a (vanishing) Carleson measure.

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4 The Case $s = 1, 2, 3, \ldots$

In this section we study boundedness, compactness, and the class $\mathcal{S}_p$ properties $2 \leq p < \infty$ for the case $s \geq 1$.

As in [14] we have the following Besov characterization. However, this lemma does not hold for $s = 0$.

Lemma 4.1 *For any positive integer s,*

$$|f(0)|^2 + \cdots + \|\partial^{s-1} f(0)\|^2 + \int_{\mathbb{B}} \|\partial^s f\|_z^2 \frac{dm(z)}{1-|z|^2} \sim \|f\|_{H^2}^2,$$

for all $f \in H^2(\partial\mathbb{B})$.

The difficulty for the Hardy spaces is explained by this example, where it is shown that we can find $f_1, f_2 \in H^2(\partial\mathbb{B})$ such that $\mathcal{T}_s(f_1, f_2) \notin \mathcal{H}^1_{d,s}$.

Example 4.2 This example is based on the proof of Theorem II in [12]. First consider the case when $s = 1$ and $d = 1$. Let

$$f_1(z) = \sum_{k=1}^{\infty} \frac{1}{k} z^{2^k} \quad \text{and} \quad f_2(z) = 1.$$

Then $f_1, f_2 \in H^2(\partial\mathbb{D})$, and since the series $f_1(z)$ is lacunary, then

$$\|\mathcal{T}_1(f_1, f_2)\|_{1,1,1} = \int_{\mathbb{D}} |f_1'(z)|\, dm(z) = \infty.$$

This is a consequence of a result about lacunary series by Zygmund (see [12]). Namely, if $n_{k+1}/n_k > \lambda$ for some $\lambda > 1$, and if $h(z) = \sum_{k=0}^{\infty} c_k z^{n_k}$ satisfies $\int_0^1 |h'(re^{i\theta})|\, dr < \infty$ for some θ, then $\sum_{k=0}^{\infty} |c_k| < \infty$.

In the general case, $d \geq 1$ and $s = 1, 2, \ldots$, we just change f_1 into

$$f_1(z) = \sum_{k=1}^{\infty} \frac{1}{k} z_1^{2^k},$$

and still let $f_2(z) = 1$. Then

$$\begin{aligned}
\|\mathcal{T}_s(f_1, f_2)\|_{d,s,1} &= \int_{\mathbb{B}} (1-|z_1|^2)^{s/2}(1-|z|^2)^{s/2-1} \left| \frac{\partial^s f_1}{\partial z_1^s}(z) \right| dm(z) \\
&\geq \int_{\mathbb{B}} (1-|z|^2)^{s-1} \left| \frac{\partial^s f_1}{\partial z_1^s}(z) \right| dm(z).
\end{aligned}$$

By Theorem 2.17 in [18] there is a constant $C > 0$ such that

$$\int_{\mathbb{B}} (1-|z|^2)^{s-1} \left| \frac{\partial^s f_1}{\partial z_1^s}(z) \right| dm(z) \geq C \int_{\mathbb{B}} \left| \frac{\partial f_1}{\partial z_1}(z) \right| dm(z)$$

and the right-hand side of the inequality above is infinite, as we can see in the initial case ($s = 1, d = 1$).

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4.1 Boundedness and Compactness

Criteria for boundedness and compactness are given in Theorem 4.5 and Theorem 4.7, respectively. To prove these theorems we need some lemmas.

For holomorphic $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$ we consider the norm $\|F\|_{CM}$ given by

$$\|F\|_{CM}^2 = \sup_{w\in\mathbb{B}} \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{2d}}\, d\mu_F(z),$$

where $d\mu_F(z) = \|F\|_z^2(1-|z|^2)^{2d-1}\, dm(z)$.

Lemma 4.3 *Let $F\colon \mathbb{B} \to \odot^s(\mathbb{C}^d)'$ be holomorphic and let k be a nonnegative integer. If the measure $d\mu_F(z) = \|F\|_z^2(1-|z|^2)^{2d-1}\, dm(z)$ is a Carleson measure, then there is a constant $C_k > 0$ such that*

$$\int_{\mathbb{B}} \|\partial^k f\|_z^2\, d\mu_F(z) \le C_k \|F\|_{CM}^2 \|f\|_{H^2}^2,$$

for all $f \in H^2(\partial\mathbb{B})$.

Proof This is clear if $k = 0$. Assume k is a positive integer. If $f \in H^2(\partial\mathbb{B})$, then $\partial^k f \in \mathcal{H}^2_{d,s}$ by Lemma 4.1. Hence, by the reproducing property in Lemma 2.2 and by Lemma 7.1 in [14],

$$\|\partial^k f\|_z \lesssim \int_{\mathbb{B}} \frac{(1-|z|^2)^{s/2}(1-|w|^2)^{s/2-1}}{|1-\langle z,w\rangle|^{d+s}}\, \|\partial^k f\|_w\, dm(w).$$

Let $0 < \varepsilon < 1$. Then by Proposition 1.4.10 in [13]

$$\|\partial^k f\|_z^2 \lesssim \int_{\mathbb{B}} \frac{(1-|w|^2)^{\varepsilon-1}}{|1-\langle z,w\rangle|^{d+\varepsilon}}\, \|\partial^k f\|_w^2\, dm(w),$$

and hence, by Lemma 4.1

$$\begin{aligned}\int_{\mathbb{B}} \|\partial^k f\|_z^2\, d\mu_F(z) &\lesssim \int_{\mathbb{B}} (1-|w|^2)^{\varepsilon-1} \|\partial^k f\|_w^2 \Big(\int_{\mathbb{B}} \frac{d\mu_F(z)}{|1-\langle z,w\rangle|^{d+\varepsilon}} \Big)\, dm(w)\\ &\lesssim \|F\|_{CM}^2 \int_{\mathbb{B}} \|\partial^k f\|_w^2 \frac{dm(w)}{1-|w|^2} \sim \|F\|_{CM}^2\|f\|_{H^2}^2. \qquad \blacksquare\end{aligned}$$

We need to consider subspaces of $\mathcal{H}^2_{u,s}$, $u > d - s$, namely $\mathcal{B}^2_{u,s}$ which consists of elements $F = \partial^s f$, where $f\colon \mathbb{B} \to \mathbb{C}$ is holomorphic and $\|F\|_{u,s,2} < \infty$.

Lemma 4.4 *Let*

$$X = \Big\{ S \in \mathcal{H}^2_{3d,s} : \|S\|_{3d,s,2} = \sup_{\|\partial^s f\|_{d,s,2}=1} |\langle \partial^s f, S\rangle_{2d,s,2}| \Big\}.$$

Then $(\mathcal{B}^2_{d,s})' \simeq X$ with respect to the pairing $\langle \partial^s f, S\rangle_{2d,s,2}$. That is, for any bounded linear functional l on $\mathcal{B}^2_{d,s}$ there is an element $S \in X$ such that $l(\partial^s f) = \langle \partial^s f, S\rangle_{2d,s,2}$ and $\|l\| \simeq \|S\|_{3d,s,2}$.

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Proof Let $l \in (\mathcal{B}^2_{d,s})'$. Extend l to $\tilde{l} \in (\mathcal{H}^2_{d,s})'$ with $\|\tilde{l}\| = \|l\|$ and $\tilde{l}(\partial^s f) = l(\partial^s f)$ by the Hahn–Banach Theorem. Then by Lemma 2.3 there is an element $S \in \mathcal{H}^2_{3d,s}$ with $\|\tilde{l}\| \simeq \|S\|_{3d,s,2}$ so $\|l\| \simeq \|S\|_{3d,s,2}$. In this sense $(\mathcal{B}^2_{d,s})'$ can be embedded continuously in $\mathcal{H}^2_{3d,s}$ and can therefore be viewed as a subspace of $\mathcal{H}^2_{3d,s}$. Hence,

$$(\mathcal{B}^2_{d,s})' \simeq \Big\{ S \in \mathcal{H}^2_{3d,s} : \|S\|_{3d,s,2} = \sup_{\|\partial^s f\|_{d,s,2}=1} |\langle \partial^s f, S\rangle_{2d,s,2}| \Big\}$$

with respect to the pairing $\langle \partial^s f, S\rangle_{2d,s,2}$. ■

Now we can prove the criterion for boundedness.

Theorem 4.5 *The Hankel form H^s_F is bounded if and only if*

$$d\mu_F(z) = \|F\|^2_z (1-|z|^2)^{2d-1}\, dm(z)$$

is a Carleson measure with equivalent norms.

Proof First assume that $d\mu_F$ is a Carleson measure. It suffices to prove that for $k > 0$

$$\Big| \int_{\mathbb{B}} \langle \partial^k f_1 \otimes \partial^{s-k} f_2, F\rangle_z (1-|z|^2)^{d-1}\, dm(z) \Big| \lesssim \|F\|_{CM} \|f_1\|_{H^2} \|f_2\|_{H^2}.$$

This is a direct consequence of Lemma 4.1 and Lemma 4.3, since

$$\begin{aligned}
&\Big| \int_{\mathbb{B}} \langle \partial^k f_1 \otimes \partial^{s-k} f_2, F\rangle_z (1-|z|^2)^{d-1}\, dm(z) \Big| \\
&\qquad \leq \int_{\mathbb{B}} \|\partial^k f_1\|_z \|\partial^{s-k} f_2\|_z \|F\|_z (1-|z|^2)^{d-1}\, dm(z) \\
&\qquad \leq \Big(\int_{\mathbb{B}} \|\partial^k f_1\|^2_z \frac{dm(z)}{1-|z|^2} \Big)^{1/2} \cdot \Big(\int_{\mathbb{B}} \|\partial^{s-k} f_2\|^2_z \, d\mu_F(z) \Big)^{1/2} \\
&\qquad \leq C_{s,k} \|F\|_{CM} \|f_1\|_{H^2} \|f_2\|_{H^2}
\end{aligned}$$

for some constant $C_{s,k} > 0$.

Now assume that H^s_F is bounded. Let $G_w = \pi_{2d,s}(\varphi_w)F$ where the action $g \to \pi_{2d,s}(g)$ is defined in (2.5), and the fractional linear map is defined in (2.2). Since $\varphi_w^{-1} = \varphi_w$, then by (2.8)

$$H^s_{G_w}(f_1, f_2) = H^s_F\big(\pi_d(\varphi_w) f_1, \pi_d(\varphi_w) f_2\big), \tag{4.1}$$

where $g \to \pi_d(g)$ is the unitary action on $H^2(\partial\mathbb{B})$ defined in (2.3).

Since $\pi_{2d,s}(\varphi_w)$ is unitary on $\mathcal{H}^2_{2d,s}$ (or even on $L^2_{2d,s}$, the space of measurable F with $\|F\|_{2d,s,2} < \infty$), then

$$\begin{aligned}
\|\pi_{2d,s}(\varphi_w)F\|^2_{3d,s,2} &= \int_{\mathbb{B}} \|F\|^2_z (1-|\varphi_w(z)|^2)^d (1-|z|^2)^{d-1}\, dm(z) \\
&= \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{2d}} \|F\|^2_z (1-|z|^2)^{2d-1}\, dm(z).
\end{aligned}$$

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Hence we can make the following reformulation of $\|F\|_{CM}$:

$$\|F\|_{CM} = \sup_{w\in\mathbb{B}} \|G_w\|_{3d,s,2}. \tag{4.2}$$

It follows from (4.1) that $\|H^s_{G_w}\| = \|H^s_F\|$ and hence $H^s_{G_w}$ is bounded for any $w \in \mathbb{B}$. Define $T_{G_w}(\partial^s f) = \langle \partial^s f, G_w\rangle_{2d,s,2}$ on $\mathcal{B}^2_{d,s}$. Then $T_{G_w}(\partial^s f) = H^s_{G_w}(f,1)$, and by Lemma 4.1,

$$|T_{G_w}(\partial^s f)| \le \|H^s_{G_w}\| \cdot \|f\|_{H^2} \lesssim \|H^s_{G_w}\| \cdot \|\partial^s f\|_{d,s,2},$$

so $T_{G_w} \colon \mathcal{B}^2_{d,s} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{B}^2_{d,s}$. Hence, by Lemma 4.4 and Lemma 4.1,

$$\begin{aligned}
\|G_w\|_{3d,s,2} &\simeq \sup_{\|\partial^s f\|_{d,s,2}=1} |\langle \partial^s f, G_w\rangle_{2d,s,2}| \\
&= \sup_{\|\partial^s f\|_{d,s,2}=1} |H^s_{G_w}(f,1)| \\
&\le \sup_{\|\partial^s f\|_{d,s,2}=1} \|H^s_{G_w}\| \cdot \|f\|_{H^2} \\
&\lesssim \|H^s_{G_w}\| = \|H^s_F\|.
\end{aligned}$$

So by (4.2) $\|F\|_{CM} \lesssim \|H^s_F\|$. ■

Before we can prove the criterion for compactness we need one more lemma.

Lemma 4.6 *Let $F \colon \mathbb{B} \to \bigodot^s(\mathbb{C}^d)'$ be holomorphic, and $F_r(z) = F(rz)$ for $0 < r < 1$. If $d\mu_F(z) = \|F\|_z^2(1-|z|^2)^{2d-1}\, dm(z)$ is a vanishing Carleson measure, then*

$$\|F_r - F\|_{CM} \to 0, \quad \text{as } r \to 1^-.$$

Proof If $d\mu_F(z)$ is a vanishing Carleson measure, then

$$\lim_{|w|\to 1^-} \int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{2d}}\, d\mu_F(z) = 0.$$

Hence, this lemma is a direct consequence of the fact that

$$\int_{\mathbb{B}} \frac{(1-|w|^2)^d}{|1-\langle z,w\rangle|^{2d}}\, d\mu_{F_r}(z) \lesssim \int_{\mathbb{B}} \frac{(1-|rw|^2)^d}{|1-\langle z,rw\rangle|^{2d}}\, d\mu_F(z)$$

and dominated convergence. ■

Theorem 4.7 *The Hankel form H^s_F is compact if and only if*

$$d\mu_F(z) = \|F\|_z^2(1-|z|^2)^{2d-1}\, dm(z)$$

is a vanishing Carleson measure.

Proof First, assume that $d\mu_F(z)$ is a vanishing Carleson measure. Then by Theorem 4.5 and Lemma 4.6 $\|H^s_{F_r} - H^s_F\| \lesssim \|F_r - F\|_{CM} \to 0$ as $r \to 1^-$. Hence, it suffices to prove that $H^s_{F_r}$ is compact. But since F_r can be approximated in Carleson norm by its Taylor polynomials $P_N^{(r)}$ and $H^s_{P_N^{(r)}}$ has finite rank, then $H^s_{F_r}$ is clearly compact (see the proof of the sufficiency in [14, Theorem 1.1(b)]).

Now assume that H^s_F is compact. As in the proof of Theorem 4.5, let $G_w = \pi_{2d,s}(\varphi_w)F$. Then $d\mu_F(z)$ is a vanishing Carleson measure if and only if for any sequence $\{w_n\} \subset \mathbb{B}$ such that $|w_n| \to 1^-$ as $n \to \infty$

$$\lim_{n\to\infty} \|G_{w_n}\|_{3d,s,2} = 0. \tag{4.3}$$

Again, as in the proof of Theorem 4.5, by Lemma 4.4

$$\begin{aligned}\|G_{w_n}\|_{3d,s,2} &= \sup_{\|\partial^s f\|_{d,s,2}=1} |H^s_{G_{w_n}}(f,1)| \\ &= \sup_{\|\partial^s f\|_{d,s,2}=1} |H^s_F(\pi_d(\varphi_{w_n})f, \pi_d(\varphi_{w_n})1)|.\end{aligned}$$

The action $g \to \pi_d(g)$ is unitary on $H^2(\partial\mathbb{B})$ and $\{\pi_d(\varphi_{w_n})1\}$ is a sequence in $H^2(\partial\mathbb{B})$ converging weakly to 0. Since H^s_F is compact, there is a sequence $\{c_n\}$ of positive numbers converging to 0 such that $|H^s_F(\pi_d(\varphi_{w_n})f, \pi_d(\varphi_{w_n})1)| \leq c_n\|f\|_{H^2}$. By Lemma 4.1, $\|G_{w_n}\|_{3d,s,2} \lesssim c_n \to 0$ as $n \to \infty$, which proves (4.3). ■

4.2 Schatten–von Neumann Class

In this subsection we prove Theorem B for $s \geq 1$. For this purpose we prove two more general results: Theorem 4.12 (valid for $s \geq 1$) and Theorem 4.14 (valid for $s \geq 0$). Then Theorem B follows by letting $\alpha = \beta = 0$. The main idea is to use the interpolation theorem for families of analytic operators. To do this we first need to rewrite Hankel forms on Bergman–Sobolev-type spaces to forms on Hardy spaces.

For $t \in \mathbb{C}$, we define the radial fractional derivative of order t by

$$(1+R)^t f(z) = \sum_{m\in\mathbb{N}^d} (1+|m|)^t c(m) z^m,$$

where $f(z) = \sum_{m\in\mathbb{N}^d} c(m)z^m$ is the Taylor expansion of f. Lemma 4.8 follows by using Taylor expansion and Stirling's formula.

Lemma 4.8 *If $2\Re(t) + \alpha > -1$, then*

$$\|f\|^2_\alpha \simeq \int_{\mathbb{B}} |(1+R)^t f(z)|^2 (1-|z|^2)^{2\Re(t)+\alpha}\, dm(z),$$

for all holomorphic functions $f\colon \mathbb{B} \to \mathbb{C}$.

As a direct consequence of Lemma 4.8 we have the following lemma.

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Lemma 4.9 *Let $\alpha > -d$. Then $\|f\|_{H^2} \simeq \|(1+R)^{\alpha/2} f\|_\alpha$, for all holomorphic functions $f\colon \mathbb{B} \to \mathbb{C}$.*

By Lemma 4.9 the Hankel forms $H_F^{\alpha,\beta,s}$ given by (2.7), defined on $\mathcal{A}^2_\alpha \times \mathcal{A}^2_\beta$, can be regarded as forms defined on $H^2(\partial\mathbb{B}) \times H^2(\partial\mathbb{B})$ via

$$\tilde{H}_F^{\alpha,\beta,s}(f_1, f_2) := H_F^{\alpha,\beta,s}\big((1+R)^{\alpha/2} f_1, (1+R)^{\beta/2} f_2\big). \tag{4.4}$$

Namely, as a direct consequence of Lemma 4.9, using (4.4), we have the following result.

Lemma 4.10 *Let $\alpha, \beta > -d$ and $p \in \{2, \infty\}$. Then*

$$\|\tilde{H}_F^{\alpha,\beta,s}\|_{\mathcal{S}_p(H^2,H^2)} \simeq \|H_F^{\alpha,\beta,s}\|_{\mathcal{S}_p(\mathcal{A}^2_\alpha,\mathcal{A}^2_\beta)}.$$

Remark 4.11 We can extend (4.4) to complex numbers α and β. In this case, if $\Re(\alpha), \Re(\beta) > -d$, then

$$\|\tilde{H}_F^{\alpha,\beta,s}\|_{\mathcal{S}_p(H^2,H^2)} = \|\tilde{H}_F^{\Re(\alpha),\Re(\beta),s}\|_{\mathcal{S}_p(H^2,H^2)}$$

for $p \in \{2, \infty\}$, by unitary operators.

Theorem 4.12 *Let $2 \le p < \infty$ and $\alpha, \beta > -1/p$. Then $\tilde{H}_F^{\alpha,\beta,s} \in \mathcal{S}_p(H^2, H^2)$ if $F \in \mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s}$. Moreover, $\|\tilde{H}_F^{\alpha,\beta,s}\|_{\mathcal{S}_p(H^2,H^2)} \lesssim \|F\|_{\frac{1}{2}p(\alpha+\beta)+pd,s,p}$.*

Remark 4.13 The proof of this theorem is based on the techniques used to prove Theorem 1 in [8].

Proof Put $\alpha_1 = \alpha - \frac{p-2}{2p}$, $\beta_1 = \beta - \frac{p-2}{2p}$, $\alpha_2 = \alpha + \frac{1}{p}$, and $\beta_2 = \beta + \frac{1}{p}$. Clearly $\alpha_1, \beta_1 > -1/2$ and $\alpha_2, \beta_2 > 0$. We will use interpolation for the analytic families of operators. For this purpose consider for $0 \le \Re(z) \le 1$ the forms $\tilde{H}_F^{\alpha_z,\beta_z,s}$ given by (4.4), where $\alpha_z = \alpha_1 + z(\alpha_2 - \alpha_1)$ and $\beta_z = \beta_1 + z(\beta_2 - \beta_1)$. Now we can define the analytic family of operators $\{\Gamma(z)\}$ on the strip $0 \le \Re(z) \le 1$ into operators from the intersection $\mathcal{H}^2_{\alpha_1+\beta_1+2d,s} \cap \mathcal{H}^\infty_{\frac{1}{2}(\alpha_2+\beta_2)+d,s}$ into $\mathcal{S}_2 + \mathcal{S}_\infty$, where $\Gamma(z)F = \tilde{H}_F^{\alpha_z,\beta_z,s}$. Consider $\Re(z) = 0$. By Remark 4.11, Lemma 4.10, and Lemma 2.5, if $F \in \mathcal{H}^2_{\alpha_1+\beta_1+2d,s}$, then $\|\tilde{H}_F^{\alpha_z,\beta_z,s}\|_{\mathcal{S}_2} \simeq \|H_F^{\alpha_1,\beta_1,s}\|_{\mathcal{S}_2} \simeq \|F\|_{\alpha_1+\beta_1+2d,s,2}$. Consider $\Re(z) = 1$. By Remark 4.11, Lemma 4.10, and Lemma 2.7, if $F \in \mathcal{H}^\infty_{\frac{1}{2}(\alpha_2+\beta_2)+d,s}$, then

$$\|\tilde{H}_F^{\alpha_z,\beta_z,s}\|_{\mathcal{S}_\infty} \simeq \|H_F^{\alpha_2,\beta_2,s}\|_{\mathcal{S}_\infty} \lesssim \|F\|_{\frac{1}{2}(\alpha_2+\beta_2)+d,s,\infty}.$$

Now we claim that there is a constant $C(d, s)$ such that

$$\|\Gamma(z)F\|_{\mathcal{S}_2} \le C(d, s)\|F\|_{\alpha_1+\beta_1+2d,s,2} \tag{4.5}$$

for $0 \le \Re(z) \le 1$ and for all $F \in \mathcal{H}^2_{\alpha_1+\beta_1+2d,s}$. Accepting temporarily the claim, since $\mathcal{S}_2 \subset \mathcal{S}_\infty$ continuously and since $\mathcal{H}^2_{\alpha_1+\beta_1+2d,s} \cap \mathcal{H}^\infty_{\frac{1}{2}(\alpha_2+\beta_2)+d,s} \subset \mathcal{H}^2_{\alpha_1+\beta_1+2d,s}$

continuously, then Γ is bounded on the strip $0 \le \Re(z) \le 1$. Hence we can apply the interpolation theorem for the analytic families of operators (see Theorem 2.12). We obtain for fixed $0 < \theta < 1$ that $\Gamma(\theta)$ is bounded from $(\mathcal{H}^2_{\alpha_1+\beta_1+2d,s}, \mathcal{H}^\infty_{\frac{1}{2}(\alpha_2+\beta_2)+d,s})_{[\theta]}$ into $(\mathcal{S}_2, \mathcal{S}_\infty)_{[\theta]}$. Put $\theta = (p-2)/p$. Using Lemma 2.4, we get

$$\left(\mathcal{H}^2_{\alpha_1+\beta_1+2d,s}, \mathcal{H}^\infty_{\frac{1}{2}(\alpha_2+\beta_2)+d,s}\right)_{[1-\frac{2}{p}]} = \mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s},$$

and hence $\|H_F^{\alpha,\beta,s}\|_{\mathcal{S}_p} \lesssim \|F\|_{\frac{1}{2}p(\alpha+\beta)+pd,s,p}$, since $\alpha_\theta = \alpha$ and $\beta_\theta = \beta$ when $\theta = (p-2)/p$.

Now we go back to the claim (4.5). We may assume that z is real, and we therefore put $z = \theta \in [0,1]$. By Lemma 4.10, $\|\Gamma(\theta)F\|_{\mathcal{S}_2(H^2,H^2)} \simeq \|H_F^{\alpha_\theta,\beta_\theta,s}\|_{\mathcal{S}_2(\mathcal{A}^2_{\alpha_\theta},\mathcal{A}^2_{\beta_\theta})}$, and since $\alpha_\theta > \alpha_1 > -1/2$, $\beta_\theta > \beta_1 > -1/2$, then

$$\begin{aligned}\|H_F^{\alpha_\theta,\beta_\theta,s}\|_{\mathcal{S}_2(\mathcal{A}^2_{\alpha_\theta},\mathcal{A}^2_{\beta_\theta})} &\le C(d,s)\sqrt{s!}\|F\|_{\alpha_\theta+\beta_\theta+2d,s,2}\\ &\le C(d,s)'\|F\|_{\alpha_1+\beta_1+2d,s,2},\end{aligned}$$

by Lemma 2.5, where $C(d,s)$ is the constant in (2.9). ∎

Theorem 4.14 *Let $2 \le p < \infty$ and $\alpha, \beta \ge 0$. Then $F \in \mathcal{H}^p_{\frac{1}{2}p(\alpha+\beta)+pd,s}$ if*

$$H_F^{\alpha,\beta,s} \in \mathcal{S}_p(H^2,H^2).$$

Moreover, $\|F\|_{\frac{1}{2}p(\alpha+\beta)+pd,s,p} \lesssim \|H_F^{\alpha,\beta,s}\|_{\mathcal{S}_p(H^2,H^2)}$.

Proof Consider $\tilde{\mathfrak{T}}_s^{\alpha,\beta}$ defined by (2.6). By Lemma 2.9 it remains to prove that

$$\tilde{\mathfrak{T}}_s^{0,0}(H_F^{0,0,s}) = F$$

if $H_F^{0,0,s} \in \mathcal{S}_p(H^2,H^2)$ for $2 \le p < \infty$. Let $H_F^{0,0,s} \in \mathcal{S}_p$ and let $F_r(z) = F(rz)$, for $r \in (0,1)$. Since $H_F^{0,0,s}$ is compact, then $\|F\|_z^2(1-|z|^2)^{2d-1}\,dm(z)$ is a vanishing Carleson measure by Theorem 4.7, and hence $\|F_r - F\|_{CM} \to 0$ as $r \to 1^-$ by Lemma 4.6. Then $F_r \to F$ pointwise and also by Theorem 4.5 we have $\|H_{F_r}^{0,0,s} - H_F^{0,0,s}\|_{\mathcal{S}_\infty} \to 0$ as $r \to 1^-$. Hence, by Lemma 2.9,

$$\tilde{\mathfrak{T}}_s^{0,0}(H_F^{0,0,s}) = \lim_{r\to 1^-} \tilde{\mathfrak{T}}_s^{0,0}(H_{F_r}^{0,0,s}) = \lim_{r\to 1^-} F_r = F. \qquad ∎$$

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