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THE SINGULAR CONGRUENCE AND THE MAXIMAL QUOTIENT SEMIGROUP

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It is a well known result (see [4, p. 108]) that if R is a ring and $Q(R)$ its maximal right quotient ring, then $Q(R)$ is (von Neumann) regular if and only if every large right ideal of R is dense. This condition is equivalent to saying that the singular ideal of R is zero. In this note we show that the condition loses its magic in the theory of semigroups.

Throughout we let S denote a semigroup with 0 and 1. A right ideal D is *dense* if and only if for all $x_1, x_2, x \in S$ with $x_1 \neq x_2$, there exists $d \in D$ such that $x_1 d \neq x_2 d$ and $xd \in D$. A right ideal L is *large* if and only if $L \cap I \neq 0$ for every nonzero right ideal I. A dense right ideal is easily seen to be large. Set $J(S)=\{x : x^r \text{ is large}\}$ where $x^r=\{s : xs=0\}$. It can be shown that $J(S)$ is an ideal (two-sided) of S, and it is called the *singular ideal* of S.

A semigroup T containing S as a subsemigroup is called a *right quotient semigroup* of S if for every $t_1, t_2, t \in T$ with $t_1 \neq t_2$, there exists $s \in S$ such that $t_1 s \neq t_2 s$ and $ts \in S$. Let S^Δ denote the set of all dense right ideals of S, and let $\mathrm{Hom}_S(D, S)$ denote the set of all right S-homomorphisms of $D \in S^\Delta$ into S. Set $Q(S) = \bigcup_{D \in S^\Delta} \mathrm{Hom}_S(D, S)$, where we set $q_1 = q_2$ ($q_1, q_2 \in Q(S)$) if and only if q_1 agrees with q_2 on some dense right ideal. It was shown in [5] that $Q(S)$ is the maximal right quotient semigroup of S. The embedding of S into $Q(S)$ is done by considering an element of S as a left multiplication on S.

We need one last definition. S is *regular* if and only if for each $a \in S$, there exists $b \in S$ such that $aba=a$. It is easy to see that if S is regular, then for each $a \in S$ there exists $x \in S$ such that $axa=a$ and $xax=x$.

The following theorem which we state without proof is due to Johnson [3].

THEOREM. *If $Q(S)$ is regular, then $J(S)=0$.*

Define the relation ψ by $a\psi b$ if and only if $ax=bx$ for all x in some large right ideal. ψ is called the *singular congruence* of S.

PROPOSITION. *ψ is a congruence relation.*

Proof. ψ is clearly an equivalence relation so that we only need show that ψ is left and right compatible [1, p. 16].

Let $a\psi b$ and $x \in S$. Assume $as=bs$ for all $s \in L$, where L is large. Consider $x^{-1}L=\{y \in S : xy \in L\}$. $x^{-1}L$ is large and $axy=bxy$ for all $y \in x^{-1}L$. Hence $ax\psi bx$ and ψ is right compatible. Left compatibility is obvious.

PROPOSITION. $\psi=i$ (*the identity relation*) *if and only if every large right ideal is dense.*

Proof. The "if" part is clear from the definition of a dense right ideal.

Assume $\psi=i$ and let L be a large right ideal. Let $x_1 \neq x_2$, $x \in S$ and consider $x^{-1}L$. Now $x^{-1}L$ is large which implies that $L^*=x^{-1}L \cap L$ is also large. Since $\psi=i$, there exists $a \in L^* \subseteq L$ such that $x_1 a \neq x_2 a$. Also $a \in L^* \subseteq x^{-1}L$ implies that $xa \in L$. Thus L is dense.

If $\psi=i$, then $J(S)=0$ but the converse is not true as will be seen below. The following examples also show that $\psi=i$ is neither a necessary nor sufficient condition for $Q(S)$ to be regular.

EXAMPLE 1. Let S be a semilattice of two groups with 0 and 1 adjoined (see [1, p. 128]). Thus $S=G_\alpha \cup G_\beta \cup 0 \cup 1$. Assume $\alpha<\beta$. In [6] we showed that $Q(S)$ is regular. But $\psi \neq i$ since the ideal $L=S\backslash 1$ is large but not dense.

EXAMPLE 2. Let T be a Baer–Levi semigroup as defined in [2, p. 82]. Thus T is a right cancellative, right simple semigroup without idempotents. Adjoin a 0 and 1, and set $S=T \cup 0 \cup 1$.

The only nonzero right ideal of S is $D=T \cup 0$ and thus D is the only proper large right ideal of S. We assert that D is dense. It suffices to show that if $s_1, s_2 \in S$ with $s_1 \neq s_2$, then there exists $d \in D$ such that $s_1 d \neq s_2 d$. The only question arises when $s_1=1$ and $s_2 \in T$. By Lemma 8.4 of [2], the equation $xy=y$ holds for no elements $x, y \in T$. Thus $s_2 d \neq d=1d$ for $d \in T$. Therefore D is dense and $\psi=i$.

Now let $a \in T$. We claim that a is not a regular element of $Q(S)$. Assume it is. Then there exists $q \in \mathrm{Hom}_S(D, S)$ such that $aqa=a$ and $qaq=q$ (recall that a is considered as a left multiplication). Since qaq agrees with q on D, we have $(qaq)(a)=q(a)$. But $(qaq)(a)=(qa)(q(a))=q(aq(a))=q(a)q(a)$. Therefore $q(a)$ is idempotent. Since T contains no idempotents, we must have $q(a)=0$ or $q(a)=1$.

If $q(a)=0$, then $aqa=a$ implies that $0=aq(a)a=(aqa)(a)=a(a)=a^2$ which is a contradiction. Assume $q(a)=1$. Since T is right simple, $aT=T$ so that there exists $y \in T$ such that $ay=a$. Hence $y=1y=q(a)y=q(ay)=q(a)=1$. This again is a contradiction. Therefore $Q(S)$ is not a regular semigroup.

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