Canad. J. Math. Vol. **61** (6), 2009 pp. 1357–1374

# On a Class of Landsberg Metrics in Finsler Geometry

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*Abstract.* In this paper, we study a long existing open problem on Landsberg metrics in Finsler geometry. We consider Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. We show that a *regular* Finsler metric in this form is Landsbergian if and only if it is Berwaldian. We further show that there is a two-parameter family of functions, $\phi = \phi(s)$, for which there are a Riemannian metric $\alpha$ and a 1-form $\beta$ on a manifold $M$ such that the scalar function $F = \alpha\phi(\beta/\alpha)$ on $TM$ is an almost regular Landsberg metric, but not a Berwald metric.

## 1 Introduction

In Finsler geometry, we consider scalar functions $F = F(x, y)$ on the tangent bundle $TM$ of a manifold $M$. The restriction $F_x := F|_{T_xM}$ on each tangent space $T_xM$ is a (Minkowski) norm, so that the length of a curve is well-defined. Thus Finsler manifolds are smooth metric spaces. If $g = g_{ij}(x)dx^i \otimes dx^j$ is a Riemannian metric on $M$, then $F := \sqrt{g_{ij}(x)y^i y^j}$ is a Finsler metric. Finsler metrics are generalized Riemannian metrics. Riemannian metrics have two special properties:

(i) The geodesic coefficients $G^i := \frac{1}{2}\Gamma^i_{jk}(x)y^i y^j$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x$. Here $G^i$ appear in the geodesic equations

$$\frac{d^2x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0.$$

(ii) All the tangent spaces $(T_xM, F_x)$ are linearly isometric to each other as Euclidean spaces.

The first property is crucial to showing that the exponential map $\exp_x\colon T_xM \to M$ is $C^\infty$ at the origin. The second property reveals another important property of Riemannian spaces, that is, the infinitesimal geometry at every point is Euclidean.

However, Finsler metrics do not share the same properties as Riemannian metrics, in general. Those with the first property are called *Berwald metrics*. On a Berwald manifold $(M, F)$, all tangent spaces $(T_xM, F_x)$ are linearly isometric as norm spaces. In this sense, a Berwald manifold is modeled on a single norm space. The local structures of Berwald metrics have been completely determined by Z. I. Szabo [8]. A typical non-trivial example of Berwald metrics on $R^3$ is

$$F(x, y) = \sqrt{e^{\rho(s,t)}(u^2 + v^2) + w^2 + \sqrt{e^{2\rho(s,t)}(u^2 + v^2)^2 + w^4}},$$

Received by the editors May 15, 2007.

Supported by the National Natural Science Foundation of China(10671214) and by the National Science Foundation (DMS-0810159)

AMS subject classification: Primary: 53B40; secondary: 53C60.

where $x = (s, t, r) \in R^3$ and $y = (u, v, w) \in T_xR^3$.

For a Finsler metric $F$ on a manifold $M$ the Minkowski norm $F_x := F|_{T_xM}$ on $T_xM$ induces a Riemannian metric $\hat{g}_x := g_{ij}(x, y)dy^i \otimes dy^j$ on $T_xM \setminus \{0\}$. It is obvious that if $F$ is a Berwald metric, then all $(T_xM \setminus \{0\}, \hat{g}_x)$ are isometric as Riemannian manifolds. Thus it is a natural problem to understand Finsler manifolds having all tangent spaces $(T_xM \setminus \{0\}, \hat{g}_x)$ isometric.

This leads to the notion of Landsberg metric. By definition, Landsberg metrics are defined by

$$L_{jkl} := -\tfrac{1}{2} F F_{y^i} [G^i]_{y^j y^k y^l} = 0. \tag{1.1}$$

It is known that on a Landsberg manifold $(M, F)$, all $(T_xM \setminus \{0\}, \hat{g}_x)$ are isometric. Clearly, (1.1) always holds for Berwald metrics. However, the classification of Landsberg metrics is far from complete.

The following is a long-standing open problem in Finsler geometry:

*Is there any Landsberg metric which is not Berwaldian?*

People wish to find some explicit examples if the answer is "yes". Recently, R. Bryant claimed that to have classified two-dimensional Landsberg metrics.[1] There are many Landsberg metrics which are not Berwaldian. The author does not know if his examples can be expressed explicitly in terms of elementary functions.

In order to find explicit examples of Landsberg metrics, we consider $(\alpha, \beta)$-metrics. The so-called $(\alpha, \beta)$-metrics are Finsler metrics defined by a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ and a 1-form $\beta = b_i y^i$ on a manifold $M$. They are "computable" Finsler metrics. Thus it is natural to search for Landsberg metrics in this class.

An $(\alpha, \beta)$-metric is a scalar function on $TM$ defined by

$$F := \alpha \phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a $C^\infty$ function on $(-b_o, b_o)$, $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form. It can be shown that for any Riemannian metric $\alpha$ and any 1-form $\beta$ on $M$ with $\|\beta_x\|_\alpha < b_o$ the function $F = \alpha\phi(\beta/\alpha)$ is a (positive definite) Finsler metric if and only if $\phi$ satisfies

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad (|s| \leq \rho < b_o), \tag{1.2}$$

(Lemma 2.1). Such $(\alpha, \beta)$-metrics are said to be *regular*.

Randers metrics are special $(\alpha, \beta)$-metrics defined by $\phi = 1 + s$, *i.e.*, $F = \alpha + \beta$. It is well known that a Randers metric is a Landsberg metric if and only if it is a Berwald metric [5, 7]. In this paper, we shall prove the following.

***Theorem 1.1*** *Assume $n \geq 3$. Let $F = \alpha\phi(\beta/\alpha)$ be a regular $(\alpha, \beta)$-metric on an $n$-dimensional manifold, where $\phi = \phi(s)$ is a $C^\infty$ function on $(-b_o, b_o)$ such that $\phi \neq k_1\sqrt{1 + k_2 s^2}$ for any constants $k_1 > 0$ and $k_2$. Then $F$ is a Landsberg metric if and only if $\beta$ is parallel with respect to $\alpha$; in this case, $F$ is a Berwald metric.*

[1] Personal communication.

Note that if $\phi = k_1\sqrt{1+k_2 s^2}$, then $F = \alpha\phi(\beta/\alpha) = k_1\sqrt{\alpha^2+k_2\beta^2}$ is a Riemannian metric. If $(\alpha,\beta)$-metrics are allowed to be singular in two extremal directions, then there are some non-trivial solutions. Let $\phi = \phi(s)$ satisfy (1.2). A function $F = \alpha\phi(\beta/\alpha)$ is called an *almost regular* $(\alpha,\beta)$*-metric* if $\beta$ satisfies that $\|\beta_x\|_\alpha \le b_o, \forall x \in M$. An almost regular $(\alpha,\beta)$-metric $F = \alpha\phi(\beta/\alpha)$ might be singular (even undefined) in the two extremal directions $y \in T_xM$ with $\beta(x,y) = \pm b_o\alpha(x,y)$.

To state our result, we shall first introduce some notations. Let $b_{i|j}$ denote the coefficients of the covariant derivative of $\beta = b_i y^i$ with respect to $\alpha = \sqrt{a_{ij}y^iy^j}$. Let

$$r_{ij} := \tfrac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \tfrac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_j := b^i r_{ij}, \qquad s_j := b^i s_{ij},$$

where $b^i := a^{ij}b_j$. Let $b(x) := \|\beta_x\|_\alpha$. It is easy to see that $\beta$ is closed if and only if $s_{ij} = 0$.

***Theorem 1.2*** *Assume $n \ge 3$. Let $F = \alpha\phi(\beta/\alpha)$ be an almost regular $(\alpha,\beta)$-metric, where $\phi = \phi(s)$ is defined on $(-b_o, b_o)$ such that $\phi \ne k_1\sqrt{1+k_2s^2}$ for any constants $k_1$ and $k_2$. Let $b(x) := \|\beta_x\|_\alpha \ne 0$. Then $F$ is a Landsberg metric if and only if either $\beta$ is parallel with respect to $\alpha$, in this case, $F$ is a Berwald metric, or $\phi$ is given by*

$$\phi(s) = c_4 \exp\Big(\int_0^s \frac{c_1\sqrt{1-(t/b_o)^2} + c_3 t}{1 + t(c_1\sqrt{1-(t/b_o)^2} + c_3 t)}\, dt\Big), \tag{1.3}$$

*and $\beta$ satisfies the following equations:*

$$b = b_o, \tag{1.4}$$

$$s_{ij} = 0, \tag{1.5}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j), \tag{1.6}$$

*where $c_1, c_3, c_4$ are constants with $c_1 \ne 0$, $1 + c_3 b_o > 0$, and $c_4 > 0$, and $k = k(x)$ is a scalar function. Moreover, $F$ is not a Berwald metric if and only if $k \ne 0$.*

The function $\phi$ in (1.3) can be expressed in terms of elementary functions. See (7.2) below.

Letting $b_o = 1$, $c_1 = g \ne 0$ and $c_3 = 0$ in (1.3), we obtain a special family of functions

$$\phi = c_4 \exp\Big(\int_0^s \frac{g\sqrt{1-t^2}}{1 + gt\sqrt{1-t^2}}\, dt\Big). \tag{1.7}$$

Now $\phi$ can be expressed in terms of elementary functions. See (7.3) below. For the function $\phi$ in (1.7), a Riemannian metric $\alpha$, and a 1-form $\beta$ with $b(x) := \|\beta_x\|_\alpha \le 1$, by Theorem 1.2, the almost regular $(\alpha,\beta)$-metric $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric if and only if $b \equiv 1$, $s_{ij} = 0$, and $r_{ij} = k(a_{ij} - b_ib_j)$. When $k \ne 0$ and $n > 2$, $F$ is not a Berwald metric. We notice that this special family of Landsberg metrics were actually introduced by G. S. Asanov [1, 2].

There are many Riemannian metrics $\alpha$ and 1-forms $\beta$ satisfying (1.5) and (1.6) with $k \ne 0$. Below is a simple example.

***Example 1.1*** At a point $\mathbf{x} = (x, y, z) \in R^3$ and in the direction

$$\mathbf{y} = (u, v, w) \in T_{\mathbf{x}} R^3,$$

define $\alpha = \alpha(\mathbf{x}, \mathbf{y})$ and $\beta = \beta(\mathbf{x}, \mathbf{y})$ by $\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)}$ and $\beta := u$. Then $\alpha$ and $\beta$ satisfy (1.5) and (1.6) with $b = \|\beta_{\mathbf{x}}\|_\alpha \equiv 1$ and $k = 1$. Let

$$F := \alpha \exp\Big( \int_0^{\beta/\alpha} \frac{c_1\sqrt{1-t^2} + c_3 t}{1 + t(c_1\sqrt{1-t^2} + c_3 t)}\, dt \Big).$$

Then $F$ is an almost regular Landsberg metric, but not a Berwald metric. This metric is singular in two directions $\mathbf{y} = (\pm 1, 0, 0) \in T_{\mathbf{x}} R^3$ at any point $\mathbf{x}$.

## 2 Landsberg Curvature

For a Finsler metric $F = F(x, y)$ on a manifold $M$, the spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a vector field on $TM$, where $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{g^{il}}{4}\big\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \big\}, \tag{2.1}$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$ and $(g^{ij}) = (g_{ij})^{-1}$. By definition, $F$ is called a *Berwald metric* if $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ for any $x \in M$. On a Berwald manifold $(M, F)$, all tangent spaces $T_x M$ with the induced Minkowski norm $F_x$ are linearly isometric. Moreover, $F$ is affinely equivalent to a Riemannian metric $g$, namely, $F$ and $g$ have the same spray. Using this fact, one can determine the local structure of a Berwald metric [8].

The Landsberg tensor $L = L_{ijk} dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{jkl} := -\tfrac{1}{2} F F_{y^i} [G^i]_{y^j y^k y^l}. \tag{2.2}$$

Clearly, if $G^i = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i}|_x \in T_x M$ for any $x \in M$, then $L_{jkl} = 0$. A Finsler metric is called a *Landsberg metric* if $L_{jkl} = 0$. It is known that on a Landsberg manifold $M$, all tangent spaces $T_x M$ with the induced Riemannian metric $\hat{g}_x = g_{ij}(x, y) dy^i \otimes dy^j$ are isometric.

Now we consider $(\alpha, \beta)$-metrics. Let $\phi = \phi(s)$ be a $C^\infty$ function on $(-b_o, b_o)$. Let $\alpha = \sqrt{a_{ij} y^i y^j}$ be a Riemannian metric and $\beta = b_i y^i$ a 1-form on a manifold $M$. The norm of $\beta$ with respect to $\alpha$ is given by

$$\|\beta_x\|_\alpha := \sqrt{a^{ij}(x) b_i(x) b_j(x)}.$$

Define

$$F := \alpha \phi(s), \quad s = \frac{\beta}{\alpha}.$$

https://doi.org/10.4153/CJM-2009-064-9 Published online by Cambridge University Press

By a direct computation, we obtain

$$FF_{y^i} = g_{ij}y^j = \phi(\phi - s\phi')y_i + \phi\phi'\alpha b_i \tag{2.3}$$

and

$$\det\left(g_{ij}\right) = \phi^{n+1}\left(\phi - s\phi'\right)^{n-2}\left[(\phi - s\phi') + (\|\beta_x\|_\alpha^2 - s^2)\phi''\right]\det\left(a_{ij}\right). \tag{2.4}$$

Using the continuity, one can easily show the following.

***Lemma 2.1*** *Let $\phi = \phi(s)$ be a $C^\infty$* positive *function on $(-b_o, b_o)$. Let $F = \alpha\phi(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. Then for any Riemannian metric $\alpha$ and any 1-form $\beta$ with $\|\beta_x\|_\alpha < 1$, the function scalar function $F = \alpha\phi(\beta/\alpha)$ is a Finsler metric if and only if $\phi = \phi(s)$ satisfies* (1.2).

**Proof** Let $\phi = \phi(s)$ be a $C^\infty$ function on $(-b_o, b_o)$ satisfying (1.2). For any $s$ with $|s| < b_o$, taking $\rho = |s|$ in (1.2), we obtain

$$\phi(s) - s\phi'(s) > 0, \quad (|s| < b_o). \tag{2.5}$$

Let $\phi_\lambda(s) := (1-\lambda) + \lambda\phi(s)$. Then $\phi_\lambda$ satisfies (1.2) and (2.5) for any $0 \le \lambda \le 1$.

Let $\alpha$ be a Riemannian metric and $\beta$ be a 1-form. Let $b := \|\beta_x\|_\alpha$. Fix an arbitrary tangent vector $y \in T_xM$. Then $s := \beta(x,y)/\alpha(x,y)$ satisfies $|s| < b_o$. Let $F^\lambda := \alpha\phi_\lambda(\beta/\alpha)$ and $g^\lambda_{ij} := \frac{1}{2}[(F^\lambda)^2]_{y^iy^j}$. By (2.4),

$$\det(g^\lambda_{ij}(x,y)) = [(1-\lambda) + \lambda\phi]^{n+1}[(1-\lambda) + \lambda(\phi - s\phi')]^{n-2} \times \{(1-\lambda) + \lambda[(\phi - s\phi') + (\|\beta_x\|_\alpha^2 - s^2)\phi'']\}\det(a_{ij}).$$

Clearly, $\det(g^\lambda_{ij}(x,y)) \ne 0$. Since $(g^0_{ij}) = (a_{ij})$ is positive definite, by the continuity, we conclude that $(g^1_{ij}(x,y)) = (g_{ij}(x,y))$ is positive definite. Thus $F$ is a regular positive definite Finsler metric.

Conversely, suppose that $F = \alpha\phi(\beta/\alpha)$ is a regular Finsler metric. First, for an arbitrary number $b$ with $0 \le b < b_o$, one can choose $\{\alpha, \beta\}$ such that $\|\beta_{x_o}\|_\alpha = b$ at some point $x_o \in M$. For an arbitrary number $s$ with $|s| \le b$, we can always find a vector $y \in T_{x_o}M$ such that $\beta(x_o, y) = s\alpha(x_o, y)$. By assumption, $F(x_o, y) = \alpha\phi(s) > 0$, so we conclude that $\phi(s) > 0$. By another assumption, $\det(g_{ij}(x_o, y)) > 0$, we conclude from (2.4) that $\phi(s) - s\phi'(s) \ne 0$, provided that $n > 2$. Since $\phi(0) > 0$, and $s$ and $b$ are arbitrary with $|s| \le b < b_o$, we must have $\phi(s) - s\phi'(s) > 0$, $(|s| < b_o)$. Now by (2.4) we conclude that $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0$, $(|s| \le b < b_o)$. If $n = 2$, we still get the above inequality from (2.4). This proves the lemma. ■

To compute the Landsberg curvature of an $(\alpha, \beta)$-metric $F = \alpha\phi(\beta/\alpha)$, we first find the spray coefficients of $F$. Let $r_{i0} := r_{ij}y^j$, $s_{i0} := s_{ij}y^j$, $r_0 := r_jy^j$, and $s_0 := s_jy^j$.

Let $G^i = G^i(x,y)$ and $\bar{G}^i = \bar{G}^i(x,y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. By (2.1), we obtain the following identity.

$$G^i = \bar{G}^i + \alpha Q s^i{}_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\left\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\right\}, \tag{2.6}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{Q - sQ'}{2(1 + sQ + (b^2 - s^2)Q')}.$$

Note that the coefficients of $G^i$ depend on $Q$ directly and $\phi$ is given by

$$\phi = \exp\Big(\int \frac{Q}{1 + sQ}\, ds\Big).$$

The formula (2.6) has been given in a different form in [3, 6].

Clearly, if $\beta$ is parallel with respect to $\alpha$ ($r_{ij} = 0$ and $s_{ij} = 0$), then $G^i = \bar{G}^i$ are quadratic in $y$ and $F$ is a Berwald metric. However, as we see in Theorem 1.3, for a positively almost regular Berwald $(\alpha, \beta)$-metric, $\beta$ is not necessarily parallel with respect to $\alpha$.

By a direct and lengthy computation using (2.2) and (2.3), we obtain the following form of the expression for $L_{jkl}$.

$$L_{jkl} = -\frac{\rho}{6\alpha^5}\{h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3E_j h_{kl} + 3E_k h_{jl} + 3E_l h_{jk}\}, \tag{2.7}$$

where $\rho := \phi(\phi - s\phi')$, and

$$h_j := \alpha b_j - s y_j, \quad h_{jk} := \alpha^2 a_{jk} - y_j y_k, \quad C_j := (X_4 r_{00} + Y_4 \alpha s_0) h_j + 3\Lambda J_j,$$
$$E_j := (X_6 r_{00} + Y_6 \alpha s_0) h_j + 3\mu J_j, \quad J_j := \alpha^2 (s_{j0} + \Gamma r_{j0} + \Pi \alpha s_j) - (\Gamma r_{00} + \Pi \alpha s_0) y_j,$$

where

$$X_4 := \frac{1}{2\Delta^2}\{-2\Delta Q''' + 3(Q - sQ')Q'' + 3(b^2 - s^2)(Q'')^2\},$$
$$X_6 := \frac{1}{2\Delta^2}\{(Q - sQ')^2 + (2(s + b^2 Q) - (b^2 - s^2)(Q - sQ'))Q''\},$$
$$Y_4 := -2QX_4 + \frac{3Q'Q''}{\Delta}, \quad Y_6 := -2QX_6 + \frac{(Q - sQ')Q'}{\Delta},$$
$$\Lambda := -Q'', \quad \mu := -\frac{1}{3}(Q - sQ'),$$
$$\Gamma := \frac{1}{\Delta}, \quad \Pi := -\frac{Q}{\Delta},$$

where $\Delta := 1 + sQ + (b^2 - s^2)Q'$.

## 3 Some ODEs

We need the following lemmas.

***Lemma 3.1*** *If a function $\phi = \phi(s)$ satisfies $\mu = 0$, then $\phi = c_1\sqrt{1 + c_2 s^2}$, where $c_1$ and $c_2$ are constants with $c_1 > 0$.*

***Lemma 3.2*** *If a function $\phi = \phi(s)$ satisfies $\Lambda = 0$, then*

$$\phi = c_3 \exp\Big( \int_0^s \frac{c_1 + c_2 t}{1 + c_1 t + c_2 t^2}\, dt \Big),$$

*where $c_1, c_2$, and $c_3$ are constants with $c_3 > 0$.*

**Proof** $\Lambda = 0$ implies that $Q'' = 0$. Thus $Q = c_1 + c_2 s$. Then

$$\phi' = \frac{Q}{1 + sQ}\phi = \frac{c_1 + c_2 s}{1 + c_1 s + c_2 s^2}\phi.$$

This determines $\phi$. ■

Our central term is given as follows:

$$\mu X_4 - \Lambda X_6 = \frac{(Q - sQ')Q''' + 3s(Q'')^2}{3(1 + sQ + (b^2 - s^2)Q')}. \tag{3.1}$$

We have the following.

***Lemma 3.3*** *If $\phi = \phi(s)$ satisfies*

$$\mu X_4 - \Lambda X_6 = 0, \tag{3.2}$$

*then either*

$$\phi = c_4 \exp\Big( \int \frac{c_1\sqrt{1 + c_2 s^2} + c_3 s}{1 + s(c_1\sqrt{1 + c_2 s^2} + c_3 s)}\, ds \Big), \tag{3.3}$$

*where $c_1, c_2, c_3$, and $c_4$ are constants with $c_1 \neq 0$ and $c_4 > 0$, or*

$$\phi = \begin{cases} c_3 s^{\frac{c_1}{1+c_1}} (1 + c_1 + c_2 s^2)^{\frac{1}{2(1+c_1)}} & \text{if } c_1 \neq -1, \\ c_3 s e^{\frac{1}{2c_2 s^2}} & \text{if } c_1 = -1, \end{cases} \tag{3.4}$$

*where $c_1, c_2$, and $c_3$ are constants with $c_1 \neq 0$ and $c_3 > 0$.*

**Proof** By (3.1), we get $(Q - sQ')Q''' + 3s(Q'')^2 = 0$. Note that $Q - sQ' \neq 0$, *i.e.*, $F$ is non-Riemannian. Then $[(Q - sQ')^{-3}Q'']' = 0$. Thus

$$(Q - sQ')^{-3}Q'' = k_1 \quad (Q - sQ')^{-3}(-sQ'') = -k_1 s.$$

Let $H := Q - sQ'$. We get $H^{-3}H' = -k_1 s$, $(H^{-2})' = 2k_1 s$, $H^{-2} = k_1 s^2 + k_2$. Since $H^{-2} > 0$ for $0 < s < b_o$, we can see that $k_2 \geq 0$ and $k_1 > 0$ when $k_2 = 0$.

***Case I:*** If $k_2 > 0$, take $c_1 = \pm 1/\sqrt{k_2} \neq 0$ and $c_2 = k_1/k_2$. We get

$$H = \frac{1}{c_1\sqrt{1 + c_2 s^2}}.$$

Then

$$Q = c_1\sqrt{1 + c_2 s^2} + c_3 s. \tag{3.5}$$

Solving (3.5) we obtain (3.3).

***Case II:*** If $k_2 = 0$ (hence $k_1 > 0$), take $c_1 = \pm 1/(2\sqrt{k_1}) \neq 0$. We get $H = \frac{2c_1}{s}$. Then

$$Q = \frac{c_1}{s} + c_2 s. \tag{3.6}$$

Solving the equation, we obtain (3.4). ■

**Lemma 3.4** *Suppose that there is a number $d$ such that $\phi$ satisfies*

$$db^2 s\Gamma + b^2\Pi - s = 0. \tag{3.7}$$

*Then*

$$Q = \frac{k_1(b^2 - s^2)}{s} + \frac{db^2 - 1}{2s}, \tag{3.8}$$

*where $k_1$ is a number independent of $s$. Thus $\phi$ satisfies*

$$\{2sX_4 + 3\Lambda\Gamma\}db^2 + \{b^2 Y_4 + 3\Lambda\} = 0, \quad \{2sX_6 + 3\mu\Gamma\}db^2 + \{b^2 Y_6 + 3\mu\} = 0,$$

**Proof** Equation (3.7) can written as

$$\frac{sdb^2 - b^2 Q}{1 + sQ + (b^2 - s^2)Q'} - s = 0.$$

It is easy to solve the above ODE and get (3.8). The remaining argument is straightforward using the formulas for $X_i, Y_i$ $\mu$, $\Lambda$, and $\Gamma$. ■

Note that if we write $Q$ in (3.8) in the following form $Q = c_1/s + c_2 s$, then $d = (1 + 2c_1 + 2c_2 b^2)/b^2$.

**Lemma 3.5** *Let $(k, c) \neq 0$. Let $\phi$ be given in (3.3) or $Q$ be given in (3.5). Then*

$$\{(b^2 - s^2)X_4 - 3\Lambda\Gamma s\}k + \{s^2 X_4 + 3\Lambda\Gamma s\}c = 0, \tag{3.9}$$

$$\{(b^2 - s^2)X_6 - 3\mu\Gamma s\}k + \{s^2 X_6 + 3\mu\Gamma s\}c = 0, \tag{3.10}$$

*if and only if $c_2 = -1/b^2$ and $c = 0$. Moreover, $\phi$ does not satisfy (3.7).*

**Proof** Plugging $Q = c_1\sqrt{1 + c_2 s^2} + c_3 s$ into (3.10) and setting $s = 0$, we get

$$\frac{kb^2 c_1^2(c_2 b^2 + 1)}{2(1 + c_3 b^2)^2} = 0. \tag{3.11}$$

It is also easy to see that at $s = 0$,

$$sX_6 + 3\mu\Gamma = -\frac{c_1}{1 + c_3 b^2} \neq 0.$$

Thus $sX_6 + 3\mu\Gamma \neq 0$ for small $s$. It follows from (3.10) that $k \neq 0$. Then by (3.11) we get $c_2 = -1/b^2$. Conversely, if $c_2 = 1/b^2$ and $c = 0$, then $Q = c_1\sqrt{1 - (s/b)^2} + c_3 s$ satisfies (3.9) and (3.10). ■

***Lemma 3.6*** *Let $\phi$ be given in* (3.4) *or $Q$ be given in* (3.6). *Then* (3.9) *and* (3.10) *hold if and only if*

$$c = -\frac{1 + c_1 + c_2 b^2}{c_1} k. \tag{3.12}$$

*Furthermore,* (3.7) *holds if and only if*

$$d = \frac{1 + 2c_1 + 2c_2 b^2}{b^2}. \tag{3.13}$$

**Proof** Plugging $Q = c_1/s + c_2 s$ into (3.9) and (3.10) yields

$$\frac{6c_1 b^2((1 + c_1 + c_2 b^2)k + c_1 c)}{(s^2 + 2c_1 s^2 - c_1 b^2 + c_2 b^2 s^2)^2} = 0, \quad \frac{2s^2 c_1 b^2((1 + c_1 + c_2 b^2)k + c_1 c)}{(s^2 + 2c_1 s^2 - c_1 b^2 + c_2 b^2 s^2)^2} = 0.$$

Thus (3.9) and (3.10) hold if and only if $c$ is given by (3.12). Plugging $Q = c_1/s + c_2 s$ into (3.7) yields

$$\frac{s^3(1 + 2c_1 + 2c_2 b^2 - db^2)}{s^2 + 2c_1 s^2 - c_1 b^2 + c_2 b^2 s^2} = 0.$$

Thus (3.7) holds if and only if $d$ is given by (3.13). ■

## 4 Landsberg Metrics

In this section, we are going to derive a sufficient and necessary condition for an $(\alpha, \beta)$-metric to be Landsbergian. First note that

$$h_j b^j = \alpha(b^2 - s^2), \quad h_j y^j = 0,$$
$$h_{jk} b^k = \alpha h_j, \quad h_{jk} y^k = 0, \quad h_{jk} b^j b^k = \alpha^2 (b^2 - s^2),$$
$$C_j y^j = 0, \quad E_j y^j = 0.$$

Let

$$J := b^j J_j = \alpha\{\alpha(s_0 + \Gamma r_0) - (\Gamma r_{00} + \Pi \alpha s_0)s\},$$
$$C := b^j C_j = (X_4 r_{00} + Y_4 \alpha s_0)\alpha(b^2 - s^2) + 3\Lambda J,$$
$$E := b^j E_j = (X_6 r_{00} + Y_6 \alpha s_0)\alpha(b^2 - s^2) + 3\mu J.$$

***Lemma 4.1*** *Assume $n \geq 3$. The following two conditions are equivalent.*

(i) $L_{jkl} = 0$.
(ii) $C_j = 0$ *and* $E_j = 0$.

**Proof** (i) $\Rightarrow$ (ii). Assume that $L_{jkl} = 0$. Then by (2.7),

$$h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3E_j h_{kl} + 3E_k h_{jl} + 3E_l h_{kj} = 0. \tag{4.1}$$

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Contracting (4.1) with $b^k$ and $b^l$ yields

$$2\alpha(b^2 - s^2)h_jC + 6\alpha h_jE + \alpha^2(b^2 - s^2)^2C_j + 3\alpha^2(b^2 - s^2)E_j = 0,$$

and contracting the above equation with $b^j$ yields

$$3\alpha^2(b^2 - s^2)^2C + 9\alpha^2(b^2 - s^2)E = 0.$$

By the above two equations, we get

$$(b^2 - s^2)C_j + 3E_j = 0. \tag{4.2}$$

Let $W_{jk} := (b^2 - s^2)h_{jk} - h_jh_k$. We have $W_{jk}y^k = 0$ and $W_{jk}b^k = 0$. Using (4.2), we can see that (4.1) is equivalent to the following equation:

$$W_{jk}C_l + W_{jl}C_k + W_{kl}C_j = 0. \tag{4.3}$$

Contracting (4.3) with $b^l$ yields $W_{jk}C = 0$. Since $n \geq 3$, $W_{jk} \neq 0$. We obtain that $C = 0$. Contracting (4.3) with $a^{kl}$ yields that $C_j = 0$. By (4.2), we get $E_j = 0$.

(ii) $\Rightarrow$ (i). Obvious by (2.7). ■

## 5 A Sufficient Condition

In this section, we shall prove the following.

***Proposition 5.1*** *Let $F = \alpha\phi(\beta/\alpha)$ be a positively almost regular $(\alpha, \beta)$-metric on a manifold. Suppose that $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ satisfy*

$$s_{ij} = \frac{1}{b^2}\{b_is_j - b_js_i\}, \tag{5.1}$$

$$r_{ij} = k(b^2a_{ij} - b_ib_j) + cb_ib_j + d(b_is_j + b_js_i), \tag{5.2}$$

*where $k = k(x)$, $c = c(x)$, and $d = d(x)$ are scalar functions. Assume that $\phi = \phi(s)$ satisfies the following ODEs:*

$$\{(b^2 - s^2)X_4 - 3\Lambda\Gamma s\}k + \{s^2X_4 + 3\Lambda\Gamma s\}c = 0, \tag{5.3}$$

$$\{(b^2 - s^2)X_6 - 3\mu\Gamma s\}k + \{s^2X_6 + 3\mu\Gamma s\}c = 0. \tag{5.4}$$

*If $s_0 \neq 0$, we further assume that $\phi = \phi(s)$ satisfies the following additional ODEs:*

$$db^2\Gamma s + b^2\Pi - s = 0, \tag{5.5}$$

$$(2sX_4 + 3\Lambda\Gamma)db^2 + (b^2Y_4 + 3\Lambda) = 0, \tag{5.6}$$

$$(2sX_6 + 3\mu\Gamma)db^2 + (b^2Y_6 + 3\mu) = 0. \tag{5.7}$$

*Then $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric.*

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**Proof** By a direct computation, we get

$$\begin{aligned}
C_j &= \alpha^2\{((b^2-s^2)X_4 - 3s\Lambda\Gamma)k + (s^2X_4 + 3\Lambda\Gamma s)c\}h_j \\
&\quad + \frac{\alpha s_0}{b^2}\{(2sX_4 + 3\Lambda\Gamma)db^2 + (b^2Y_4 + 3\Lambda)\}h_j \\
&\quad - \frac{3\Lambda\alpha}{b^2}\{db^2\Gamma s + b^2\Pi - s\}(s_0y_j - \alpha^2 s_j), \\
E_j &= \alpha^2\{((b^2-s^2)X_6 - 3s\mu\Gamma)k + (s^2X_6 + 3\mu\Gamma s)c\}h_j \\
&\quad + \frac{\alpha s_0}{b^2}\{(2sX_6 + 3\mu\Gamma)db^2 + (b^2Y_6 + 3\mu)\}h_j \\
&\quad - \frac{3\mu\alpha}{b^2}\{db^2\Gamma s + b^2\Pi - s\}(s_0y_j - \alpha^2 s_j).
\end{aligned}$$

Under our assumption, we always have $C_j = 0$ and $E_j = 0$. ■

## 6 A Necessary Condition (Assuming $n \geq 3$)

We are going to study the equations $C_j = 0$ and $E_j = 0$. The difficulty is how to deal with the terms involving $\phi(\beta/\alpha)$. To overcome this difficulty, we change the $y$-coordinates $(y^i)$ at a point to "polar" coordinates $(s, y^a)$, where $i = 1, \dots, n$ and $\alpha = 2, \dots, n$.

Fix an arbitrary point $x \in M$. Take an orthonormal basis $\{e_i\}$ at $x$ such that

$$\alpha = \sqrt{\sum_{i=1}^{n}(y^i)^2}, \quad \beta = by^1.$$

Fix an arbitrary number $s$ with $|s| \leq b$. It follows from $\beta = s\alpha$ that $y^1 = \frac{s}{\sqrt{b^2-s^2}}\bar{\alpha}$, where

$$\bar{\alpha} := \sqrt{\sum_{\alpha=2}^{n}(y^\alpha)^2}.$$

Then

$$\alpha = \frac{b}{\sqrt{b^2-s^2}}\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2-s^2}}\bar{\alpha}.$$

Let

$$\bar{r}_{10} := \sum_{\alpha=2}^{n} r_{1\alpha}y^\alpha, \quad \bar{r}_{00} := \sum_{\alpha,\beta=2}^{n} r_{\alpha\beta}y^\alpha y^\beta, \quad \bar{s}_0 := \sum_{\alpha=2}^{n} s_\alpha y^\alpha.$$

We have $\bar{s}_0 = b\bar{s}_{10}$ and $s_1 = bs_{11} = 0$.

Before we derive some equations on $r_{ij}$ and $s_{ij}$, let us state the following trivial fact.

***Lemma 6.1*** *If $\bar{\alpha} = \sqrt{\sum_{a=2}^{n}(y^a)^2}$ satisfies $\varphi + \psi\bar{\alpha} = 0$, where $\varphi = \varphi(\bar{y})$ and $\psi = \psi(\bar{y})$ are homogeneous polynomials in $\bar{y} = (y^a)$, then $\varphi = 0$ and $\psi = 0$.*

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**Proof** First, we assume that $\deg\varphi = $ even. Then $\deg\psi = $ odd. We have

$$\varphi(\bar{y}) \pm \psi(\bar{y})\bar{\alpha}(\bar{y}) = 0.$$

Thus $\varphi(\bar{y}) = 0$ and $\psi(\bar{y}) = 0$. If $\deg\varphi = $ odd, by a similar argument, we still get the same conclusion. ∎

Although the expressions for $C_j$ and $E_j$ are similar, we shall first study the equation $E_j = 0$ because $\mu \neq 0$ under our assumption on $\phi$.

By a direct computation, one can show that $E_1 = 0$ is equivalent to the following two equations:

$$\{(b^2 - s^2)X_6 - 3s\mu\Gamma\}\bar{r}_{00} + s\{sX_6 + 3\mu\Gamma\}r_{11}\bar{\alpha}^2 = 0, \tag{6.1}$$

$$(b^2 - s^2)\{(2sX_6 + 3\mu\Gamma)\bar{r}_{10} + (b^2Y_6 + 3\mu)\bar{s}_{10}\} - 3s\mu\{s\Gamma\bar{r}_{10} + (b^2\Pi - s)\bar{s}_{10}\} = 0, \tag{6.2}$$

and $E_a = 0$ is equivalent to the following two equations:

$$s\{((b^2 - s^2)X_6 - 3s\mu\Gamma)\bar{r}_{00} + s(sX_6 + 3\mu\Gamma)r_{11}\bar{\alpha}^2\}y^a - 3b^2\mu\{\Gamma(\bar{r}_{0a}\bar{\alpha}^2 - \bar{r}_{00}y^a) - \bar{s}_{0a}\bar{\alpha}^2\} = 0, \tag{6.3}$$

$$\begin{aligned} s\{(b^2 - s^2)[(2sX_6 + 3\mu\Gamma)\bar{r}_{10} + (b^2Y_6 + 3\mu)\bar{s}_{10}] \\ - 3\mu s[\Gamma s\bar{r}_{10} + (b^2\Pi - s)\bar{s}_{10}]\}y^a \\ = 3\mu b^2\{[\Gamma s r_{1a} + (b^2\Pi - s)s_{1a}]\bar{\alpha}^2 - [\Gamma s\bar{r}_{10} + (b^2\Pi - s)\bar{s}_{10}]y^a\}. \end{aligned} \tag{6.4}$$

Contracting (6.3) with $y^a$ yields (6.1) and contracting (6.4) with $y^a$ yields (6.2).

***Lemma 6.2*** *Let $n \geq 3$. Assume that $\phi \neq k_1\sqrt{1 + k_2s^2}$ for any constants $k_1, k_2$. If $C_j = 0$ and $E_j = 0$ at a point $x$, then*

$$s_{ab} = 0, \tag{6.5}$$

$$r_{ab} = kb^2\delta_{ab}, \tag{6.6}$$

$$r_{11} = cb^2, \tag{6.7}$$

*where $k$ and $c$ are numbers, and $\phi = \phi(s)$ satisfies* (5.3) *and* (5.4).

**Proof** Using (6.1), we get from (6.3) that

$$3\mu b^2\{\Gamma(\bar{r}_{0a}\bar{\alpha}^2 - \bar{r}_{00}y^a) - \bar{s}_{0a}\bar{\alpha}^2\} = 0. \tag{6.8}$$

By Lemma 3.1, $\mu \neq 0$. We get from (6.8)

$$\Gamma(\bar{r}_{a0}\bar{\alpha}^2 - \bar{r}_{00}y^a) - \bar{s}_{0a}\bar{\alpha}^2 = 0. \tag{6.9}$$

Since $n \geq 3$, (6.9) implies (6.5) and (6.6).

Letting $c := r_{11}/b^2$, we obtain (6.7). Plugging (6.6) and (6.7) into (6.1) yields (5.4). Similarly, $C_j = 0$ implies (5.3). Since $\Lambda = 0$ for a large class of functions (Lemma 3.2), one should first use (6.9) to simplify $C_a = 0$. Then $C_j = 0$ implies (5.3). ■

Now we assume that $\phi \neq k_1\sqrt{1+k_2s^2}$ for any constants $k_1$ and $k_2$, and it satisfies $C_j = 0$ and $E_j = 0$. By (6.2) and $\mu \neq 0$, we get from (6.4) that

$$(s\Gamma r_{1a} + (b^2\Pi - s)s_{1a})\bar{\alpha}^2 - (s\Gamma \bar{r}_{10} + (b^2\Pi - s)\bar{s}_{10})y^a = 0. \tag{6.10}$$

Since $n \geq 3$, it follows from (6.10) that

$$s\Gamma r_{1a} + (b^2\Pi - s)s_{1a} = 0. \tag{6.11}$$

Using (6.11), equation (6.4) can be reduced to $\{2sX_6 + 3\mu\Gamma\}r_{1a} + \{b^2Y_6 + 3\mu\}s_{1a} = 0$. By a similar argument for $C_j = 0$ using (6.11), we get

$$\{2sX_4 + 3\Lambda\Gamma\}r_{1a} + \{b^2Y_4 + 3\Lambda\}s_{1a} = 0.$$

***Lemma 6.3*** *Let $n \geq 3$. Assume that $\phi \neq k_1\sqrt{1+k_2s^2}$ for any constants $k_1$ and $k_2$ and that $C_j = 0$ and $E_j = 0$ at a point $x$. Suppose that $\phi = \phi(s)$ does not satisfy* (5.5) *for any number $d$. Then $r_{1a} = 0$, $s_{1a} = 0$.*

**Proof** By comparing (6.11) and (5.5) and using the fact $\Gamma \not\equiv 0$, we conclude that $r_{1a} = 0$ and $s_{1a} = 0$. ■

***Lemma 6.4*** *Let $n \geq 3$. Assume that $\phi \neq k_1\sqrt{1+k_2s^2}$ for any constants $k_1$ and $k_2$ and that $C_j = 0$ and $E_j = 0$ at a point $x$. Suppose that $\phi$ satisfies* (5.5) *for some number $d$. Then*

$$r_{1a} = db^2 s_{1a}. \tag{6.12}$$

**Proof** By (5.5), we have $(b^2\Pi - s)/s\Gamma = -db^2$. Then (6.12) follows from (6.11). ■

Let us summarize what we have proved.

***Proposition 6.5*** *Assume $n \geq 3$. Let $F = \alpha\phi(\beta/\alpha)$ be a positively almost regular $(\alpha, \beta)$-metric on a manifold of dimension $n \geq 3$. Assume that $\phi \neq k_1\sqrt{1+k_2s^2}$ for any constants $k_1$ and $k_2$. Suppose that $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric. Then $\beta$ satisfies* (5.1) *and* (5.2) *and $\phi$ satisfies* (5.3) *and* (5.4). *If $s_0 \neq 0$, then $\phi = \phi(s)$ satisfies three additional ODEs:* (5.5), (5.6) *and* (5.7).

**Proof** By Lemma 6.2, $\beta$ satisfies $s_{ab} = 0$, $r_{ab} = kb^2\delta_{ab}$, $r_{11} = cb^2$, and $\phi$ satisfies (5.3) and (5.4).

***Case I:*** If $\phi$ does not satisfy (5.5) for any $d = d(x)$, then by Lemma 6.3, $r_{1a} = 0$, $s_{1a} = 0$. In this case, we can write the above equations together in the form (5.1) and (5.2) for any scalar function $d = d(x)$.

***Case II:*** If there is a scalar function $d = d(x)$ such that $\phi$ satisfies (5.5), then $\phi$ satisfies (5.6) and (5.7) by Lemma 3.4. By Lemma 6.4, we have $r_{1a} = db^2 s_{1a}$. In this case, we can write the above equations together in the form (5.1) and (5.2). ■

## 7 Proof of Theorems 1.1 and 1.2

We have obtained a sufficient and necessary condition for a positively almost regular $(\alpha, \beta)$-metric $F = \alpha\phi(\beta/\alpha)$ to be Landsbergian. See Propositions 5.1 and 6.5 above. To find non-trivial Landsberg $(\alpha, \beta)$-metrics, one has to solve equations (5.3) and (5.4), and when $s_0 \neq 0$, one has to solve the additional equations (5.5), (5.6), and (5.7).

A natural question arises: are there non-trivial solutions $\phi = \phi(s)$ to these ODEs? Luckily, we can solve them. After solving these equations, we shall prove Theorem 1.2. Theorem 1.1 follows from Theorem 1.2 because there are no regular Landsberg $(\alpha, \beta)$-metrics which are not Berwaldian.

***Lemma 7.1*** *Let $n \geq 3$. Assume that $C_j = 0$ and $E_j = 0$. If $\mu X_4 - \Lambda X_6 \neq 0$, then $\beta$ is parallel with respect to $\alpha$.*

**Proof** By Proposition 6.5, $\phi$ satisfies (5.3) and (5.4). It follows from (5.3) and (5.4) that $(\mu X_4 - \Lambda X_6)\{(b^2 - s^2)k + s^2 c\} = 0$. We get $(b^2 - s^2)k + s^2 c = 0$, by which we conclude that $k = 0$ and $c = 0$. Then (5.1) and (5.2) are reduced to the following identities:

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i), \quad r_{ij} = d(b_i s_j + b_j s_i).$$

We claim that $s_i = 0$. If this is not true, then $\phi$ satisfies (5.5) by Proposition 6.5. Then $Q$ is given by (3.8). By (3.1), one easily can verify that $\phi$ satisfies $\mu X_4 - \Lambda X_6 = 0$. This contradicts the assumption. ■

**Proof of Theorem 1.2** ($\Rightarrow$) Suppose that $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric. By Proposition 6.5, $\beta$ satisfies (5.1) and (5.2) and $\phi$ satisfies (5.3) and (5.4). If $s_0 \neq 0$, $\phi$ also satisfies (5.5), (5.6), and (5.7). However, we do not solve (5.3) and (5.4) directly. Instead, we consider the equation (3.2) which is obtained from (5.3) and (5.4).

First suppose that $\phi$ does not satisfy (3.2), *i.e.*, $\mu X_4 - \Lambda X_6 \neq 0$. Then by Lemma 7.1, $k = 0$, $c = 0$, and $\beta$ is parallel with respect to $\alpha$. In this case, $F$ is a Berwald metric.

Now suppose that $\phi$ satisfies (3.2), *i.e.*, $\mu X_4 - \Lambda X_6 = 0$. Since $F = \alpha\phi(\beta/\alpha)$ is almost regular, by Lemma 3.3 $\phi$ is given by (3.3). Hence $Q$ is given by (3.5). By Lemma 3.5, one can see that $\phi$ does not satisfy (5.5). By virtue of Proposition 6.5, we conclude that $s_0 = 0$.

If $(k, c) = (0, 0)$, then it follows from (5.1) and (5.2) that $\beta$ is parallel with respect to $\alpha$.

Now assume that $(k, c) \neq 0$. By Lemma 3.5, $\phi$ satisfies (5.3) and (5.4) if and only if $c_2 = -1/b^2$ and $c = 0$. Note that the norm of $\beta$ with respect to $\alpha$ is a constant,

$$b(x) = \|\beta_x\|_\alpha = 1/\sqrt{-c_2} = \text{constant}.$$

Since our function $\phi$ is $C^\infty$ on $(0, b_o)$, we must have $b \equiv b_o$. Then $\phi$ takes the form

$$\phi = c_4 \exp\left( \int \frac{c_1\sqrt{1 - (s/b_o)^2} + c_3 s}{1 + s(c_1\sqrt{1 - (s/b_o)^2} + c_3 s)} \right) ds, \tag{7.1}$$

where $c_1, c_3$ and $c_4$ are constants with $c_1 \neq 0$ and $c_4 > 0$.

($\Leftarrow$) Assume that $\beta$ and $\phi$ satisfy the conditions in Theorem 1.2. If $\beta$ is parallel, we are done. Thus suppose that $\beta$ is not parallel with respect to $\alpha$. By assumption, $\beta$ satisfies (5.1) and (5.2) with $c = 0, d = 0, s_0 = 0$, and $\phi$ is given by (3.3) with $c_2 = -1/b^2$. By Lemma 3.5, $\phi$ satisfies (5.3) and (5.4). By Proposition 5.1, we see that $F$ is a Landsberg metric.

In Lemma 8.1 below we will show that the Landsberg $(\alpha,\beta)$-metrics in Theorem 1.2 are not Berwaldian if $k \neq 0$. ■

Let $b_o = 1$. Using the substitution $s = \sin(t)$, $|t| < \pi/2$, we obtain the following formula for the function $\phi$ in (7.1).

$$\phi = c_4\sqrt{1 + s\left[c_1\sqrt{1-s^2} + c_3 s\right]}\,\exp\left(\frac{c_1 \arctan\psi}{\sqrt{(2+c_3)^2 - (c_1^2 + c_3^2)}}\right), \tag{7.2}$$

where $\psi$ is given by either of the following formulas:

$$\psi = \frac{\big(c_3 r + (2+c_3)(c_1+r)\big)\,s + \big(r(c_1+r) - (2+c_3)c_3\big)\sqrt{1-s^2}}{\big(c_3 s + (c_1+r)\sqrt{1-s^2}\big)\sqrt{(2+c_3)^2 - r^2}},$$

$$\psi = \frac{\big((2+c_3)c_3 + r(r-c_1)\big)\,s + \big(c_3 r - (2+c_3)(r-c_1)\big)\sqrt{1-s^2}}{\big(c_3\sqrt{1-s^2} + (r-c_1)s\big)\sqrt{(2+c_3)^2 - r^2}},$$

where $r := \sqrt{c_1^2 + c_3^2}$.

When $c_1 = g$ and $c_3 = 0$,

$$\phi = c_4\sqrt{1 + gs\sqrt{1-s^2}}\,\exp\left(\frac{g\arctan\psi}{\sqrt{4-g^2}}\right), \tag{7.3}$$

where $\psi$ is given by

$$\psi = \begin{cases} \dfrac{2s + g\sqrt{1-s^2}}{\sqrt{1-s^2}\sqrt{4-g^2}} & \text{if } g > 0,\\[2ex] -\dfrac{sg + 2\sqrt{1-s^2}}{s\sqrt{4-g^2}} & \text{if } g < 0. \end{cases}$$

Comparing it with [1, 2], we see that the $(\alpha,\beta)$-metric defined by (1.4), (1.5), (1.6), and (1.3) with $c_1 = g$ and $c_3 = 0$ is the metric constructed by G. Asanov from a completely different approach.

If we allow more singularity for Finsler metrics, we get more Landsberg metrics. A function $F = \alpha\phi(\beta/\alpha)$ is called a *positively almost regular* $(\alpha,\beta)$*-metric* if $\phi$ is a $C^\infty$ function on $(0, b_o)$ satisfying

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad (0 < s \le b < b_o).$$

and the 1-form $\beta$ satisfies that $\beta(x,y) \le b_o\alpha(x,y)$ if $\beta(x,y) > 0$. A positively almost regular $(\alpha,\beta)$-metric might be singular (even undefined) in the directions $y \in T_xM$ with $\beta(x,y) = b_o\alpha(x,y)$ or $\beta(x,y) < 0$.

***Theorem 7.2*** *Assume $n \geq 3$. Let $F = \alpha\phi(\beta/\alpha)$ be a positively almost regular $(\alpha, \beta)$-metric, where $\phi = \phi(s)$ is a function on $(0, b_o)$ such that $\phi \neq k_1\sqrt{1 + k_2 s^2}$ for any constants $k_1 > 0$ and $k_2$. Let $b(x) := \|\beta_x\|_\alpha \neq 0$. Assume that $b(x) =$ constant or $db(x) \neq 0$ on $M$. Then $F$ is a Landsberg metric if and only if $F$ has one of the following forms.*

(i) *$F$ is given in Theorem 1.2.*

(ii) *$\phi$ is given by*

$$\phi(s) = \begin{cases} c_3 s^{\frac{c_1}{(1+c_1)}}(1 + c_1 + c_2 s^2)^{\frac{1}{2(1+c_1)}} & \text{if } c_1 \neq -1, \\ c_3 s e^{\frac{1}{2c_2 s^2}} & \text{if } c_1 = -1, \end{cases} \tag{7.4}$$

*where $c_1, c_2, c_3$ are constants with $c_1 \neq 0$ and $c_3 > 0$ and $\beta$ satisfies the equations*

$$s_{ij} = \frac{1}{b^2}\Big\{ b_i s_j - b_j s_i \Big\} \tag{7.5}$$

$$r_{ij} = k\Big\{ (b^2 a_{ij} - b_i b_j) - \frac{1 + c_1 + c_2 b^2}{c_1} b_i b_j \Big\} + \frac{1 + 2c_1 + 2c_2 b^2}{b^2}(b_i s_j + b_j s_i), \tag{7.6}$$

*where $k = k(x)$ is a scalar function. This metric is always a positively almost regular Berwald metric.*

**Proof** Suppose that $F = \alpha\phi(\beta/\alpha)$ is a Landsberg metric. By Proposition 6.5, $\beta$ satisfies (5.1) and (5.2) and $\phi$ satisfies (5.3) and (5.4). If $s_0 \neq 0$, $\phi$ also satisfies (5.5), (5.6) and (5.7).

As shown in the proof of Theorem 1.2, if $\mu X_4 - \Lambda X_6 \neq 0$, then $F$ is Berwald metric. Assume that $\mu X_4 - \Lambda X_6 = 0$. By Lemma 3.3, $\phi$ is either in the form (3.3) or (3.4). If $\phi$ is given by (3.3), we have shown that $F$ is as given in Theorem 1.2.

Assume that $\phi$ is given by (3.4) or $Q$ is given by (3.6). By Lemma 3.6, $\phi$ satisfies (5.3) and (5.4) if and only if $c = -(1 + c_1 + c_2 b^2)k/c_1$. We see no restriction on $c_1$ and $c_2$ in this case. Moreover, $\phi$ satisfies (5.5) if and only if $d = (1+2c_1+2c_2 b^2)/b^2$. In this case, $\phi$ also satisfies (5.6) and (5.7) by Lemma 3.4. We have proved Theorem 7.2(ii).

Conversely, if $\beta$ satisfies (7.5) and (7.6) and $\phi = \phi(s)$ is given by (7.4), then $\beta$ satisfies (5.1) and (5.2) with $c = -(1 + c_1 + c_2 b^2)k/c_1$ and $d = (1 + 2c_1 + 2c_2 b^2)/b^2$. By Lemma 3.6, $\phi$ satisfies (5.3), (5.4) and (5.5). Thus $F$ is a Landsberg metric by Proposition 5.1.

In Lemma 8.2 below we will show that the Landsberg $(\alpha, \beta)$-metrics in Theorem 7.2(ii) are Berwaldian. ■

Theorem 7.2 also characterizes positively almost regular Berwald $(\alpha, \beta)$-metrics in dimension $n \geq 3$.

## 8 Spray Coefficients

To check whether or not an $(\alpha, \beta)$-metric is a Berwald metric, by definition it suffices to look at its spray coefficients. In this section, we shall compute the spray coefficients for the $(\alpha, \beta)$-metrics in Theorems 1.2 and 7.2.

***Lemma 8.1*** *Assume $n \geq 3$. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a positively almost regular $(\alpha, \beta)$-metric in Theorem* 1.2*, where $\phi = \phi(s)$ is given by* (1.3) *and $\beta$ satisfies* (1.4), (1.5), *and* (1.6)*. Then $F$ is not a Berwald metric if $k \neq 0$.*

**Proof** For the function $\phi$ in (1.3),

$$Q = \frac{\phi'}{\phi - s\phi'} = c_1\sqrt{1 - (s/b_o)^2} + c_3 s. \tag{8.1}$$

It follows from (8.1) that

$$\frac{Q'}{Q - sQ'} = \frac{-c_1 s + c_3 b_o^2\sqrt{1 - (s/b_o)^2}}{c_1 b_o^2},$$
$$\Theta = \frac{c_1}{2\sqrt{1 - (s/b_o)^2}(1 + c_3 b_o^2)}.$$

By (1.6), we get $r_{00} = k(b_o^2\alpha^2 - \beta^2)$. Plugging these into (2.6) yields

$$G^i = \bar{G}^i + \frac{c_1 k\sqrt{\alpha^2 - (\beta/b_o)^2}}{2(1 + c_3 b_o^2)}\Big\{ b_o^2 y^i - \beta b^i + \frac{c_3 b_o^2}{c_1}\sqrt{\alpha^2 - (\beta/b_o)^2} b^i \Big\}.$$

Thus if $k \neq 0$, then $G^i$ is not quadratic in $y$. Hence $F$ is not a Berwald metric. ■

***Lemma 8.2*** *Assume $n \geq 3$. Let $F = \alpha\phi(s)$, $s = \beta/\alpha$ be a positively almost regular $(\alpha, \beta)$-metric in Theorem* 7.2*, where $\phi = \phi(s)$ is given in* (7.4) *and $\beta$ satisfies* (7.5) *and* (7.6)*. Then $F$ must be a Berwald metric.*

**Proof** For the function $\phi$ in (7.4),

$$Q = \frac{c_1}{s} + c_2 s. \tag{8.2}$$

It follows from (8.2) that

$$\frac{Q'}{Q - sQ'} = -\frac{c_1 - c_2 s^2}{2c_1 s},$$
$$\Theta = \frac{-c_1 s}{c_1 b^2 - (1 + 2c_1 + c_2 b^2)s^2}.$$

By (7.5) and (7.6), we get

$$\alpha Q s_0^i = \frac{\alpha}{b^2 s}(c_1 + c_2 s^2)(s_0 b^i - s^i\beta),$$
$$\Theta\{-2\alpha Q s_0 + r_{00}\} = -\frac{\alpha}{b^2}(kb^2\beta - 2c_1 s_0).$$

Plugging these into (2.6) yields

$$G^i = \bar{G}^i + \frac{1}{2c_1 b^2}\{4c_1 c_2\beta s_0 + kb^2(c_1\alpha^2 - c_2\beta^2)\}b^i$$
$$- \frac{1}{b^2}(kb^2\beta - 2c_1 s_0)y^i - \frac{1}{b^2}(c_1\alpha^2 + c_2\beta^2)s^i.$$

Clearly, $G^i$ are quadratic in $y$. Thus $F$ is always a Berwald metric. ■

**Acknowledgment** The author is grateful to G. S. Asanov and D. Bao for valuable discussions on this topic.

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