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MULTIPLE SOLUTIONS FOR SOME NEUMANN PROBLEMS IN EXTERIOR DOMAINS

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In this paper, we show that if $q(x)$ satisfies suitable conditions, then the Neumann problem $-\Delta u+u = q(x)|u|^{p-2}u$ in Ω has at least two solutions of which one is positive and the other changes sign.

1. INTRODUCTION

Throughout this article, let $N = m + n$, where m and n are nonnegative integers with $m \geqslant 3$. For $x = (x_1, \dots, x_N) = (x_1, \dots, x_m, x_{m+1}, \dots, x_N) \in \mathbb{R}^N$, let $Px = (x_1, \dots, x_m) \in \mathbb{R}^m$ and $Qx = (x_{m+1}, \dots, x_N) \in \mathbb{R}^n$. Consider the Neumann boundary value problem

$$(1) \qquad \begin{cases} -\Delta u + u = q(x)\,|u|^{p-2}\,u & \text{in } \Omega; \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \left(\mathbb{R}^m \backslash (\overline{\Omega^m})\right) \times \mathbb{R}^n$, Ω^m is a smooth bounded domain in $\mathbb{R}^m$, $2 < p < 2^* = (2N)/(N-2)$, η is the outward unit normal to $\partial\Omega$ and $q(x)$ is a bounded continuous function in Ω. Moreover, $q(x)$ satisfies the following hypotheses:

- $(q1)$ $q(x)$ is a positive function in Ω, $\inf\{q(x) | x \in \Omega\} > 0$ and $q(x) = q(y)$ for any $Px = Py$;
- $(q2)$ there exists a positive number q_∞ such that $\lim\limits_{|Px| \to \infty} q(x) = q_\infty$ and $q(x) \not\equiv q_\infty$ in Ω.

If Ω^c is bounded ($n = 0$ in our case), Esteban [**5, 6**] proved the existence of the "ground state solution" of Equation (1) provided that $q(x) \equiv 1$. In the case q is not a constant function, Cao [**3**] and Hsu and Lin [**9**] proved the multiplicity of the solutions of Equation (1). In this article, we assert that Equation (1) still has the same results of Hsu and Lin [**9**] even if Ω^c is unbounded. First, we use the concentration-compactness argument of Lions [**11, 12, 13, 14**] to obtain the "ground state solution", and then combine it with some ideas of Zhu [**16**] to show the existence of another solution which changes sign.

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2. Preliminaries

Associated with Equation (1), we consider the energy functionals a, b and J, for $u \in H^1(\Omega)$

$$a(u) = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx;$$
$$b(u) = \int_\Omega q(z)|u|^p\, dx;$$
$$J(u) = \frac{1}{2}a(u) - \frac{1}{p}b(u).$$

Define

$$\alpha = \inf_{u\in \mathbf{M}(\Omega)} J(u),$$

where

$$\mathbf{M}(\Omega) = \left\{u \in H^1(\Omega)\backslash\{0\} \mid a(u) = b(u)\right\}.$$

It is well known that there is a positive radially symmetric smooth solution w of Equation (2)

$$\text{(2)}\qquad \begin{cases} -\bigtriangleup u + u = q_\infty\, |u|^{p-2}\, u & \text{in } \mathbb{R}^N; \\ u \in H^1\left(\mathbb{R}^N\right). \end{cases}$$

We also define

$$a^\infty(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx;$$
$$b^\infty(u) = \int_{\mathbb{R}^N} q_\infty |u|^p\, dx;$$
$$J^\infty(u) = \frac{1}{2}a^\infty(u) - \frac{1}{p}b^\infty(u);$$
$$\alpha^\infty = \inf_{u\in \mathbf{M}^\infty(\mathbb{R}^N)} J^\infty(u),$$

where

$$\mathbf{M}^\infty(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N)\backslash\{0\} \mid a^\infty(u) = b^\infty(u)\right\}.$$

Recall the fact that

$$w\left(|x|\right)|x|^{(N-1)/2}\exp\left(|x|\right) \to \bar{c} > 0 \text{ as } |x| \to \infty,$$

where $\bar{c}$ is some constant. (See Bahri and Li [**1**], Bahri and Lions [**2**], Gidas, Ni and Nirenberg [**7, 8**] and Kwong [**10**].) In particular, we have

(i) there exists a constant $C_0 > 0$ such that

$$w(x) \leqslant C_0 \exp\left(-|x|\right) \text{ for all } x \in \mathbb{R}^N;$$

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(ii) for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$w(x) \geqslant C_\varepsilon \exp(-(1+\varepsilon)|x|) \text{ for all } x \in \mathbb{R}^N.$$

We need the following definition and lemmas to prove the main theorems.

DEFINITION 1: For $\beta \in \mathbb{R}$, a sequence $\{u_k\}$ is a $(\mathrm{PS})_\beta$-sequence in $H^1(\Omega)$ for J if $J(u_k) = \beta + o(1)$ and $J'(u_k) = o(1)$ strongly in $H^1(\Omega)$ as $k \to \infty$.

LEMMA 2. *Let $\beta \in \mathbb{R}$ and let $\{u_k\}$ be a (PS)$_\beta$-sequence in $H^1(\Omega)$ for J, then $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$. Moreover,*

$$a(u_k) = b(u_k) + o(1) = \frac{2p}{p-2}\beta + o(1)$$

and $\beta \geqslant 0$.

PROOF: For n sufficiently large, we have

$$|\beta| + 1 + \sqrt{a(u_k)} \geqslant J(u_k) - \frac{1}{p}\langle J'(u_k), u_k\rangle = \Big(\frac{1}{2} - \frac{1}{p}\Big)a(u_k).$$

It follows that $\{u_k\}$ is bounded in $H^1(\Omega)$. Since $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$, then $\langle J'(u_k), u_k\rangle = o(1)$ as $k \to \infty$. Thus,

$$\beta + o(1) = J(u_k) = \Big(\frac{1}{2} - \frac{1}{p}\Big)a(u_k) + o(1) = \Big(\frac{1}{2} - \frac{1}{p}\Big)b(u_k) + o(1),$$

that is, $a(u_k) = b(u_k) + o(1) = (2p/p-2)\beta + o(1)$ and $\beta \geqslant 0$. ∎

LEMMA 3.

(i) *For each $u \in H^1(\Omega)\backslash\{0\}$, there exists an $s_u > 0$ such that $s_u u \in \mathrm{M}(\Omega)$;*

(ii) *Let $\{u_k\}$ be a (PS)$_\beta$-sequence in $H^1(\Omega)$ for J with $\beta > 0$. Then there is a sequence $\{s_k\}$ in $\mathbb{R}^+$ such that $\{s_k u_k\} \subset \mathrm{M}(\Omega)$, $s_k = 1 + o(1)$ and $J(s_k u_k) = \beta + o(1)$. In particular, the statement holds for J^∞.*

PROOF: See Chen, Lee and Wang [4]. ∎

LEMMA 4. *There exists a $c > 0$ such that $\|u\|_{H^1(\Omega)} \geqslant c > 0$ for each $u \in \mathrm{M}(\Omega)$.*

PROOF: See Chen, Lee and Wang [4]. ∎

LEMMA 5. *Let $u \in \mathrm{M}(\Omega)$ satisfy $J(u) = \min_{v \in \mathrm{M}(\Omega)} J(v) = \alpha$. Then u is a nonzero solution of Equation* (1).

PROOF: We define $g(v) = a(v) - b(v)$ for $v \in H^1(\Omega)\backslash\{0\}$. Note that $\langle g'(u), u\rangle = (2-p)a(u) \neq 0$. Since the minimum of J is achieved at u and is constrained on $\mathrm{M}(\Omega)$, by the Lagrange multiplier theorem, there exists a $\lambda \in \mathbb{R}$ such that $J'(u) = \lambda g'(u)$ in $H^1(\Omega)$. Then we have

$$0 = \langle J'(u), u\rangle = \lambda\langle g'(u), u\rangle,$$

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that is, $\lambda = 0$. Hence, $J'(u) = 0$ and u is a nonzero solution of Equation (1) in Ω such that $J(u) = \alpha$. ∎

Define $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

Lemma 6. *Let u be a solution of Equation (1) that changes sign. Then $J(u) \geqslant 2\alpha$. In particular, the result holds for J^∞.*

Proof: Since u is a solution of Equation (1) that changes sign, then u^- is nonnegative and nonzero. Multiply Equation (1) by u^- and integrate to obtain

$$\int_\Omega \nabla u \nabla u^- + u u^- = \int_\Omega q(x)|u|^{p-2} u u^-,$$

that is, $u^- \in \mathrm{M}(\Omega)$ and $J(u^-) \geqslant \alpha$. Similarly, $J(u^+) \geqslant \alpha$. Hence,

$$J(u) = J(u^+) + J(u^-) \geqslant 2\alpha.$$

∎

Lemma 7. (Improved Decomposition Lemma) *Let $\{u_k\}$ be a $(PS)_\beta$-sequence in $H^1(\Omega)$ for J. Then there are a subsequence $\{u_k\}$, an integer $l \geqslant 0$, sequences $\{x_k^i\}_{k=1}^\infty$ in $\mathbb{R}^N$, functions $v \in H^1(\Omega)$ and $w_i \neq 0$ in $H^1(\mathbb{R}^N)$ for $1 \leqslant i \leqslant l$ such that*

$$\begin{aligned} -\triangle v + v &= q(x)|v|^{p-2} v \text{ in } \Omega; \\ -\triangle w_i + w_i &= q_\infty |w_i|^{p-2} w_i \text{ in } \mathbb{R}^N; \\ |Px_k^i| &\longrightarrow \infty \text{ for } 1 \leqslant i \leqslant l; \\ |x_k^i| &\longrightarrow \infty \text{ for } 1 \leqslant i \leqslant l; \\ u_k &= v + \sum_{i=1}^l w_i(\cdot - x_k^i) + o(1) \text{ strongly in } H^1(\mathbb{R}^N); \\ J(u_k) &= J(v) + \sum_{i=1}^l J^\infty(w_i) + o(1). \end{aligned}$$

In addition, if $u_k \geqslant 0$, then $v \geqslant 0$ and $w_i \geqslant 0$ for $1 \leqslant i \leqslant l$.

Proof: The proof can be obtained by using the arguments in Bahri and Lions [**2**] or see Tzeng and Wang [**15**]. ∎

3. Existence of the ground state solution

Lemma 8. *If $\alpha < \alpha^\infty$, then α attains a minimiser v_1, that is, there exists a ground state solution v_1 of Equation (1).*

Proof: See Cao [**3**]. ∎

Let $w_k(x) = w(x + e_k)\,|_\Omega$, where $e_k = (k, 0, \ldots, 0)$. Then we have the following lemmas.

Lemma 9. *Let Θ be a domain in $\mathbb{R}^m$. If $f : \Theta \to \mathbb{R}$ satisfies*

$$\int_\Theta \Bigl| f(x) \exp\bigl(\sigma|x|\bigr) \Bigr| \, dx < \infty \text{ for some } \sigma > 0,$$

then

$$\left(\int_\Theta f(x) \exp\bigl(-\sigma|x+e_k|\bigr)\, dx \right) \exp(\sigma k) = \int_\Theta f(x) \exp(-\sigma x_1)\, dx + o(1) \text{ as } k \to \infty,$$

or

$$\left(\int_\Theta f(x) \exp\bigl(-\sigma|x-e_k|\bigr)\, dx \right) \exp(\sigma k) = \int_\Theta f(x) \exp(\sigma x_1)\, dx + o(1) \text{ as } k \to \infty.$$

Proof: We know $\sigma|e_k| \leqslant \sigma|x| + \sigma|x+e_k|$, then

$$\Bigl| f(x) \exp\bigl(-\sigma|x+e_k|\bigr) \exp\bigl(\sigma|e_k|\bigr) \Bigr| \leqslant \Bigl| f(x) \exp\bigl(\sigma|x|\bigr) \Bigr|.$$

Since

$$-\sigma|x+e_k| + \sigma|e_k| = -\sigma \frac{\langle x, e_k \rangle}{|e_k|} + o(1) = -\sigma x_1 + o(1)$$

as $k \to \infty$, the lemma follows from the Lebesgue dominated convergence theorem. ∎

Lemma 10. *Assume that there are positive numbers δ, R_0 and C such that*

(Q) $$q(x) \geqslant q_\infty - C\exp\bigl(-\,(2+\delta)|Px|\bigr) \text{ for } |Px| \geqslant R_0.$$

Then there exists a $k_0 \in \mathbb{N}$ such that for $k \geqslant k_0$, we have

$$\sup_{s \geqslant 0} J(sw_k) < \alpha^\infty.$$

Proof: Take a $k_1 \in \mathbb{N}$ such that the N-ball $B(-e_k;1) \subset \Omega$ for $k \geqslant k_1$. Then we have

$$\begin{aligned} J(sw_k) &\leqslant \frac{s^2}{2} \int_{\mathbb{R}^N} \bigl[|\nabla w|^2 + w^2\bigr] - c\frac{s^p}{p} \int_{B(-e_k;1)} w^p(x+e_k)\, dx \\ &= \frac{s^2}{2} \int_{\mathbb{R}^N} \bigl[|\nabla w|^2 + w^2\bigr] - c\frac{s^p}{p} \int_{B(0;1)} w^p \, dx. \end{aligned}$$

Therefore, there exists an $s_1 > 0$ such that

$$J(sw_k) < 0 \text{ for } s \geqslant s_1 \text{ and } k \geqslant k_1.$$

Since J is continuous in $H^1(\Omega)$ and

$$\int_\Omega \bigl[|\nabla w_k|^2 + w_k^2\bigr] \leqslant \int_{\mathbb{R}^N} |\nabla w|^2 + w^2 < \infty \text{ for any } k \in \mathbb{N},$$

there exists an $s_0 > 0$ such that

$$J(sw_k) < \alpha^\infty \text{ for } s < s_0 \text{ and any } k \in \mathbb{N}.$$

Then we only need to prove

$$\sup_{s_0 \leqslant s \leqslant s_1} J(sw_k) < \alpha^\infty \text{ for } k \text{ sufficiently large.}$$

For $k \geqslant k_1$ and $s_0 \leqslant s \leqslant s_1$, since

$$\sup_{s \geqslant 0} J^\infty(sw) = J^\infty(w) = \alpha^\infty,$$

$$\begin{aligned}
J(sw_k) &= \frac{s^2}{2}\int_\Omega \left[\left|\nabla w(x+e_k)\right|^2 + \left|w(x+e_k)\right|^2\right] - \frac{s^p}{p}\int_\Omega q(x)\left|w(x+e_k)\right|^p \\
&= J^\infty(sw) - \frac{s^2}{2}\int_{\Omega^m \times \mathbb{R}^n}\left[\left|\nabla w(x+e_k)\right|^2 + \left|w(x+e_k)\right|^2\right] \\
&\quad + \frac{s^p}{p}\left[\int_{\Omega^m \times \mathbb{R}^n} q_\infty \left|w(x+e_k)\right|^p + \int_\Omega \big(q_\infty - q(x)\big)\left|w(x+e_k)\right|^p\right] \\
&\leqslant \alpha^\infty - \frac{s_0^2}{2}\int_{\Omega^m \times \mathbb{R}^n}\left[\left|\nabla w\ (x+e_k)\right|^2 + \left|w(x+e_k)\right|^2\right] \\
&\quad + \frac{s_1^p}{p}\left[\int_{\Omega^m \times \mathbb{R}^n} q_\infty \left|w(x+e_k)\right|^p + \int_\Omega \big(q_\infty - q(x)\big)\left|w(x+e_k)\right|^p\right].
\end{aligned}$$

(i) Let $B(0;1) \subset \mathbb{R}^n$ be the unit n-ball, then

$$\begin{aligned}
\int_{\Omega^m \times \mathbb{R}^n} \left|w(x+e_k)\right|^2 dx &\geqslant \int_{\Omega^m \times B(0;1)} C_\varepsilon^2 \exp\big(-2(1+\varepsilon)|x+e_k|\big)\, dx \\
&\geqslant C'_\varepsilon \exp\big(-2(1+\varepsilon)k\big).
\end{aligned}$$

(ii) It is easy to see that the following inequality

$$\sqrt{(a^2+b^2)} \geqslant \vartheta a + \sqrt{1-\vartheta^2}\, b$$

holds for any $a, b > 0$ and $0 \leqslant \vartheta \leqslant 1$. Take $\vartheta = 1$, and since $\Omega^m \times \mathbb{R}^n$ is unbounded, then for a small $\varepsilon > 0$, we have

$$\begin{aligned}
\int_{\Omega^m \times \mathbb{R}^n} q_\infty \left|w(x+e_k)\right|^p dx &\leqslant \int_{\Omega^m \times \mathbb{R}^n} q_\infty C_0^p \exp\big(-p|x+e_k|\big)\, dx \\
&\leqslant \int_{\Omega^m \times \mathbb{R}^n} q_\infty C_0^p \exp\Big(-p\vartheta\big|(Px+Pe_k)\big|\Big)\, dx \\
&\leqslant C'_0 \exp\big(-(p-\varepsilon)k\big).
\end{aligned}$$

(iii) It is similar to (ii) we have

$$\int_{\Omega \cap \{|Px| \leqslant R_0\}} \big(q_\infty - q(x)\big)\left|w(x+e_k)\right|^p dx \leqslant M \exp\big(-(p-\varepsilon)k\big).$$

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Since q satisfies the condition (Q), then by Lemma 9, there exists a $k_2 \geqslant k_1$ such that for $k \geqslant k_2$

$$\begin{aligned}\int_{\Omega\cap\{|Px|\geqslant R_0\}} &(q_\infty - q(x))\big|w(x+e_k)\big|^p\,dx \\ &\leqslant \int_{\Omega\cap\{|Px|\geqslant R_0\}} C\exp\big(-(2+\delta)|Px|\big)C_0^p\exp\big(-p|x+e_kt|\big)\,dx \\ &\leqslant \int_{\Omega\cap\{|Px|\geqslant R_0\}} C_1\exp\big(-(2+\delta)|Px|\big)\exp\Big(-p\vartheta\big|(Px+Pe_k)\big|\Big)\,dx \\ &\leqslant C'\exp\Big(-\min\Big\{2+\frac{\delta}{2},p\Big\}k\Big).\end{aligned}$$

By (i)–(iii) and $2 < p < 2^*$, choosing $\varepsilon > 0$, such that $2+2\varepsilon < p-\varepsilon$ and $2\varepsilon < \delta/2$, we can find a $k_0 \geqslant k_2$ such that for $k \geqslant k_0$

$$\begin{aligned}\frac{s_1^p}{p}\Big[\int_{\Omega^m\times\mathbb{R}^n} q_\infty\big|w(x+e_k)\big|^p + \int_\Omega \big(q_\infty - q(x)\big)\big|w(x+e_k)\big|^p\Big] \\ - \frac{s_0^2}{2}\int_{\Omega^m\times\mathbb{R}^n}\big|\nabla w(x+e_k)\big|^2 + \big|w(x+e_k)\big|^2 < 0.\end{aligned}$$

Hence, we have

$$\sup_{s\geqslant 0} J(sw_k) < \alpha^\infty \text{ for } k \geqslant k_0.$$

∎

Theorem 11. *Assume that q satisfies (q_1), (q_2) and the condition (Q), then Equation (1) has a positive solution v_1.*

Proof: By Lemma 3 (i), there exists an $s_k > 0$ such that $s_k w_k \in \mathrm{M}(\Omega)$, that is, $\alpha \leqslant J(s_k w_k)$. Applying Lemma 10, we have $\alpha < \alpha^\infty$. Thus, there exists a ground state solution v_1 of Equation (1). By the standard arguments and the maximum principle, $v_1 > 0$ in Ω. ∎

Remark 1. $v_1(x) \leqslant C_1\exp\big(-|x|\big)$ for $|x| \geqslant R_1$, where C_1 and R_1 are some positive constants.

Proof: See Cao [**3**]. ∎

4. Existence of the second solution

In this section, q satisfies (q_1), (q_2) and the condition $(\overline{Q})$

$$(\overline{Q}) \qquad q(x) \geqslant q_\infty + \overline{C}\exp\big(-\delta|Px|\big) \text{ for } |Px| \geqslant \overline{R_0}$$

where $\delta < 1$, $\overline{C}$ and $\overline{R_0}$ are some positive constants. Let $h(u)$ be a functional in $H^1(\Omega)$ defined by

$$h(u) = \begin{cases} \dfrac{b(u)}{a(u)} & \text{for } u \neq 0; \\ 0 & \text{for } u = 0. \end{cases}$$

Denote by

$$\mathrm{M}_0 = \{u \in H^1(\Omega) \mid h(u^+) = h(u^-) = 1\};$$
$$\mathcal{N} = \left\{u \in H^1(\Omega) \mid \left|h(u^\pm) - 1\right| < \frac{1}{2}\right\},$$

where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$.

Lemma 12.

(i) *If $u \in H^1(\Omega)$ changes sign, then there are positive numbers $s^\pm(u) = s^\pm$ such that $s^+u^+ \pm s^-u^- \in \mathrm{M}(\Omega)$;*

(ii) *There exists a $c' > 0$ such that $\|u^\pm\|_{H^1} \geqslant c' > 0$ for each $u \in \mathcal{N}$.*

Proof: (i) Since u^+ and u^- are nonzero, by Lemma 3 (i), it is easy to obtain the result.

(ii) For each $u \in \mathcal{N}$, by Lemma 3 (i), there exist $s^\pm(u) = s^\pm > 0$ such that $s^\pm u^\pm \in \mathrm{M}(\Omega)$. Then we have

$$\frac{1}{2} < (s^\pm)^{2-p} = \frac{b(u^\pm)}{a(u^\pm)} < \frac{3}{2} \text{ for each } u \in \mathcal{N}. \tag{3}$$

By Lemma 4, we have

$$\|s^\pm u^\pm\|_{H^1} \geqslant c \text{ for some } c > 0 \text{ and each } u \in \mathcal{N}.$$

Thus, by (3), we have $\|u^\pm\|_{H^1} \geqslant c/s^\pm \geqslant c' > 0$ for each $u \in \mathcal{N}$. ∎

Define

$$\gamma = \inf_{u \in \mathrm{M}_0} J(u).$$

By Lemma 12, $\gamma > 0$.

Lemma 13. *There exists a sequence $\{u_k\} \subset \mathcal{N}$ such that $J(u_k) = \gamma + o(1)$ and $J'(u_k) = o(1)$ strongly in $H^{-1}(\Omega)$.*

Proof: It is similar to the proof of Zhu [**16**]. ∎

Lemma 14. *Let f and g are real-valued functions in Ω. If $g(x) > 0$ in Ω, then we have the following inequalities.*

(i) $(f+g)^+ \geqslant f^+$;

(ii) $(f+g)^- \leqslant f^-$;

(iii) $(f-g)^+ \leqslant f^+$;

(iv) $(f-g)^- \geqslant f^-$.

Lemma 15. *Let $\{u_k\} \subset \mathcal{N}$ be a (PS)$_\gamma$-sequence in $H^1(\Omega)$ for J satisfying*

$$\alpha < \gamma < \alpha + \alpha^\infty (< 2\alpha^\infty).$$

Then there exists a $v_2 \in \mathrm{M}_0$ such that u_k converges to v_2 strongly in $H^1(\Omega)$. Moreover, v_2 is a higher energy solution of Equation (1) such that $J(v_2) = \gamma$.

PROOF: By the definition of the $(\mathrm{PS})_\gamma$-sequence in $H^1(\Omega)$ for J, it is easy to see that $\{u_k\}$ is a bounded sequence in $H^1(\Omega)$ and satisfies

$$\int_\Omega \left[|\nabla u_k^\pm|^2 + |u_k^\pm|^2\right] = \int_\Omega q(x)|u_k^\pm|^p + o(1).$$

By Lemma 12 (ii), there exists a $C > 0$ such that

$$C \leqslant \int_\Omega \left[|\nabla u_k^\pm|^2 + |u_k^\pm|^2\right] = \int_\Omega q(x)|u_k^\pm|^p + o(1).$$

By the Decomposition Lemma 7, we have $\gamma = J(v_2) + \sum_{i=1}^{l} J^\infty(w_i)$, where v_2 is a solution of Equation (1) in Ω and w_i is a solution of Equation (2) in $\mathbb{R}^N$. Since $J^\infty(w_i) \geqslant \alpha^\infty$ for each $i \in \mathbb{N}$ and $\alpha < \alpha^\infty$, we have $l \leqslant 1$. Now we want to show that $l = 0$. On the contrary, suppose $l = 1$.

(i) w_1 is a changed sign solution of Equation (2): by Lemma 6, we have $\gamma \geqslant 2\alpha^\infty$, which is a contradiction.

(ii) w_1 is a constant sign solution of Equation (2): we may assume $w_1 > 0$. By the Decomposition Lemma 7, there exists a sequence $\{x_k^1\}$ in $\mathbb{R}^N$ such that $|x_k^1| \to \infty$, and

$$\left\|u_k - v_2 - w_1(\cdot - x_k^1)\right\|_{H^1(\Omega)} = o(1) \text{ as } k \to \infty.$$

By the Sobolev continuous embedding inequality, we obtain

$$\left\|u_k - v_2 - w_1(\cdot - x_k^1)\right\|_{L^p(\Omega)} = o(1) \text{ as } k \to \infty.$$

Since $w_1 > 0$, by Lemma 14, then

$$\left\|(u_k - v_2)^-\right\|_{L^p(\Omega)} = o(1) \text{ as } k \to \infty.$$

Suppose $v_2 \equiv 0$, we obtain $\|u_k^-\|_{L^p(\Omega)} = o(1)$ as $k \to \infty$. Then

$$0 < C \leqslant \int_\Omega q(x)|u_k^-|^p = o(1),$$

which is a contradiction. Hence, $v_2 \not\equiv 0$. So we have $\gamma = J(v_2) + J^\infty(w_1) \geqslant \alpha + \alpha^\infty$, which a contradiction.

By (i) and (ii), then $l = 0$. Thus, $\|u_k - v_2\|_{H^1(\Omega)} = o(1)$ as $k \to \infty$ and $J(v_2) = \gamma$. Similarly, by Lemma 14, we obtain that v_2 is a changed sign solution of Equation (1) in Ω. By Lemma 6, $2\alpha \leqslant \gamma$. ∎

Recall that $w_k(x) = w(x + e_k)\,|_\Omega$, where $e_k = (k, 0, \dots, 0)$ and w is a positive ground state solution of Equation (2) in $\mathbb{R}^N$. Then we have the following results.

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LEMMA 16. *There are* $k_0 \in \mathbb{N}$, *real numbers* t_1^* *and* t_2^* *such that for* $k \geqslant k_0$

$$t_1^* v_1 - t_2^* w_k \in \mathbf{M}_0 \text{ and } \gamma \leqslant J(t_1^* v_1 - t_2^* w_k),$$

where $1/2 \leqslant t_1^*, t_2^* \leqslant 2$.

PROOF: The proof is similar to Zhu [**16**] or see Cao [**3**]. ∎

LEMMA 17. *There exists a* $k_0^* \in \mathbb{N}$ *such that for* $k \geqslant k_0^* \geqslant k_0$

$$\gamma \leqslant \sup_{1/2 \leqslant t_1, t_2 \leqslant 2} J(t_1 v_1 - t_2 w_k) < \alpha + \alpha^\infty,$$

where v_1 *is a gound state solution of Equation* (1) *in* Ω.

PROOF: Since v_1 is a positive solution of Equation (1) in Ω and $w_k > 0$ for each $k \in \mathbb{N}$, we have

$$\begin{aligned} J(t_1 v_1 - t_2 w_k) &= \frac{1}{2} a(t_1 v_1) + \frac{1}{2} a(t_2 w_k) - t_1 t_2 \left(\int_\Omega \nabla v_1 \nabla w_k + v_1 w_k \right) - \frac{1}{p} b(t_1 v_1 - t_2 w_k) \\ &\leqslant J(t_1 v_1) + J^\infty(t_2 w) - \frac{1}{p} b(t_1 v_1 - t_2 w_k) + \frac{1}{p} b(t_1 v_1) + \frac{1}{p} b^\infty(t_2 w_k) \end{aligned}$$

We use the inequality

$$(c_1 - c_2)^p > c_1^p + c_2^p - K(c_1^{p-1} c_2 + c_1 c_2^{p-1}),$$

for any c_1, $c_2 > 0$ and some positive constant K, then

$$\begin{aligned} \sup_{1/2 \leqslant t_1, t_2 \leqslant 2} J(t_1 v_1 - t_2 w_k) \leqslant \sup_{t_1 \geqslant 0} J(t_1 v_1) + \sup_{t_2 \geqslant 0} J^\infty(t_2 w) - \frac{1}{2^p p} \int_\Omega (q(x) - q_\infty) w_k^p \\ + K' \left(\int_\Omega v_1^{p-1} w_k + w_k^{p-1} v_1 \right) + \frac{2^p}{p} \int_{\Omega^m \times \mathbb{R}^n} q_\infty w_k^p. \end{aligned}$$

The following estimates is similar to Lemma 10.

(i) $\displaystyle\int_{\Omega^m \times \mathbb{R}^n} q_\infty w_k^p = \int_{\Omega^m \times \mathbb{R}^n} q_\infty \left| w(x + e_k) \right|^p dx \leqslant C_0' \exp\bigl(-(p - \varepsilon) k\bigr)$.

(ii) By the Hőlder inequality,

$$\begin{aligned} \int_{\Omega \cap \{|x| \leqslant R_1\}} v_1^{p-1} w_k &\leqslant \left(\int_{\Omega \cap \{|x| \leqslant R_1\}} v_1^p \right)^{(p-1)/p} \left(\int_{\Omega \cap \{|x| \leqslant R_1\}} w_k^p \right)^{1/p} \\ &\leqslant M \exp(-k). \end{aligned}$$

Applying Lemma 9, there exists a $k_1 \geqslant k_0$ such that for $k \geqslant k_1$

$$\begin{aligned} \int_{\Omega \cap \{|x| \geqslant R_1\}} v_1^{p-1} w_k &\leqslant C_1' \int_{\Omega \cap \{|x| \geqslant R_1\}} \exp\bigl(-(p-1)|x|\bigr) \exp\bigl(-|x + e_k|\bigr)\, dx \\ &\leqslant C_1'' \exp(-k). \end{aligned}$$

Similarly, we also obtain

$$\int_{\Omega\cap\{|x|\leqslant R_1\}} w_k^{p-1}v_1 \leqslant M'\exp\bigl(-(p-1)k\bigr),$$

$$\int_{\Omega\cap\{|Px|\leqslant \overline{R_0}\}} \bigl|q(x)-q_\infty\bigr|w_k^p \leqslant M''\exp\bigl(-(p-\varepsilon)k\bigr),$$

and there exists a $k_2 \geqslant k_1$ such that for $k \geqslant k_2$

$$\int_{\Omega\cap\{|x|\geqslant R_1\}} w_k^{p-1}v_1 \leqslant C'\exp(-k).$$

(iii) Since q satisfies the condition $(\overline{Q})$ and $0 < \delta < 1$, by Lemma 9, there exists a $k_3 \geqslant k_2$ such that for $k \geqslant k_3$

$$\int_{\Omega\cap\{|Px|\geqslant \overline{R_0}\}} \bigl(q(x)-q_\infty\bigr)w_k^p \geqslant C''\exp(-\delta k).$$

By (i)–(iii) and $2 < p < 2^*$, for small $\varepsilon < 1$ we can find a $k_0^* \geqslant k_3 \geqslant k_0$ such that for $k \geqslant k_0^*$

$$K'\Bigl(\int_\Omega v_1^{p-1}w_k + w_k^{p-1}v_1\Bigr) + \frac{2^p}{p}\int_{\Omega^m\times\mathbb{R}^n} q_\infty w_k^p - \frac{1}{2^p p}\int_\Omega \bigl(q(x)-q_\infty\bigr)w_k^p\,dx < 0.$$

Since $J(v_1) = \sup_{t\geqslant 0} J(tv_1)$ and $J^\infty(w) = \sup_{t\geqslant 0} J^\infty(tw)$, we have for $k \geqslant k_0^*$

$$\sup_{1/2\leqslant t_1,t_2\leqslant 2} J(t_1v_1 - t_2w_k) < J(v_1) + J^\infty(w) = \alpha + \alpha^\infty.$$

∎

Theorem 18. *Assume that q satisfies (q_1), (q_2) and the condition $(\overline{Q})$, then Equation (1) has a positive solution v_1 and a solution v_2 which changes sign.*

Proof: By Lemmas 13, 15 16 17 and Theorem 11. ∎

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