Appendix 3
Spin properties of fields and wave equations

This is not a book on field theory, so we do not wish to get involved in a comprehensive discussion of field equations. But the transformation laws for particle states examined in Section 2.4 shed an interesting light upon the problem of constructing fields for arbitrary spin-particles and upon the wave equations they satisfy.

In particular, concerning the Dirac equation, many readers will have followed the beautiful derivation by Dirac of his famous equation for spin-1/2 particles (See Dirac, 1947). Here we shall look at the Dirac equation from a different point of view which provides an alternative insight into the origin and meaning of the equation.

A3.1 Relativistic quantum fields

The essence of the physical states that were discussed in Chapter 1 is that for a particle at rest they transform *irreducibly* under rotations. It would be possible to deal with quantum field operators that also had this property (Weinberg, 1964a), i.e. spin-s fields, which have only $2s + 1$ components. This, as we shall see, is not very convenient for constructing Lagrangians and building-in symmetry properties so that, for example, we normally use a four-component field for spin-1/2 Dirac particles and a 4-vector A_μ to describe spin-1 mesons or photons etc. Thus we usually carry redundant components, and the free-field equations, other than the Klein–Gordon equation, do nothing other than place Lorentz-invariant constraints on the redundant components. It is instructive to compare the approach via irreducible fields with the conventional approach, especially in the case of the Dirac equation.

A local field is constructed by taking a linear combination of creation and annihilation operators in the form of a Fourier transform. Under an arbitrary homogeneous Lorentz transformation l and space–time transla-

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tion a^μ an N-component field is required to transform as

$$U(l,a)\Psi_n(x)U^{-1}(l,a) = \sum_m D_{nm}(l^{-1})\Psi_m(lx+a) \tag{A3.1}$$

where D_{nm} is an N-dimensional representation of the homogeneous Lorentz group. These properties make it relatively simple to write down Lorentz-invariant lagrangians and interactions.

In the following we shall briefly survey the relationship between the physical states introduced earlier and the local fields related to them. We shall see that quanta that have spin s can be embedded in many ways in a field with $N \geq 2s+1$ components. For a more detailed discussion the reader is referred to Weinberg's seminal paper.

We shall present the analysis in terms of the helicity states defined in (1.2.26). With obvious modifications one can base the discussion on the canonical states.

Let $a^\dagger(\mathbf{p},\lambda)$ be the creation operator of the state $|\mathbf{p};\lambda\rangle$ when acting on the bare vacuum. With the invariant normalization

$$\langle \mathbf{p}';\lambda'|\mathbf{p};\lambda\rangle = 2p^0\delta(\mathbf{p}'-\mathbf{p})\delta_{\lambda'\lambda} \tag{A3.2}$$

we take

$$|\mathbf{p};\lambda\rangle = a^\dagger(\mathbf{p},\lambda)|0\rangle \tag{A3.3}$$

so that a and $a^\dagger$ are annihilation and creation operators satisfying commutation or anticommutation relations

$$\left[a(\mathbf{p},\lambda), a^\dagger(\mathbf{p}',\lambda')\right]_\mp = 2p^0\delta(\mathbf{p}'-\mathbf{p})\delta_{\lambda'\lambda} \tag{A3.4}$$

according as $(-1)^{2s} = \pm 1$.

From the transformation properties of the state vector and the invariance of the vacuum, one sees via (2.1.9) that under a Lorentz transformation l

$$U(l)a^\dagger(\mathbf{p},\lambda)U^{-1}(l) = \mathscr{D}^{(s)}_{\lambda'\lambda}\left[h^{-1}(l\mathbf{p})lh(\mathbf{p})\right]a^\dagger(l\mathbf{p},\lambda') \tag{A3.5}$$

where $h(\mathbf{p})$ is given in (1.2.22).

Taking the adjoint and using the unitarity of the representations of the rotation group we get

$$U(l)a(\mathbf{p},\lambda)U^{-1}(l) = \mathscr{D}^{(s)}_{\lambda\lambda'}\left[h^{-1}(\mathbf{p})l^{-1}h(l\mathbf{p})\right]a(l\mathbf{p},\lambda') \tag{A3.6}$$

where the argument of $\mathscr{D}$ is just a Wick helicity rotation.

Because of this complicated $\mathbf{p}$-dependence, a local field built from the $a(\mathbf{p},\lambda)$ via a Fourier transform will not transform in a simple covariant fashion.

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A3.2 Irreducible relativistic quantum fields

To construct a field transforming according to (A3.1) it is necessary to split off the **p**-dependent factors appearing in (A3.5) and to absorb them into new creation operators. The problem is that $\mathscr{D}^{(s)}$ is a representation matrix of the *rotation* group, not the Lorentz group, so that we cannot simply use the property $\mathscr{D}_{ij}(l_1 l_2) = \mathscr{D}_{ik}(l_1)\mathscr{D}_{kj}(l_2)$. To proceed we require certain properties of the representations of the Lorentz group that were discussed in Appendix 2.

As explained there the finite-dimensional representations are labelled (A, B), where A is either integer $(0, 1, 2, \ldots)$ or half integer $(1/2, 3/2, \ldots)$. The simplest representations are the $(s, 0)$ and $(0, s)$ representations, of dimension $2s + 1$.

Consider now fields based upon the use of the $(s, 0)$ representation given in (A2.10). In (A3.6) we can now put

$$\begin{aligned}\mathscr{D}^{(s)}(r_{\text{Wick}}) &= \mathscr{D}^{(s,0)}(r_{\text{Wick}}) \\ &= \mathscr{D}^{(s,0)}\left[h^{-1}(\mathbf{p})\right]\mathscr{D}^{(s,0)}\left(l^{-1}\right)\mathscr{D}^{(s,0)}\left[h(l\mathbf{p})\right]. \end{aligned} \tag{A3.7}$$

If we then define

$$\mathscr{A}(\mathbf{p}, \lambda) \equiv \mathscr{D}^{(s,0)}_{\lambda\lambda'}\left[h(\mathbf{p})\right] a(\mathbf{p}, \lambda') \tag{A3.8}$$

then from (A3.6) and (A3.7) we get the simple result

$$U(l)\mathscr{A}(\mathbf{p}, \lambda)U^{-1}(l) = \mathscr{D}^{(s,0)}_{\lambda\lambda'}\left(l^{-1}\right)\mathscr{A}(l\mathbf{p}, \lambda') \tag{A3.9}$$

so that the transformation matrix is no longer a function of **p**.

In order that the field include both particles and antiparticles we must now consider the operator $b^\dagger(\mathbf{p}, \lambda)$ that creates the antiparticle of the particle which $a(\mathbf{p}, \lambda)$ annihilates. It must transform just like $a^\dagger(\mathbf{p}, \lambda)$, as given in (A3.5). However, the ordering of summation indices in (A3.6) and (A3.5) is different, so we first rewrite (A3.5) in a form analogous to (A3.6) using (A2.13).

Because $\mathscr{D}^{(s)}(r)$ is unitary, we can write

$$\mathscr{D}^{(s)}(r) = \mathscr{D}^{(s)}(r^{-1})^\dagger = \left[C\mathscr{D}^{(s)}(r^{-1})C^{-1}\right]^T \tag{A3.10}$$

where the last step follows from (A2.13), so that (A3.5) becomes

$$U(l)a^\dagger(\mathbf{p}, \lambda)U^{-1}(l) = \left\{C\mathscr{D}^{(s)}\left[h^{-1}(\mathbf{p})l^{-1}h(l\mathbf{p})\right]C^{-1}\right\}_{\lambda\lambda'} a^\dagger(l\mathbf{p}, \lambda') \tag{A3.11}$$

and the same result will hold for $b^\dagger(l\mathbf{p}, \lambda)$. We now define

$$\mathscr{B}^\dagger(l\mathbf{p}, \lambda) = \left\{\mathscr{D}^{(s,0)}[h(\mathbf{p})]C^{-1}\right\}_{\lambda\lambda'} b^\dagger(l\mathbf{p}, \lambda') \tag{A3.12}$$

from which follows, just as in (A3.9),

$$U(l)\mathscr{B}^\dagger(\mathbf{p},\lambda)U^{-1}(l) = \mathscr{D}^{(s,0)}_{\lambda\lambda'}\left(l^{-1}\right)\mathscr{B}^\dagger(l\mathbf{p},\lambda'). \tag{A3.13}$$

The local spin-s field of type $(s,0)$,

$$\phi^{(s,0)}_\lambda(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}2p^0}\left[\mathscr{A}(\mathbf{p},\lambda)e^{-ip\cdot x} + \mathscr{B}^\dagger(\mathbf{p},\lambda)e^{ip\cdot x}\right], \tag{A3.14}$$

transforms according to (A3.1) with $D_{nm} \to \mathscr{D}^{(s,0)}_{nm}$ and can be shown to satisfy causal commutation or anticommutation relations, according as $(-1)^{2s} = \pm 1$. Note that we could introduce a phase factor ξ, $|\xi| = 1$, in front of the $\mathscr{B}^\dagger$ term without altering any of the relevant properties of the local field.

By rewriting ϕ in terms of the original a and b operators, i.e.

$$\begin{aligned}\phi^{(s,0)}_\lambda(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}2p^0}\Big\{&\mathscr{D}^{(s,0)}_{\lambda\lambda'}[h(\mathbf{p})]a(\mathbf{p},\lambda')e^{-ip\cdot x}\\ &+\left[\mathscr{D}^{(s,0)}[h(\mathbf{p})]C^{-1}\right]_{\lambda\lambda'}b^\dagger(\mathbf{p},\lambda')e^{ip\cdot x}\Big\}\end{aligned} \tag{A3.15}$$

one can see, for example, that $\phi^{(s,0)}_\lambda(x)$ creates particles of momentum $\mathbf{p}$ and helicity λ' with wave function

$$\frac{1}{(2\pi)^{3/2}2p^0}\mathscr{D}^{(s,0)*}_{\lambda\lambda'}[h(\mathbf{p})]e^{ip\cdot x}.$$

The field (A3.15) obeys only the Klein–Gordon equation.

Clearly we can introduce a field $\phi^{(0,s)}_\lambda(x)$ in an analogous fashion, and it will transform according to (A3.1) with $D_{nm} \to \mathscr{D}^{(0,s)}_{nm}$. It turns out to be most useful to define $\phi^{(0,s)}_\lambda(x)$ with a phase factor $(-1)^{2s}$ in front of the creation operators:

$$\begin{aligned}\phi^{(0,s)}_\lambda(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}2p^0}\Big\{&\mathscr{D}^{(0,s)}_{\lambda\lambda'}[h(\mathbf{p})]a(\mathbf{p},\lambda')e^{-ip\cdot x}\\ &+(-1)^{2s}\left[\mathscr{D}^{(0,s)}[h(\mathbf{p})]C^{-1}\right]_{\lambda\lambda'}b^\dagger(\mathbf{p},\lambda')e^{ip\cdot x}\Big\}.\end{aligned} \tag{A3.16}$$

A3.3 Parity and field equations

We shall now see that these fields, by themselves, are not suited to a parity-conserving theory. From (A3.3) and eqn (2.3.7) of Chapter 2, we deduce that

$$\mathscr{P}a(p,\theta,\varphi;\lambda)\mathscr{P}^{-1} = \eta_{\mathscr{P}}e^{i\pi s}a(p,\pi-\theta,\varphi+2\pi;-\lambda). \tag{A3.17}$$

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Then after some labour, one finds that

$$\mathscr{P}\phi_\lambda^{(s,0)}(t,\mathbf{x})\mathscr{P}^{-1} = \eta_{\mathscr{P}}\phi_\lambda^{(0,s)}(t,-\mathbf{x}) \tag{A3.18}$$

provided that the intrinsic parity $\bar{\eta}_{\mathscr{P}}$ of the antiparticle is chosen in such a way that

$$\bar{\eta}_{\mathscr{P}} = (-1)^{2s}\eta_{\mathscr{P}}. \tag{A3.19}$$

Thus parity transforms the $(s,0)$ field into the $(0,s)$ field and we are forced to use both to set up a parity-conserving theory.

It is then helpful to combine the $(s,0)$ and $(0,s)$ fields into one $2(2s+1)$-component field

$$\psi_\alpha(x) = \begin{pmatrix} \phi^{(s,0)}(x) \\ \phi^{(0,s)}(x) \end{pmatrix}, \tag{A3.20}$$

which then transforms according to

$$U(l)\psi_\alpha(x)U^{-1}(l) = D_{\alpha\beta}^{(s)}(l^{-1})\psi_\beta(lx) \tag{A3.21}$$

where

$$D_{\alpha\beta}^{(s)}(l) = \begin{pmatrix} \mathscr{D}^{(s,0)}(l) & 0 \\ 0 & \mathscr{D}^{(0,s)}(l) \end{pmatrix} \tag{A3.22}$$

i.e. ψ transforms according to the $(s,0)\oplus(0,s)$ representation.

It can be shown that the fields $\psi_\alpha(x)$ satisfy causal commutation or anticommutation relations. The factor $(-1)^{2s}$ in (A3.16) is crucial for this.

Each field $\psi_\alpha(x)$ will clearly satisfy a Klein–Gordon equation. But there will be other equations of constraint. To see where these come from consider the matrices

$$\Pi^{(s)}(p) \equiv m^{2s}\mathscr{D}^{(s,0)}[h(\mathbf{p})]\mathscr{D}^{(0,s)}[h^{-1}(\mathbf{p})] \tag{A3.23}$$

and

$$\overline{\Pi}^{(s)}(p) \equiv m^{2s}\mathscr{D}^{(0,s)}[h(\mathbf{p})]\mathscr{D}^{(s,0)}[h^{-1}(\mathbf{p})], \tag{A3.24}$$

which will convert $\mathscr{D}^{(0,s)}[h(\mathbf{p})]$ to $\mathscr{D}^{(s,0)}[h(\mathbf{p})]$ and vice versa respectively.

Using (A2.10), (A2.11) and (1.2.22), Weinberg has shown that $\Pi^{(s)}$ and $\overline{\Pi}^{(s)}$ are homogeneous polynomials of order $2s$ in the components p^μ of the 4-vector $(E,\mathbf{p})$. Hence we can define a matrix differential operator $\overline{\Pi}^{(s)}(i\partial)$ and consider its action on $\phi_\lambda^{(s,0)}(x)$. When taken under the integral sign in (A3.15) and acting on $e^{-ip\cdot x}$, $\overline{\Pi}^{(s)}(i\partial)$ becomes $\overline{\Pi}^{(s)}(p)$ and thus converts the first $\mathscr{D}^{(s,0)}$ to $\mathscr{D}^{(0,s)}$. Acting on $e^{ip\cdot x}$, it converts the second $\mathscr{D}^{(s,0)}$ to $(-1)^{2s}\mathscr{D}^{(0,s)}$. In other words, using (A3.16),

$$\overline{\Pi}_{\nu\lambda}^{(s)}(i\partial)\phi_\lambda^{(s,0)}(x) = m^{2s}\phi_\nu^{(0,s)}(x). \tag{A3.25}$$

Similarly

$$\Pi^{(s)}_{\nu\lambda}(i\partial)\phi^{(0,s)}_{\lambda}(x) = m^{2s}\phi^{(s,0)}_{\nu}(x). \tag{A3.26}$$

Thus $\psi(x)$ satisfies the equation

$$\begin{pmatrix} 0 & \Pi^{(s)}(i\partial) \\ \overline{\Pi}^{(s)}(i\partial) & 0 \end{pmatrix} \psi(x) = m^{2s}\psi(x). \tag{A3.27}$$

A3.4 The Dirac equation

The classic example of the above construction is the Dirac equation for spin 1/2. For the (1/2, 0) representation (see eqn (A2.2)) $\hat{\mathbf{A}} \to (1/2)\boldsymbol{\sigma}$ so that

$$\mathscr{D}^{(1/2,0)}(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) = e^{-i\boldsymbol{\sigma}\cdot(\boldsymbol{\vartheta}-i\boldsymbol{\alpha})/2}. \tag{A3.28}$$

For (0, 1/2), $\hat{\mathbf{B}} \to (1/2)\boldsymbol{\sigma}$ so that

$$\mathscr{D}^{(0,1/2)}(\boldsymbol{\vartheta}, \boldsymbol{\alpha}) = e^{-i\boldsymbol{\sigma}\cdot(\boldsymbol{\vartheta}+i\boldsymbol{\alpha})/2} \tag{A3.29}$$

and the physical meanings of $\boldsymbol{\vartheta}$ and $\boldsymbol{\alpha}$ are given after equation (A2.4).

Then, using (1.2.22), (A3.23) and (A3.24), one finds

$$\Pi^{(1/2)}(p) = me^{-\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}} = p^0 - \mathbf{p}\cdot\boldsymbol{\sigma} \tag{A3.30}$$

and

$$\overline{\Pi}^{(1/2)}(p) = me^{\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}} = p^0 + \mathbf{p}\cdot\boldsymbol{\sigma} \tag{A3.31}$$

where we have used the fact that for a boost from rest to 4-momentum $(E, \mathbf{p})$, $\tanh\alpha = \beta = |\mathbf{p}|/E$.

Finally, (A3.27) for $s = 1/2$ can be recognized as the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\,\psi(x) = 0 \tag{A3.32}$$

in the representation where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{A3.33}$$

This is just a representation in which γ_5 is diagonal. Weinberg (1964a) showed how the above generalizes to a Dirac-like equation for arbitrary spin, i.e where the fields transform like $(s, 0) \oplus (0, s)$.

As mentioned earlier, these minimal fields, with the exception of the spin-1/2 case, are not those normally used in constructing lagrangians for particle interactions. For developments concerning more general fields consult the fundamental papers of Weinberg (1964a,b).