Canad. Math. Bull. Vol. **38** (3), 1995 pp. 278–285

AN EXTENSION THEOREM CONCERNING FRECHET MEASURES

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ABSTRACT. An F-measure on a Cartesian product of algebras of sets is a scalar-valued function which is a scalar measure independently in each coordinate. It is demonstrated that an F-measure on a product of algebras determines an F-measure on the product of the corresponding σ-algebras if and only if its *Fréchet* variation is finite. An analogous statement is obtained in a framework of fractional Cartesian products of algebras, and a measurement of p-variation of F-measures, based on Littlewood-type inequalities, is discussed.

0. **Introduction.** *A scalar measure on an algebra of sets is extendible to a scalar measure on the corresponding σ-algebra if and only if its total variation is finite.* In one direction, this cornerstone of classical measure theory is the assertion that a scalar measure on a σ-algebra is necessarily bounded (*e.g.*, [7, Corollary III.4.6]), and in the other, it is the Carathéodory-Hahn-Jordan theorem (*e.g.*, [7, Theorem III.5.8, Corollary III.5.9]). In this note, we establish the multidimensional version of this basic result.

By an *F-measure* we shall mean a scalar-valued function on a Cartesian product of algebras of sets that is a scalar measure independently in each coordinate. In Section 1, we prove that an F-measure on a Cartesian product of algebras is extendible to an F-measure on the Cartesian product of the corresponding σ-algebras if and only if its *Fréchet* variation is finite. The *only if* direction, based on the Nikodym boundedness principle ([7, Theorem IV.9.8] or [5, Theorem I.3.1]), has in effect already been noted (*e.g.*, [12, Theorem 4.4], [1, Theorem 4.3]), but the other direction has hitherto gone unnoticed (*cf.* [5, Theorem I.5.2], [6]).

In Section 2, we use the extension theorem established in Section 1 to obtain the corresponding result for F-measures on *fractional* Cartesian products of algebras. We then comment on the intervention of Littlewood-type inequalities and resulting measurements of *p-variations* of F-measures. These measurements, noted previously in more restrictive settings [1], have been recently shown to play key roles in non-adaptive stochastic integration [11].

1. F-measures.

DEFINITION 1.1. Let $X_1,\dots,X_n$ be sets, and let $C_1,\dots,C_n$ be algebras of subsets of $X_1,\dots,X_n$, respectively. A scalar-valued function μ on the n-fold Cartesian product

Received by the editors January 14, 1994; revised December 15, 1994.
AMS subject classification: Primary: 28A35, 46E27; secondary: 26D15.

$C_1 \times \cdots \times C_n$ is an F_n-measure if μ is a scalar measure separately in each coordinate. Such μ will be generically called *Fréchet* measures or F-measures.

The space of F_n-measures on $C_1 \times \cdots \times C_n$ is denoted by $F_n(C_1 \times \cdots \times C_n)$. If $C_1, \ldots, C_n$ are arbitrary or understood from the context, then $F_n(C_1 \times \cdots \times C_n)$ is denoted by F_n. $\mu \in F_n$ is said to be *bounded* if

$$\text{(1.1)} \qquad \sup\{|\mu(E_1 \times \cdots \times E_n)| : E_1 \times \cdots \times E_n \in C_1 \times \cdots \times C_n\} < \infty.$$

The objects which I call F_2-measures arose first in Fréchet's work [8] as bounded bilinear functions on $C([0, 1])$. In later studies, in a framework of two-fold topological products, these bilinear functionals were dubbed *bimeasures* (*e.g.*, [10]). In general multi-dimensional settings, references have occasionally been made to *multimeasures* or *polymeasures* (*e.g.*, [6]). I prefer the term *F_n-measure* in a multilinear measure-theoretic context (*e.g.*, [1]) primarily because it registers dimensions of underlying Cartesian products, which could be fractional (see next section). F of course is for Fréchet.

If C is an algebra of subsets of X, then a *C-partition* of X will mean a collection of mutually disjoint elements of C whose union is X. If $C_1, \ldots, C_n$ are algebras of subsets of $X_1, \ldots, X_n$, respectively, then $C_1 \times \cdots \times C_n$-*grid* of $X_1 \times \cdots \times X_n$ will mean n-fold Cartesian product of finite $C_1, \ldots, C_n$-partitions of $X_1, \ldots, X_n$, respectively. When $(X_1, C_1), \ldots, (X_n, C_n)$ are arbitrary or understood from the context, we refer simply to *partitions* and *grids*.

A Rademacher system indexed by a set τ is the collection of functions $\{r_\alpha\}_{\alpha \in \tau}$ defined on $\{-1, +1\}^\tau$, such that $r_\alpha(\omega) = \omega(\alpha)$ for $\alpha \in \tau$ and $\omega \in \{-1, +1\}^\tau$. If $\tau_1, \ldots, \tau_n$ are indexing sets, then $r_{\alpha_1} \otimes \cdots \otimes r_{\alpha_n}$ denotes the function on $\{-1, +1\}^{\tau_1} \times \cdots \times \{-1, +1\}^{\tau_n}$ whose value at $(\omega_1, \ldots, \omega_n)$ equals $r_{\alpha_1}(\omega_1) \cdots r_{\alpha_n}(\omega_n)$.

If $\mu \in F_n(C_1 \times \cdots \times C_n)$, then the F_n-norm (Fréchet variation) of μ is defined by

$$\text{(1.2)} \qquad \|\mu\|_{F_n} = \sup\Big\{\Big\| \sum_{E_1 \times \cdots \times E_n \in \gamma} \mu(E_1 \times \cdots \times E_n) r_{E_1} \otimes \cdots \otimes r_{E_n} \Big\|_\infty : \gamma \text{ a grid}\Big\}$$

(*cf.* [1, (4.3)]; $r_{E_1}, \ldots, r_{E_n}$ are elements of n Rademacher systems indexed respectively by the n partitions whose Cartesian product is γ).

THEOREM 1.2. *Let $C_1, \ldots, C_n$ be algebras of sets in $X_1, \ldots, X_n$, respectively, and let $\mu \in F_n(C_1 \times \cdots \times C_n)$. Then, μ is uniquely extendible to an F_n-measure on $\sigma(C_1) \times \cdots \times \sigma(C_n)$ if and only if $\|\mu\|_{F_n} < \infty$ ($\sigma(C)$ = σ-algebra generated by C). Moreover,*

$$\text{(1.3)} \qquad \|\mu\|_{F_n(C_1 \times \cdots \times C_n)} = \|\mu\|_{F_n(\sigma(C_1) \times \cdots \times \sigma(C_n))}.$$

The proof of Theorem 1.2 requires two elementary lemmas. The first, Lemma 1.3 below, appeared in [1] (Lemma 4.4 on p. 41) where the proof was too long; I include here a simpler and shorter proof. The second, Lemma 1.4, can be verified quickly in a context of harmonic analysis by use of Riesz products; a longer but elementary proof can be found in [9, pp. 167–168].

LEMMA 1.3. *Let N be an arbitrary positive integer, and let $\{a_{i_i \cdots i_n}\}_{i_1,\ldots,i_n=1}^N$ be an array of scalars. Then, for each $j \in [n]$ $(:= \{1,\ldots,n\})$, there exist subsets $T_j \subset [N]$ such that*

$$\left|\sum_{(i_1,\ldots,i_n)\in T_1\times\cdots\times T_n} a_{i_1\cdots i_n}\right| \geq \frac{1}{4^n}\left\|\sum_{(i_1,\ldots,i_n)\in[N]^n} a_{i_1\cdots i_n} r_{i_1}\otimes\cdots\otimes r_{i_n}\right\|_\infty.$$

PROOF (BY INDUCTION ON n). The case $n = 1$ is merely the statement that for every set of scalars $\{a_j : j \in [N]\}$ there exists $T \subset [N]$ such that

$$\left|\sum_{j\in T} a_j\right| \geq \frac{1}{4}\sum_{j\in[N]} |a_j|.$$

If $n > 1$ and $\{a_{i_1\cdots i_n}\}_{i_1,\ldots,i_n=1}^N$ is an array of scalars, then let $\omega_1 \in \{-1,1\}^{[N]},\ldots,\omega_n \in \{-1,1\}^{[N]}$ be such that

$$\left\|\sum_{(i_1,\ldots,i_n)\in[N]^n} a_{i_1\cdots i_n} r_{i_1}\otimes\cdots\otimes r_{i_n}\right\|_\infty = \left|\sum_{(i_1,\ldots,i_n)\in[N]^n} a_{i_1\cdots i_n} r_{i_1}(\omega_1)\cdots r_{i_n}(\omega_n)\right|. \tag{1.4}$$

By the assertion for $n = 1$, there exists $T_1 \subset [N]$ so that

$$4\left|\sum_{i_1\in T_1}\left(\sum_{(i_2,\ldots,i_n)\in[N]^n} a_{i_1\cdots i_n} r_{i_2}(\omega_2)\cdots r_{i_n}(\omega_n)\right)\right| \tag{1.5}$$

majorizes (1.4). Now reverse the two summations in (1.5) and apply the induction hypothesis to obtain $T_2 \subset [N],\ldots,T_n \subset [N]$ so that (1.5) is majorized by

$$4^{n-1}\left|\sum_{i_2\in T_2,\ldots,i_n\in T_n}\left(4\sum_{i_1\in T_1} a_{i_1\ldots i_n}\right)\right|.$$

■

LEMMA 1.4. *Suppose $\{a_{ij} : (i,j) \in \mathbb{N}^2\}$ is an array of scalars such that $\sup\{\|\sum_{i\in S,j\in T} a_{ij} r_i\otimes r_j\|_\infty : S$ and T finite subsets of $\mathbb{N}\} < \infty$. Then,*

$$\sum_{i=1}^\infty\sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty\sum_{i=1}^\infty a_{ij}.$$

PROOF OF THEOREM 1.2. If $\mu \in F_n(C_1\times\cdots\times C_n)$ is a restriction of an F_n-measure on $\sigma(C_1)\times\cdots\times\sigma(C_n)$, then an inductive application of the Nikodym boundedness principle implies μ is bounded. Then, a standard argument based on Lemma 1.3 implies $\|\mu\|_{F_n} < \infty$.

Conversely, we show by induction on n that if $\mu \in F_n(C_1\times\cdots\times C_n)$ and $\|\mu\|_{F_n} < \infty$, then there exists an extension of μ to an F_n-measure on $\mathfrak{C}_1\times\cdots\times\mathfrak{C}_n$, where $\mathfrak{C}_i = \sigma(C_i)$ $(i \in [n])$. It is evident that such an extension is necessarily unique.

The case $n = 1$ is standard. Let $n > 1$, and assume the assertion holds in the case $n-1$. Let μ be an F_n-measure on $C_1\times\cdots\times C_n$. Then, for each $A_2\times\cdots\times A_n \in C_2\times\cdots\times C_n$,

https://doi.org/10.4153/CMB-1995-041-0 Published online by Cambridge University Press

$\mu(\cdot \times A_2 \times \cdots \times A_n)$ is extendible to a scalar measure on $\mathfrak{C}_1$. Denote this extension also by μ. Note

(1.6) $\sup\{|\mu(A_1 \times A_2 \times \cdots \times A_n)| : A_1 \times A_2 \times \cdots \times A_n \in \mathfrak{C}_1 \times C_2 \times \cdots \times C_n\} < \infty.$

CLAIM. *For each $A \in \mathfrak{C}_1$, $\mu(A \times \cdot \times \cdots \times \cdot) \in F_{n-1}(C_2 \times \cdots \times C_n)$.*

PROOF OF CLAIM. Let

$$(1.7) \qquad \Omega = \{A : A \in \mathfrak{C}_1, \mu(A \times \cdot \times \cdots \times \cdot) \in F_{n-1}(C_2 \times \cdots \times C_n)\}.$$

Clearly, Ω is an algebra containing C_1. We will show that Ω is a σ-algebra. Suppose $E_i \in \Omega$ ($i \in \mathbb{N}$) and that the E_i's are mutually disjoint. Let $E = \bigcup_i E_i$. To verify $E \in \Omega$, we need to establish that $\mu(E \times \cdot \times \cdots \times \cdot)$ is countably additive separately in each of the $n-1$ coordinates. Fix $B_2 \in C_2, \ldots, B_{n-1} \in C_{n-1}$. Let $\{F_j\}_j$ be a sequence of mutually disjoint elements in C_n such that $\bigcup_j F_j \in C_n$. We claim that

$$(1.8) \qquad \mu\Big(E \times B_2 \times \cdots \times B_{n-1} \times \bigcup_j F_j\Big) = \sum_{j=1}^{\infty} \mu(E \times B_2 \times \cdots \times B_{n-1} \times F_j).$$

Since μ is a scalar measure in its first coordinate,

$$(1.9) \qquad \mu\Big(E \times B_2 \times \cdots \times B_{n-1} \times \bigcup_j F_j\Big) = \sum_{i=1}^{\infty} \mu\Big(E_i \times B_2 \times \cdots \times B_{n-1} \times \bigcup_j F_j\Big).$$

Since $E_i \in \Omega$ for each $i \in \mathbb{N}$,

$$(1.10) \quad \sum_{i=1}^{\infty} \mu\Big(E_i \times B_2 \times \cdots \times B_{n-1} \times \bigcup_j F_j\Big) = \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty} \mu(E_i \times B_2 \times \cdots \times B_{n-1} \times F_j)\Big).$$

By (1.6) and Lemma 1.3,

$$\sup\Big\{\Big\|\sum_{E_1 \times \cdots \times E_n \in \gamma} \mu(E_1 \times \cdots \times E_n) r_{E_1} \otimes \cdots \otimes r_{E_n}\Big\|_\infty : \gamma \text{ is a } \mathfrak{C}_1 \times C_2 \times \cdots \times C_n\text{-grid}\Big\}$$

is finite. Therefore, by Lemma 1.4, we can reverse the order of summation on the right hand side of (1.10). Therefore, since μ is a measure in the first coordinate, we obtain

(1.11)

$$\begin{aligned}\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty} \mu(E_i \times B_2 \times \cdots \times B_{n-1} \times F_j)\Big) &= \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty} \mu(E_1 \times B_2 \times \cdots \times B_{n-1} \times F_j)\Big)\\ &= \sum_{j=1}^{\infty} \mu(E \times B_2 \times \cdots \times B_{n-1} \times F_j)\Big),\end{aligned}$$

thus establishing (1.8). ■

The induction hypothesis and the Claim imply that for each $A \in \mathfrak{C}_1$, $\mu(A \times \cdot \times \cdots \times \cdot)$ is extendible to an F_{n-1}-measure on $\mathfrak{C}_2 \times \cdots \times \mathfrak{C}_n$. Denote this extension also by μ.

To verify that for every $A_2 \times \cdots \times A_n \in \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_n$, $\mu(\cdot \times A_2 \times \cdots \times A_n)$ is a scalar measure on $\mathfrak{C}_1$, first fix $B_2 \times \cdots \times B_{n-1} \in C_2 \times \cdots \times C_{n-1}$, and let

$$(1.12) \quad \Omega_n = \{A : A \in \mathfrak{C}_n, \mu(\cdot \times B_2 \times \cdots \times B_{n-1} \times A) \text{ is a scalar measure on } \mathfrak{C}_1\}.$$

The argument establishing that Ω_n is a σ-algebra is similar to the argument used in the proof of the Claim. We continue by recursion to treat remaining coordinates, obtaining that for all $A_2 \times \cdots \times A_n \in \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_n$, $\mu(\cdot \times A_2 \times \cdots \times A_n)$ is a scalar measure on $\mathfrak{C}_1$.

To verify (1.3), approximate $\mu(A_1 \times \cdots \times A_n)$, where $A_1 \times \cdots \times A_n \in \mathfrak{C}_1 \times \cdots \times \mathfrak{C}_n$, by $\mu(E_1 \times \cdots \times E_n)$, where $E_1 \times \cdots \times E_n \in C_1 \times \cdots \times C_n$. ■

2. *F*-**measures in fractional dimensions.** The question concerning the extension of $\mu \in F_n(C_1 \times \cdots \times C_n)$ to an F_n-measure on $\sigma(C_1) \times \cdots \times \sigma(C_n)$, considered in Section 1, is an instance of a general question in a framework of *fractional* Cartesian products of algebras.

If Y is a set and $S \subset [n]$, then Y^S denotes the Cartesian product of Y whose coordinates are indexed by S; slightly abusing notation, we shall write Y^n for $Y^{[n]}$. We denote by π_S the canonical projection from Y^n onto Y^S, *i.e.*, $\pi_S(y_1, \ldots, y_n) = (y_j : j \in S)$. We shall sometimes use also the notation $\pi_S(y) = y|_S$. To simplify notation, we shall consider Cartesian products of (X, C), where X is a set and C is an algebra of subsets of X (in place of $(X_1, C_1), \ldots, (X_n, C_n)$, considered in Section 1).

DEFINITION 2.1. Let V be a cover of $[n]$, *i.e.*, $S \subset [n]$ for $S \in V$ and $\bigcup\{S : S \in V\} = [n]$. Let X be a set and let C be an algebra of subsets of X. Then, $\mu \in F_n(C^n)$ is an F_V-measure on C^n if for every $S \in V$ and for all $\times\{A_i : i \in S^c\} \in C^{S^c}$,

$$(2.1) \qquad \mu(A_1 \times \cdots \times A_n), \quad \times\{A_i : i \in S\} \in C^S,$$

defines a scalar measure on $\alpha(C^S)$ ($\alpha(\cdot) =$ algebra generated by $\cdot$). The set of F_V-measures on C^n is denoted by $F_V(C^V)$, or simply by F_V.

A basic problem is to identify the "largest" domain on which a finitely additive function on C^n determines an F-measure. For example, if $\mu \in F_1\big(\alpha(C^n)\big)$ then μ determines an F_1-measure on $\sigma(C^n)$ if and only if its *total variation*

$$(2.2) \qquad \sup\Big\{\sum_{c\in\gamma} |\mu(c)| : C^n\text{-grid } \gamma\Big\} \quad \Big(= \sup\Big\{\Big\|\sum_{c\in\gamma} \mu(c) r_c\Big\|_\infty : C^n\text{-grid } \gamma\Big\}\Big)$$

is finite (this of course is classical). At the other end, if $\mu \in F_n(C^n)$ then μ determines an F_n-measure on $\sigma(C)^n$ if and only if $\|\mu\|_{F_n}$ is finite (Theorem 1.2). Definition 2.1 deals with the intermediate cases between these two extremes. To be precise, let $V = \{S_j\}_{j=1}^m$ be a cover of $[n]$, and consider the collection of sets

$$(2.3) \qquad \alpha(C)^V := \{\pi_{S_1}^{-1}[c_1] \cap \cdots \cap \pi_{S_m}^{-1}[c_m]) : c_1 \in \alpha(C^{S_1}), \ldots, c_m \in \alpha(C^{S_m})\},$$

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whose elements will be called V-cubes. (The simplest non-trivial case, $n = 3$ and $V = \{(1,2),(2,3),(1,3)\}$, is an effective illustration for the discussion that follows.) Observe that if $c_1 \in \alpha(C^{S_1}),\ldots,c_m \in \alpha(C^{S_m})$, then

$$\pi_{S_1}^{-1}[c_1]\cap\cdots\cap\pi_{S_m}^{-1}[c_m] = \bigcap\{\pi_1(c_i) : 1 \in S_i\} \times \cdots \times \bigcap\{\pi_n(c_i) : n \in S_i\}. \tag{2.4}$$

where $\pi_1,\ldots,\pi_n$ denote the canonical projections from X^n onto X. Let $\mu \in F_n(C^n)$. By finite additivity, we extend the domain of μ to $\alpha(C)^V$, and then define

$$\begin{aligned}\tilde{\mu}(c_1 \times \cdots \times c_m) &= \mu(\pi_{S_1}^{-1}[c_1]\cap\cdots\cap\pi_{S_m}^{-1}[c_m])\\ &\left(= \mu\Big(\bigcap\{\pi_1(c_i) : 1 \in S_i\} \times \cdots \times \bigcap\{\pi_n(c_i) : n \in S_i\}\Big)\right),\\ &c_1 \in \alpha(C^{S_1}),\ldots,c_m \in \alpha(C)^{S_m}.\end{aligned} \tag{2.5}$$

Then, $\tilde{\mu}$ is well defined and $\tilde{\mu} \in F_m\big(\alpha(C^{S_1}) \times \cdots \times \alpha(C^{S_m})\big)$ if and only if $\mu \in F_V(C^V)$. Denote by $F_V\big(\sigma(C)^V\big)$ the class consisting of $\mu \in F_V(C^V)$ extendible to $\sigma(C)^V$ so that

$$\mu(\pi_{S_1}^{-1}[E_1]\cap\cdots\cap\pi_{S_m}^{-1}[E_m]), \quad E_1 \in \sigma(C^{S_1}),\ldots,E_m \in \sigma(C^{S_m}), \tag{2.6}$$

determines an F_m-measure on $\sigma(C^{S_1}) \times \cdots \times \sigma(C^{S_m})$.

By Theorem 1.2, if $\tilde{\mu} \in F_m\big(\alpha(C^{S_1}) \times \cdots \times \alpha(C^{S_m})\big)$, then $\tilde{\mu}$ determines an element in $F_m\big(\sigma(C^{S_1}) \times \cdots \times \sigma(C^{S_m})\big)$ (denoted also by $\tilde{\mu}$) if and only if

$$\begin{aligned}\|\tilde{\mu}\|_{F_m} = \sup\Big\{\Big\|\sum_{c_1\times\cdots\times c_m\in\gamma} &\mu\Big(\bigcap\{\pi_1(c_i) : 1 \in S_i\} \times \cdots\\ &\times \bigcap\{\pi_n(c_i) : n \in S_i\}\Big) r_{c_1} \otimes \cdots \otimes r_{c_m}\Big\|_\infty :\\ &\gamma = \gamma_1 \times \cdots \times \gamma_m, \text{ where } \gamma_j \text{ is a } \alpha(C^{S_j})\text{-grid}, j \in [m]\Big\}\end{aligned} \tag{2.7}$$

is finite. In (2.7), by passing to refinements of partitions, we can assume that the γ_j's are generated by the same C-partition of X, say τ, *i.e.*, $\gamma = \tau^{S_1} \times \cdots \times \tau^{S_m}$. In this case, if $c_1 \times \cdots \times c_m \in \gamma$ then

$$\bigcap\{\pi_1(c_i) : 1 \in S_1\} \times\cdots\times \bigcap\{\pi_n(c_i) : n \in S_i\} = \begin{cases} d & d \in \tau^n, c_j = d|_{S_j}, j \in [m] \\ \emptyset & \text{otherwise}\end{cases} \tag{2.8}$$

(notation: for $d = d_1 \times \cdots \times d_n \in \tau^n$, $d|_{S_j} = \times\{d_i : i \in S_j\}$). Define

$$\|\mu\|_{F_V} = \sup\Big\{\Big\|\sum_{d\in\gamma} \mu(d) r_{d|_{S_1}} \otimes \cdots \otimes r_{d|_{S_m}}\Big\|_\infty : \gamma \text{ a } C^n\text{-grid of } X^n\Big\}, \tag{2.9}$$

and deduce that $\|\mu\|_{F_V} = \|\tilde{\mu}\|_{F_m}$ ($r_{d|_{S_j}}$ is an element of a Rademacher system indexed by $\tau^{S_j}, j \in [m]$). For example, if $n = 3$ and $V = \{(1,2),(2,3),(1,3)\}$ then

(2.10)
$$\|\mu\|_{F_V} = \sup\Big\{\Big\|\sum_{A\times B\times C\in\tau^3} \mu(A \times B \times C) r_{A\times B} \otimes r_{B\times C} \otimes r_{A\times C}\Big\|_\infty : \tau \text{ a } C\text{-partition of } X\Big\},$$

where the sup-norm is evaluated on $\{-1,+1\}^{\tau^2} \times \{-1,+1\}^{\tau^2} \times \{-1,+1\}^{\tau^2}$.

It follows from (2.4) that if $\tilde{\mu}$ is extendible to an F_m-measure on $\sigma(C^{S_1}) \times \cdots \times \sigma(C^{S_m})$ then it is determined by its values on V-cubes, *i.e.*, if

$$\pi_{S_1}^{-1}[E_1] \cap \cdots \cap \pi_{S_m}^{-1}[E_m] = \pi_{S_1}^{-1}[F_1] \cap \cdots \cap \pi_{S_m}^{-1}[F_m]$$

then $\tilde{\mu}(E_1 \times \cdots \times E_m) = \tilde{\mu}(F_1 \times \cdots \times F_m)$. Therefore, if $\tilde{\mu} \in F_m\big(\sigma(C^{S_1}) \times \cdots \times \sigma(C^{S_m})\big)$ then we can unambiguously write

$$\mu(\pi_{S_1}^{-1}[E_1] \cap \cdots \cap \pi_{S_m}^{-1}[E_m]) = \tilde{\mu}(E_1 \times \cdots \times E_m). \tag{2.11}$$

We summarize:

THEOREM 2.2. *If $\mu \in F_V(C^V)$, then $\mu \in F_V\big(\sigma(C)^V\big)$ if and only if $\|\mu\|_{F_V} < \infty$.*

REMARK. Suppose C is infinite. The fundamental observation, that the inclusion $F_1\big(\sigma(C^2)\big) \subset F_2\big(\sigma(C)^2\big)$ is proper, was noted independently in various contexts during the 1930's (*e.g.* [4], [9]). A key to this observation was that if $\mu \in F_2\big(\sigma(C)^2\big)$ were extendible to an element in $F_1\big(\sigma(C^2)\big)$ then its *total variation*, defined in (2.2), would be finite. Indeed, after noting that there exists μ with finite F_2-variation and infinite total variation, Littlewood [9] proceeded to derive his $4/3$-inequality(ies), conveying that the p-variation of *every* $\mu \in F_2\big(\sigma(C)^2\big)$ is finite if and only if $p \geq 4/3$. In particular, let $\mu \in F_n\big(\sigma(C)^n\big)$, and define the p-variation of μ by

$$\|\mu\|_{(p)} = \sup\Big\{\sum_{c\in\gamma} |\mu(\mathbf{c})|^p : \gamma \ C^n\text{-grid of } X^n\Big\}. \tag{2.12}$$

Define the Littlewood exponent of μ (*e.g.*, [2]) by

$$\ell_\mu = \inf\{p : \|\mu\|_{(p)} < \infty\}. \tag{2.13}$$

Then, Littlewood's inequalities are equivalent to the statement

$$\sup\big\{\ell_\mu : \mu \in F_2\big(\sigma(C)^2\big)\big\} = 4/3. \tag{2.14}$$

In the general case, let $V = \{S_j\}_{j=1}^m$ be a cover of $[n]$, and consider the linear programming problem.

Maximize $x_1 + \cdots + x_n = e$ subject to the constraints that each $x_i \geq 0$ and $\sum_{i\in S_j} x_i \leq 1$ for each $j \in [m]$.

Let the optimal value solving this problem be $e = e(V)$. Combining the "fractional" version of Littlewood's inequalities (*e.g.*, [1]) with the result in [3], asserting that $e(V)$ is the *combinatorial dimension* of $C^{(V)} = \big\{\big(\pi_{S_1}(\mathbf{c}), \ldots, \pi_{S_m}(\mathbf{c})\big) : \mathbf{c} \in C^m\big\}$, we deduce

THEOREM 2.3. $\sup\big\{\ell_\mu : \mu \in F_V\big(\sigma(C)^V\big)\big\} = \frac{2e(V)}{e(V)+1}$.

If U and V are two covers of $[n]$, then $U < V$ means that for every $T \in U$ there exists $S \in V$ such that $T \subset S$. It is easy to see that if $U < V$, then $e(U) \geq e(V)$ and $F_U\big(\sigma(C)^U\big) \supset F_V\big(\sigma(C)^V\big)$. Theorem 2.3 implies that if $e(U) > e(V)$ then the preceding inclusion is proper.

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https://doi.org/10.4153/CMB-1995-041-0 Published online by Cambridge University Press