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A NOTE ON SEMIHEREDITARY RINGS

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It's well known (see Endo [1]) that for a commutative ring A, if A is semihereditary then w. gl. dim. $A \leq 1$. It seems worth recording the noncommutative version of this.

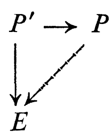
PROPOSITION. *If A ring A is left semihereditary then* w. gl. dim. $A \leq 1$.

For the proof we need the following notion:

DEFINITION (Maddox [2]). A left A-module E is absolutely pure if and only if for every left A-module F containing E as a submodule, E is pure in F in the sense of P. M. Cohn [3].

The following then holds:

THEOREM (Megibben [4]). *A left A-module E is absolutely pure if and only if every diagram*



can be completed to a commutative diagram where P is a projective left A-module and P′ is a finitely generated submodule of P.

We now prove the proposition. Let G be a flat right A-module and T a submodule of G. Then since Q/Z is Z-injective

$$\mathrm{Hom}_Z(G, Q/Z) \to \mathrm{Hom}_Z(T, Q/Z) \to 0$$

is exact. But a result of Lambek [5] says $\mathrm{Hom}_Z(G, Q/Z)$ is injective and so absolutely pure. Thus by Megibben's theorem $\mathrm{Hom}_Z(T, Q/Z)$ is absolutely pure. Now let P' be a finitely generated submodule of a projective left A-module P.

Then

$$\mathrm{Hom}_A(P, \mathrm{Hom}_Z(T, Q/Z)) \to \mathrm{Hom}_A(P', \mathrm{Hom}_Z(T, Q/Z)) \to 0$$

is exact by Megibben's proposition.

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But then using the natural isomorphism

$$\mathrm{Hom}_A(P, \mathrm{Hom}_Z(T, Q/Z) \cong \mathrm{Hom}_Z(T \otimes_A P, Q/Z)$$

$$\mathrm{Hom}_A(P', \mathrm{Hom}_Z(T, Q/Z)) \cong \mathrm{Hom}_Z(T \otimes_A P', Q/Z)$$

we get

$$\mathrm{Hom}_Z(T \otimes_A P, Q/Z) \to \mathrm{Hom}_Z(T \otimes_A P', Q/Z) \to 0$$

exact. But Q/Z is a faithfully injective Z-module so

$$0 \to T \otimes_A P' \to T \otimes_A P$$

is exact. But this means T is a flat right A-module. This completes the proof.

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