Math. Proc. Camb. Phil. Soc. (2025), **178**, 15–29
doi:10.1017/S0305004124000392
First published online 8 January 2025

Uniformity of quasi-integral points of bounded degree in higher-dimensional orbits

BY YU YASUFUKU†
Department of Mathematics, School of Education, Waseda University, 1-6-1 *Nishi-Waseda, Shinjuku-ku, Tokyo* 169-8050, *Japan.*
e-mail: yasufuku@waseda.jp

(*Received* 11 *May* 2023*; revised* 02 *December* 2024)

Abstract

Given a self-morphism ϕ on a projective variety defined over a number field k, we prove two results which bound the largest iterate of ϕ whose evaluation at P is quasi-integral with respect to a divisor D, uniformly across P defined over a field of bounded degree over k. The first result applies when the pullback of D by some iterate of ϕ breaks up into enough irreducible components which are numerical multiples of each other. The proof uses Le's algebraic-point version of a result of Ji–Yan–Yu, which is based on Schmidt subspace theorem. The second result applies more generally but relies on a deep conjecture by Vojta for algebraic points. The second result is an extension of a recent result of Matsuzawa, based on the theory of asymptotic multiplicity. Both results are generalisations of Hsia–Silverman, which treated the case of morphisms on $\mathbb{P}^1$.

2020 Mathematics Subject Classification: 37P55 (Primary); 11J87, 11J97, 37P15, 14G40 (Secondary)

1. *Introduction*

A celebrated theorem of Silverman [24] states that there are only finitely many integral points in an orbit under a rational function defined over a number field, as long as the second iterate of the function is not a polynomial. Hsia–Silverman [9] then made this theorem explicit and uniform: the upper bound for the number of integral points is given by an explicit formula in terms of the height of the initial point, and the largest iterate which can be integral is given uniformly independent of the initial point. The results of Hsia–Silverman have already been generalised in many directions. Just to name a few, Mello [21] has considered integral points in orbits under several (not necessarily commuting) rational functions and obtained similar upper bounds, and Hindes [7] has proved that the number of integral points in an orbit is uniformly bounded in a suitable one-parameter family of initial points and rational functions and that the average over such a family is in fact zero.

In this paper, we consider another generalisation of Hsia–Silverman, namely to morphisms on higher-dimensional projective spaces. The author [29] has considered some cases

†This work was supported in part by JSPS Kekenhi Grants 19K03412 and 24K06696

of higher-dimensional analogs of Silverman's result [24], so in some sense this paper is the combination of the two generalisations of [24]: namely, Hsia–Silverman [9] and the author [29]. Higher-dimensional Diophantine geometry is in general quite difficult; the Bombieri–Lang conjecture, which states the sparsity of rational points on varieties of general type, is already open in dimension 2. There has been plethora of recent research surrounding uniformity of rational points on abelian varieties (e.g. [3, 5]), but still, working with higher-dimensional varieties often forces us to either restrict to certain cases or assume various deep conjectures.

We prove two results in this paper on uniformity of quasi-integral points in higher-dimensional orbits. The results are uniform in two senses: independent of the initial point of the orbit, and independent of a specific number field as long as the degree is bounded. The first result applies when the pullback of a divisor by some iterate of ϕ breaks up into enough irreducible components which are numerical multiples of each other. For this result, we use Le's algebraic-point version [13] of the recent result of Ji–Yan–Yu [10] which is a type of Schmidt subspace theorem for divisors in subgeneral position. The second result applies more generally, but it assumes the algebraic-point version of Vojta's conjecture. This conjecture, in the special form of the *abc* conjecture and the *abcd* conjecture, was used by Looper [15, 16] to show uniform boundedness of preperiodic points for polynomials. The second result is more dynamical in nature, as it uses arithmetic degrees introduced by Kawaguchi–Silverman as well as the theory of asymptotic multiplicities of forward orbits, which were used in a recent result of Matsuzawa [20].

We now state our main results precisely. For the first result, we follow [13] to define $C(m, M, \delta)$ as

$$C(m,M,\delta)=\begin{cases}(\delta m-\delta M+1)(\delta M+1) & M\le \frac{m}{2}\\ (\frac{\delta m}{2}+1)^2 & M>\frac{m}{2} \text{ and } \delta m \text{ is even}\\ (\lfloor\frac{\delta m}{2}\rfloor+1)(\lfloor\frac{\delta m}{2}\rfloor+2) & M>\frac{m}{2} \text{ and } \delta m \text{ is odd}\end{cases} \tag{1·1}$$

for natural numbers m, M, δ, where $\lfloor\ \rfloor$ is the floor function. Throughout the paper, we use the notation of $\phi^{\circ t}$ for the tth iterate of ϕ, and the degree of ϕ is the polarisation degree (not the topological degree which counts the number of preimages). The first result is as follows.

THEOREM 1. *Let m, M, δ be natural numbers, d be a natural number at least 2, and let ϵ be a positive real number at most 1. Let k be a number field, and suppose that:*

(i) $\phi : \mathbb{P}^M \longrightarrow \mathbb{P}^M$ is a morphism of degree d defined over k;

(ii) D is a (possibly reducible) hypersurface in $\mathbb{P}^M$;

(iii) there exists a natural number t such that

$$\left(\phi^{\circ t}\right)^* D = D_1 + \cdots + D_q + D',$$

where $D_1, \ldots, D_q$ are in m-subgeneral position, D' is an effective divisor, and

$$\frac{\epsilon d^t(\deg D) - (\deg D')}{\max_i \deg D_i} > C(m, M, \delta);$$

(iv) S is a finite set of places of k including all archimedean ones, and $\mathcal{R}_\epsilon$ is a set of (D, S, ϵ)-quasi-integral points.

Then

$$\{\phi^{\circ n}(P) \in \mathcal{R}_\epsilon \backslash |D| : n \in \mathbb{N}^{\geq t}, [K:k] \leq \delta, P \in \mathbb{P}^M(K)\} \tag{1·2}$$

and

$$\{P \in \mathbb{P}^M(K) : [K:k] \leq \delta, \phi^{\circ n}(P) \in \mathcal{R}_\epsilon \backslash |D| \text{ for some } n \in \mathbb{N}^{\geq t}\} \tag{1·3}$$

are both finite sets, and moreover, there exists $N = N(\phi, S, \epsilon, \delta, m)$ such that whenever $\phi^{\circ n}(P) \in \mathcal{R}_\epsilon \backslash |D|$ for $P \in \mathbb{P}^M(K)$ with $[K:k] \leq \delta$ and $n \in \mathbb{N}$, n is at most N.

The definitions of m-subgeneral positions and quasi-integral points will be given in the next section. We only note here that the $\epsilon = 1$ case of (D, S, ϵ)-quasi-integral points corresponds to the usual notion of (D, S)-integral points.

By conjugating with a linear polynomial with a highly divisible leading coefficient, one can force the first N iterates to be integral (cf. [**25**, proposition 3·46]). Therefore, N has to depend on ϕ (and thus also on d). N also has to depend on S, since enlarging S makes more iterates integral.

Theorem 1 for the case of $\mathbb{P}^1$ does not quite recover the result of Hsia–Silverman [**9**], as there are non-polynomial maps not satisfying hypothesis (iii). On the other hand, we can modify the proof in the case of $\mathbb{P}^1$ to obtain a stronger statement, from which the results of Hsia–Silverman easily follow. With this modification, Theorem 1 for the case of $\mathbb{P}^1$ easily implies that the 'average' number of integral points in orbits is zero as first proved by Gunther–Hindes [**6**], because finiteness of (1·3) implies that orbits of points with sufficiently large height contain *no* integral points. See Remark 6 for more details. Also, we discuss in Remark 7 the inter-relationship among the three conclusions of the theorem, and we discuss some generalisations to projective varieties in Remark 8.

Example 2. Let $\mathfrak{R}_S$ be the subset of $\overline{\mathbb{Q}}$ consisting of elements which are integral over the ring of S-integers of k. Let $L_1, \ldots, L_q \in k[X_0, \ldots, X_M]$ be linear forms in general position, $F_1, \ldots, F_M \in k[X_0, \ldots, X_M]$ be homogeneous of degree d, and let $F_0 \in k[X_0, \ldots, X_M]$ be homogeneous of degree $d - q$. Let $\phi = [L_1 \cdots L_q \cdot F_0 : F_1 : \cdots : F_M]$, and for any $P \in \mathbb{P}^M(\overline{\mathbb{Q}})$, we write $\phi^{\circ n}(P) = [a_0{}^{\circ n} : \cdots : a_M{}^{\circ n}]$. Then Theorem 1 shows that as long as $q > C(M, M, \delta)$,

$$\left\{\phi^{\circ n}(P) : n \in \mathbb{N}, [K:k] \leq \delta, P \in \mathbb{P}^M(K), \frac{a_1{}^{\circ n}}{a_0{}^{\circ n}}, \ldots, \frac{a_M{}^{\circ n}}{a_0{}^{\circ n}} \in \mathfrak{R}_S\right\}$$

and

$$\left\{P \in \mathbb{P}^M(K) : [K:k] \leq \delta, \text{ there exists } n \in \mathbb{N} \text{ such that } \frac{a_1{}^{\circ n}}{a_0{}^{\circ n}}, \ldots, \frac{a_M{}^{\circ n}}{a_0{}^{\circ n}} \in \mathfrak{R}_S\right\}$$

are both finite sets, and there exists a uniform bound for n (independently of P) for which $a_1{}^{\circ n}/a_0{}^{\circ n}, \ldots, a_M{}^{\circ n}/a_0{}^{\circ n} \in \mathfrak{R}_S$. In particular, for quadratic points (the case of $\delta = 2$), we get these conclusions if $q > (M+1)^2$.

Next, we state our second result, this time with a less restrictive hypothesis but under assuming a deep conjecture from Diophantine geometry.

THEOREM 3. *Let $\phi : \mathbb{P}^M \longrightarrow \mathbb{P}^M$ be a surjective morphism of degree $d \geq 2$ defined over a number field k, and let S be a finite subset of M_k. Let D be a nontrivial effective divisor on $\mathbb{P}^M$ defined over k, ϵ be a positive real number at most 1, and let $\mathcal{R}_\epsilon$ be a set of (D, S, ϵ)-quasi-integral points. Assume that Vojta's General Conjecture (Conjecture 9) holds for a certain blowup of projective spaces determined by ϕ and D. Then there exist a Zariski-closed $Z = Z(D, S, \epsilon) \subsetneq \mathbb{P}^M$ and a constant N such that if P satisfies:*

(i) $P \in \mathbb{P}^M(K)$ with $[K : k] \leq \delta$;

(ii) P is not a preperiodic point under ϕ;

(iii) for every ϕ-periodic irreducible subvariety Y with $Y \cap D \neq \emptyset$, $e_{\phi,+}(Y) < \alpha_\phi(P)$;

(iv) $\phi^{\circ n}(P) \in \mathcal{R}_\epsilon$,

then either $\phi^{\circ n}(P) \in Z$ or n is bounded above by N.

Here, $\alpha_\phi(P)$ is the arithmetic degree and $e_{\phi,+}(Y)$ is the asymptotic multiplicity of the forward orbit of the scheme point Y: the precise definitions will be given in Section 3. For the case of $\mathbb{P}^1$, there exist no nontrivial blowups, and Vojta's General conjecture is known to be equivalent to the *abc* conjecture ([**27**]). Moreover, since Z is a finite set in this case, the proof in Section 3 will show the existence of a uniform bound N without the exceptional set Z. This way, we obtain the algebraic-point version of Hsia–Silverman [**9**, theorem 11]. Also, in Remark 11, we show an example which demonstrates that the hypothesis $e_{\phi,+}(Y) < \alpha_\phi(P)$ is necessary, and in Remark 12, we discuss a generalisation of this theorem to polarised morphisms on projective varieties. In the final remark, we consider whether the exceptional set Z is really necessary in Theorem 3.

2. *Notations, and the Proof of Theorem 1*

Let k be a number field, and let M_k be the set of places of k. For each $v \in M_k$, we define the normalised absolute value $|\cdot|_v$ as follows: for $k = \mathbb{Q}$, $|\cdot|_\infty$ is the usual absolute value on $\mathbb{R}$, and for each prime p, we define the normalised p-adic absolute value $|\cdot|_p$ by setting $|p|_p = 1/p$. For a general number field k, we define $|\cdot|_v$ to be the $[k_v : \mathbb{Q}_v]/[k : \mathbb{Q}]$-th power of the v-adic absolute value which restricts to one of the normalised absolute values on $\mathbb{Q}$. With this normalisation, the product formula

$$\prod_{v \in M_k} |x|_v = 1$$

holds for all $x \in k^*$.

The height of a point $P = [a_0 : \cdots : a_M] \in \mathbb{P}^M(k)$ is defined by

$$h(P) = \sum_{v \in M_k} \log \max_i |a_i|_v.$$

For an effective divisor D on $\mathbb{P}^M$ defined by a homogeneous polynomial $F \in k[x_0, \ldots, x_M]$, the *Weil height* is defined simply by

$$h(D, P) = (\deg F) h(P),$$

and the *local height* is defined by

$$\lambda_v(D,P) = \log \frac{|F|_v \cdot (\max_i |a_i|_v)^{\deg F}}{|F(a_0,\ldots,a_M)|_v}$$

for $v \in M_k$ and $P \in (\mathbb{P}^M \backslash D)(k)$, where $|F|_v$ is the maximum of the v-adic absolute values of the coefficients of F. The notions of Weil heights and local heights can be generalised to any projective variety X, and

$$h(D,-) - \sum_{v \in M_k} \lambda_v(D,-)$$

is a bounded function on $(X\backslash D)(\overline{\mathbb{Q}})$. For more details, see for example [**2, 8**].

We can now define the notion of quasi-integral points.

Definition 4. Let X be a projective variety defined over a number field k, and D be an effective divisor defined over k. Let S be a finite set of places of k, and let ϵ be a positive real number at most 1. A set of (D, S, ϵ)-quasi-integral points is a set of the form

$$\{Q \in X(\overline{\mathbb{Q}}) : \sum_{v \in S} \sum_{\substack{w \in M_{k(Q)} \\ w|v}} \lambda_w(D,Q) \geq \epsilon h(D,Q)\},$$

where $k(Q)$ is the field of definition of Q.

A set of $(D, S, 1)$-quasi-integral points corresponds to the notion of (D, S)-integral points. In particular, when $D = (X_0 = 0)$ is the hyperplane in $\mathbb{P}^M$,

$$\{[1 : a_1 : \cdots : a_M] : a_i \in \mathfrak{R}_S\}$$

is a set of (D, S)-integral points, where $\mathfrak{R}_S$ is the subset of $\overline{\mathbb{Q}}$ consisting of elements which are integral over the the ring of S-integers of k.

To prove Theorem 1, we use the following theorem of [**13**], which is an algebraic-point version of [**10**]. We recall that effective divisors $D_1,\ldots,D_q$ are in *m-subgeneral position* if for any subset $I \subseteq \{1,\ldots,q\}$ with $|I| \leq m+1$, we have

$$\dim \bigcap_{i \in I} |D_i| \leq m - |I|.$$

THEOREM 5 *([**13**], theorem 2]). Let m, M, δ be natural numbers, and let $C(m, M, \delta)$ as in (1·1). Let $D_1,\ldots,D_q$ be effective nontrivial Cartier divisors on $\mathbb{P}^M$ which are defined over a number field k and are in m-subgeneral position. Then given $\epsilon' > 0$ and a finite set S of places of k including all archimedean ones,*

$$\sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \sum_{i=1}^{q} \frac{\lambda_w(D_i,P)}{\deg D_i} < (C(m,M,\delta)+\epsilon')h(P)$$

holds for all but finitely many points $P \in \mathbb{P}^M(\bar{k}) \setminus \bigcup_{i=1}^{q} |D_i|$ satisfying $[k(P):k] \leq \delta$.

Proof of Theorem 1. By a standard Weil height property, there exists a constant C_1 such that

$$h(\phi(Q)) \geq dh(Q) - C_1$$

for all $Q \in \mathbb{P}^M(\overline{\mathbb{Q}})$. As $d > 1$, we can rearrange terms to obtain

$$h(\phi^{\circ t}(Q)) - \frac{C_1}{d-1} \geq d^t \left(h(Q) - \frac{C_1}{d-1}\right), \tag{2·1}$$

so by letting $C_2 = \dfrac{(d^t-1)C_1}{d-1}$, we have

$$h(\phi^{\circ t}(Q)) \geq d^t h(Q) - C_2 \tag{2·2}$$

for all $Q \in \mathbb{P}^M(\overline{\mathbb{Q}})$. By a standard property of local heights and by the fact that the Néron–Severi rank of $\mathbb{P}^M$ is one, it also follows that there exist constants C_3 and $C_3{}'$ such that

$$\sum_{v \in S} \sum_{\substack{w \in M_{k(Q)} \\ w|v}} \lambda_w(D', Q) \leq h(D', Q) + C_3{}' \leq (\deg D')h(Q) + C_3 \tag{2·3}$$

for all $Q \in (\mathbb{P}^M \backslash D')(\overline{\mathbb{Q}})$. There also exists a constant C_4 such that

$$\sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D, \phi^{\circ t}(Q)) \leq \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w((\phi^{\circ t})^* D, Q) + C_4 \tag{2·4}$$

for all $Q \in \mathbb{P}^M(\overline{\mathbb{Q}}) \setminus |(\phi^{\circ t})^{-1} D|$.

Now, suppose that $[K:k] \leq \delta$, $P \in \mathbb{P}^M(K)$, and $\phi^{\circ n}(P) \in \mathcal{R}_\epsilon \backslash \phi^{\circ t}(|D'| \cup \bigcup_{i=1}^q |D_i|)$ with $n \geq t$. By definition and by equations (2·3) and (2·4),

$$\begin{aligned}
\epsilon(\deg D)\cdot h(\phi^{\circ n}(P)) = \epsilon h(D, \phi^{\circ n}(P)) &\leq \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D, \phi^{\circ n}(P)) \\
&\leq \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w((\phi^{\circ t})^* D, \phi^{\circ n-t}(P)) + C_4 \\
&= \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q + D', \phi^{\circ n-t}(P)) + C_4. \\
&\leq \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P)) + (\deg D')h(\phi^{\circ n-t}(P)) + C_3 + C_4.
\end{aligned}$$

Therefore, combining with (2·2), we have

$$\begin{aligned}
\epsilon(\deg D)\left(d^t h(\phi^{\circ n-t}(P)) - C_2\right) \leq \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P)) \\
+ (\deg D')h(\phi^{\circ n-t}(P)) + C_3 + C_4,
\end{aligned} \tag{2·5}$$

hence by letting $C_5 = \epsilon(\deg D)C_2 + C_3 + C_4$, we have

$$(\epsilon(\deg D)d^t - (\deg D'))h(\phi^{\circ n-t}(P)) \le \sum_{v\in S} \sum_{\substack{w\in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P)) + C_5. \tag{2·6}$$

By assumption,

$$\epsilon'' := \frac{1}{2}\left(\frac{\epsilon(\deg D)d^t - (\deg D')}{\max_{i=1,\ldots q} \deg D_i} - C(m, M, \delta)\right)$$

is positive, and we have

$$\begin{aligned}
&\sum_{v\in S} \sum_{\substack{w\in M_{k(P)} \\ w|v}} \sum_{i=1}^{q} \frac{\lambda_w(D_i, \phi^{\circ n-t}(P))}{\deg D_i} \\
&\qquad \ge \sum_{v\in S} \sum_{\substack{w\in M_{k(P)} \\ w|v}} \frac{\lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P))}{\max_{i=1,\ldots q} \deg D_i} \\
&\qquad \ge \frac{\epsilon(\deg D)d^t - (\deg D')}{\max_{i=1,\ldots q} \deg D_i} h(\phi^{\circ n-t}(P)) - \frac{C_5}{\max_{i=1,\ldots q} \deg D_i} \qquad (\because (2.6)) \\
&\qquad > \left(C(m, M, \delta) + \epsilon''\right) h(\phi^{\circ n-t}(P)) - \frac{C_5}{\max_{i=1,\ldots q} \deg D_i}.
\end{aligned}$$

Therefore, applying Theorem 5 with $\epsilon' = \epsilon''/2$, $\phi^{\circ n-t}(P)$ lies either in the finite set of exceptions coming from Le's theorem or in the finite set of height bounded above by $2C_5/\left(\epsilon'' \max_{i=1,\ldots q} \deg D_i\right)$. By taking the image under $\phi^{\circ t}$, it follows that $\phi^{\circ n}(P)$ lies in a finite set $\mathcal{F} = \mathcal{F}(\phi, S, \epsilon, \delta, m)$. In particular, for a non-preperiodic P, the *number* of n's for which $\phi^{\circ n}(P) \in \mathcal{R}_\epsilon \backslash \phi^{\circ t}(|D'| \cup \bigcup_{i=1}^{q} |D_i|)$ is uniformly bounded above by $t + \#\mathcal{F}$.

By the Northcott property, there exists α such that any $Q \in \mathbb{P}^M(K)$ with $[K:k] \le \delta$ and $h(Q) > C_1/(d-1)$ satisfies $h(Q) \ge \alpha$. Therefore, by letting β be the maximum height of the finite set $\mathcal{F}$, it follows from (2·1) that any time $\phi^{\circ n}(P)$ is in $\mathcal{R}_\epsilon \backslash \phi^{\circ t}(|D'| \cup \bigcup_{i=1}^{q} |D_i|)$ with $h(P) > C_1/(d-1)$,

$$\beta \ge h(\phi^{\circ n}(P)) \ge d^n(h(P) - \frac{C_1}{d-1}) + \frac{C_1}{d-1} \ge d^n(\alpha - \frac{C_1}{d-1}) + \frac{C_1}{d-1} \tag{2·7}$$

is satisfied, so we obtain

$$n \le \log_d \left(\frac{\beta - \frac{C_1}{d-1}}{\alpha - \frac{C_1}{d-1}}\right).$$

On the other hand, if P is not preperiodic, then after at most

$$\#\{Q \in \mathbb{P}^M(K) : [K:k] \le \delta, h(Q) \le \frac{C_1}{d-1}\}$$

iterations, the point will have height at least $C_1/(d-1)$. So it follows that after at most

$$t + \log_d\left(\frac{\beta - \frac{C_1}{d-1}}{\alpha - \frac{C_1}{d-1}}\right) + \#\{Q \in \mathbb{P}^M(K) : [K:k] \leq \delta, h(Q) \leq \frac{C_1}{d-1}\}$$

iterations, no orbit point will be in $\mathcal{R}_\epsilon$. It is also immediate that whenever $\phi^{\circ n}(P)$ is in $\mathcal{R}_\epsilon \backslash \phi^{\circ t}(|D'| \cup \bigcup_{i=1}^q |D_i|)$ for some n, (2·7) implies that the height of P is bounded above by $\max(\beta, C_1/(d-1))$, so such a P comes from a finite set when $P \in \mathbb{P}^M(K)$ with $[K:k] \leq \delta$. This finishes all parts of the proof.

Remark 6. As noted in the introduction, Theorem 1 for the case of $\mathbb{P}^1$ does not recover Hsia–Silverman [9]. Indeed, for example for $\phi(x) = x/(x-1)^d$, $\phi^{-1}(\infty) = \{1\}$, $\phi^{-1}(0) = \{\infty, 0\}$, and the ramification points are $1, \infty, 1/(1-d)$. Since $(x-1)^d - x$ does not have a root in $\mathbb{Q}$, the preimages of 1 are not in $\mathbb{Q}$, so from here on the points in the preimage tree are not in $\mathbb{Q}$. So in particular, they are not ramified. Therefore, the number of preimage points of ∞ by $\phi^{\circ t}$ is d^{t-1}, so $\deg D'$ is $d^t - d^{t-1} = ((d-1)/d) \cdot d^t$. Hence, when $\epsilon \leq (d-1)/d$, one can never satisfy hypothesis (iii).

On the other hand, by modifying the proof argument slightly, we recover the result of Hsia–Silverman. By [25, lemma 3·52], for any small $\epsilon'' > 0$, there exists t such that the maximal multiplicity of points in $(\phi^{\circ t})^*(\infty)$ is at most $\epsilon'' d^t$. Then instead of using divisor D' and obtaining (2·5), letting $D_1 + \cdots + D_q$ be the reduced induced closed subscheme of $(\phi^{\circ t})^*(\infty)$ (i.e. just making the multiplicity of each preimage point to 1), we have

$$\epsilon\left(d^t h(\phi^{\circ n-t}(P)) - C_2\right) \leq \epsilon'' d^t \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P)) + C_3 + C_4.$$

Therefore,

$$\sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D_1 + \cdots + D_q, \phi^{\circ n-t}(P)) \geq \frac{\epsilon}{\epsilon''} h(\phi^{\circ n-t}(P)) - \frac{\epsilon C_2 + C_3 + C_4}{\epsilon'' d^t}$$

and thus by choosing ϵ'' so that $\epsilon/\epsilon'' > C(1, 1, \delta)$, the rest of the argument goes through and we obtain the result of Hsia–Silverman [9, theorem 11].

Remark 7. We observe that the three conclusions of the theorem are related, but none implies another directly. Indeed, the set of initial points (1·3) can be finite while larger and larger iterate could be in $\mathcal{R}_\epsilon$ to make (1·2) infinite and force nonexistence of a uniform bound N. Similarly, even if the set (1·2) of integral points in orbits is finite, this may occur with larger and larger iterate with (infinitely many) different initial points. Of course, as is evident from the proof, finiteness of either (1·2) or (1·3) will imply the finiteness of the other as well as the existence of the uniform bound N, by the Northcott property and by the fact that every morphism on projective space is polarised.

Remark 8. Since Le's theorem holds for projective varieties as long as there exists an ample divisor A such that each D_i is numerically equivalent to a multiple of A, Theorem 1 also holds under this setting as long as the divisor D' is also numerically equivalent to a multiple of A (we define the 'degree' by the multiple of A to which each D_i is numerically equivalent). In particular, the theorem holds for projective varieties with Néron–Severi rank one.

3. *Proof of Theorem 3*

In this section, we apply results of Matsuzawa [20] to obtain a similar result as Theorem 1, but with different hypotheses. Since Matsuzawa's result assumes Vojta's conjecture, the result of this section will also be conditional on this conjecture. Matsuzawa used the so-called "Main Conjecture" of Vojta [27, conjecture 3·4·3], which works over a fixed number field. In contrast, since we would like to obtain an algebraic-point version, here we assume the so-called "General Conjecture" of Vojta [27, conjecture 5·2·6], which applies for algebraic points but with a discriminant term on the right-hand side. In the following, we define the *absolute logarithmic discriminant* by

$$\mathrm{disc}(P) = \frac{1}{[k(P):\mathbb{Q}]} \log |D_{k(P)/\mathbb{Q}}|,$$

where $D_{L/\mathbb{Q}}$ is the (usual) discriminant of the number field L.

CONJECTURE 9 (Vojta's General Conjecture). *Let X be a smooth projective variety defined over a number field k, with K_X as a canonical divisor. Let D be a normal-crossings divisor on X and A be an ample divisor both defined over k, and let S be a finite subset of M_k. Let δ be a natural number. Then given $\epsilon > 0$, there exist a Zariski-closed $Z = Z(\epsilon) \subsetneq X$ and a constant $c_{X,k}$ such that*

$$\sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w|v}} \lambda_w(D, P) + h(K_X, P) < \epsilon h(A, P) + c_{X,k}\mathrm{disc}(P)$$

for $P \in X(K) \setminus Z$ with $[K:k] \leq \delta$.

In [27, conjecture 5·2·6], Vojta originally set $c_{X,k}$ to be $\dim X$. Later in [28, conjecture 2·1], Vojta set $c_{X,k}$ to be simply 1. On the other hand, Masser [18] has shown a counterexample for $c_{X,k} = 1$, and Levin [14] has constructed a surface for which $c_{X,k}$ needs to be at least $3/2$. In [14], Levin also discusses why the same statement for all algebraic points at once does not hold, i.e. why cannot let $\delta = \infty$. Perhaps, $c_{X,k} = \dim X$ is suitable, but the exact constant seems not quite settled, and it will be sufficient for our purposes as long as $c_{X,k}$ does not depend on P (it could even depend on ϵ, S and D), thus we state this conjecture in this fashion with $c_{X,k}$ in this paper. We note that this conjecture is extremely deep: for example the conjecture for Fermat curves easily implies the *abc* conjecture.

The *arithmetic degree* $\alpha_\phi(P)$, introduced by Kawaguchi–Silverman [12], is

$$\alpha_\phi(P) = \lim_{n\to\infty} \max\left(h(\phi^{\circ n}(P)),\ 1\right)^{1/n},$$

when the limit exists.

Finally, we also need the notion of 'asymptotic' multiplicity of the forward orbit of a scheme point.

Definition 10. Let $\phi : X \longrightarrow X$ be a finite flat self-morphism of a separated scheme of finite type over a number field. For a scheme point x, we define the multiplicity of ϕ at x as

$$e_\phi(x) = l_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/\phi^*\mathfrak{m}_{\phi(x)}\mathcal{O}_{X,x}),$$

 https://doi.org/10.1017/S0305004124000392

where $\mathfrak{m}_{\phi(x)}$ is the maximal ideal of $\mathcal{O}_{X,\phi(x)}$ and $l_{\mathcal{O}_{X,x}}$ stands for the length as an $\mathcal{O}_{X,x}$-module. We then define the asymptotic multiplicity as

$$e_{\phi,+}(x) = \lim_{n\to\infty} e_{\phi^{\circ n}}(x)^{1/n}.$$

Favre [4, theorem 2·5·8] has shown that this limit exists. When Y is an irreducible subvariety with the generic point η, we also write $e_{\phi,+}(Y)$ for $e_{\phi,+}(\eta)$.

By Matsuzawa [20, theorem 4·8], the function

$$x \mapsto \max\{e_{\phi,+}(\eta) : \eta \text{ is a } \phi\text{-periodic scheme point with } x \in \overline{\{\eta\}}\}$$

is well-defined and upper-semicontinuous. In particular,

$$\max_{x\in D} \max\{e_{\phi,+}(\eta) : \eta \text{ is a } \phi\text{-periodic scheme point with } x \in \overline{\{\eta\}}\} \tag{3·1}$$

exists, and hypothesis (iii) of Theorem 3 is satisfied if and only if $\alpha_\phi(P)$ is bigger than (3·1).

We are now ready to prove Theorem 3.

Proof of Theorem 3. The proof is based on the proofs of Lemma 7.17 and Theorem 1.16 of Matsuzawa [20], but one needs to take care to make N independent of P and its field of definition. First, letting H be an ample divisor on $\mathbb{P}^M$ (e.g. a hyperplane), there exists a constant C_6 such that

$$h(H,-) - h(K_{\mathbb{P}^M},-) \leq C_6 h(H,-). \tag{3·2}$$

This is an immediate consequence of the ampleness of H, but in this setting of $\mathbb{P}^M$, we can let C_6 be $M+2$.

Next, let e be the quantity defined by (3·1). For morphisms on normal projective varieties, the set of all arithmetic degrees is finite by Kawaguchi–Silverman [11] (in fact, for polarised morphisms as in this case, it is clear from their proof that the arithmetic degree can only be 1 or $\deg\phi$). In particular,

$$\{\alpha_\phi(Q) : Q \in \mathbb{P}^M(K), [K:k] \leq \delta, \alpha_\phi(Q) > e\}$$

is a finite set, and we let the minimum of this set be r (if this set is empty, then the statement of Theorem 3 will also be vacuous). Letting $\epsilon_1 = (r-e)/2 > 0$, we have

$$e + \epsilon_1 < r,$$

so for sufficiently large t,

$$\left(\frac{e+\epsilon_1}{r}\right)^t < \frac{\epsilon\cdot(\deg D)}{2C_6}. \tag{3·3}$$

Moreover, from [20, corollary 7·13], for all sufficiently large t, the log canonical threshold of $(\phi^{\circ t})^{-1}(D)$ is at least $(1/(e+\epsilon_1))^t$. So from here on, we fix a large enough t satisfying this as well as (3·3). Applying Conjecture 9 (instead of the conjecture for $\delta = 1$ without the discriminant term) to the log resolution X of $(\mathbb{P}^M, (\phi^{\circ t})^{-1}(D))$, the proof of [20, proposition 7·15] shows that there exist a Zariski-closed $Z_0 \subsetneq \mathbb{P}^M$ and a constant C_7 (depending on t, but recall that we have already fixed a large enough t) such that

$$\begin{aligned}\sum_{v\in S}\sum_{\substack{w\in M_{k(Q)}\\ w|v}}\lambda_w(D,\phi^{\circ t}(Q)) &\le \sum_{v\in S}\sum_{\substack{w\in M_{k(Q)}\\ w|v}}\lambda_w((\phi^{\circ t})^{-1}D,Q)+C_7\\ &\le (e+\epsilon_1)^t(h(H,Q)-h(K_{\mathbb{P}^M},Q)+c_{X,k}\mathrm{disc}(Q))+C_7\\ &\le (e+\epsilon_1)^tC_6h(H,Q)+(e+\epsilon_1)^tc_{X,k}\mathrm{disc}(Q)+C_7 \qquad (\because (3.2))\end{aligned} \tag{3·4}$$

for all Q of degree bounded by δ lying outside of Z_0.

From Mahler [17], there exists a constant C_8 so that

$$(2\delta-2)h(Q)\ge \mathrm{disc}(Q)-C_8. \tag{3·5}$$

More explicitly Silverman [23, theorem 2] has shown that we can take $C_8=\log\delta$. Since ϕ is a morphism of degree d, we also have

$$h(\phi(Q))\ge dh(Q)-C_9 \tag{3·6}$$

for some constant $C_9\ge 0$. By setting $C_{10}=C_9/(d-1)$, we have

$$h(\phi(Q))-C_{10}\ge d(h(Q)-C_{10}),$$

so

$$h(\phi^{\circ m}(Q))>d^m\cdot(h(Q)-C_{10})+C_{10}\ge d^m\cdot(h(Q)-C_{10}) \tag{3·7}$$

for any m.

Now, let P and n satisfy the hypotheses of the theorem. By hypothesis (iii) and the fact that e is equal to (3·1), we have $e<\alpha_\phi(P)$. By our definition of r and (3·3),

$$\left(\frac{e+\epsilon_1}{\alpha_\phi(P)}\right)^t<\frac{\epsilon\cdot(\deg D)}{2C_6} \tag{3·8}$$

holds. Moreover,

$$\alpha_\phi(P)\le d.$$

This is proved by Silverman [26, proposition 12] for rational maps on projective spaces, and proved for general projective varieties by Matsuzawa [19, theorem 1·4]. In all, we have

$$e+\epsilon_1<r\le\alpha_\phi(P)\le d. \tag{3·9}$$

Therefore, we have

$$\begin{aligned}(e+\epsilon_1)^th(H,\phi^{\circ n-t}(P)) &\le \frac{\epsilon\cdot(\deg D)}{2C_6}\alpha_\phi(P)^th(H,\phi^{\circ n-t}(P)) \qquad (\because (3.8))\\ &\le \frac{\epsilon\cdot(\deg D)}{2C_6}d^th(H,\phi^{\circ n-t}(P))\\ &\le \frac{\epsilon\cdot(\deg D)}{2C_6}\left(h(H,\phi^{\circ n}(P))+d^tC_{10}\right) \qquad (\because (3.7))\end{aligned} \tag{3·10}$$

We now combine all of these together. We divide the argument into two cases, depending on whether disc(P) is bigger than the following quantity or not:

$$\max\left(\frac{1}{[k:\mathbb{Q}]}\log|\mathrm{D}_{k/\mathbb{Q}}|,\ \ \delta C_8+\delta(2\delta-2)C_{10},\ \ 1\right). \tag{3·11}$$

Case I: disc$(P) >$ (3·11)

In this case, $k(P)\neq k$, so in particular, $\delta>1$. The assumption implies

$$\frac{1}{2\delta-2}\mathrm{disc}(P)-\frac{C_8}{2\delta-2}>\frac{1}{2\delta}\mathrm{disc}(P)+C_{10},$$

thus by Mahler's inequality (3·5),

$$h(P)>\frac{1}{2\delta}\mathrm{disc}(P)+C_{10}.$$

Hence, by (3·7), we have

$$h(\phi^{\circ m}(P))>d^m\frac{1}{2\delta}\mathrm{disc}(P). \tag{3·12}$$

We now compute from (3·4):

$$\begin{aligned}
\epsilon &\leq \frac{\displaystyle\sum_{v\in S}\sum_{\substack{w\in M_{k(P)}\\ w|v}}\lambda_w(D,\phi^{\circ n}(P))}{h(D,\phi^{\circ n}(P))}\\
&=\frac{\displaystyle\sum_{v\in S}\sum_{\substack{w\in M_{k(P)}\\ w|v}}\lambda_w(D,\phi^{\circ n}(P))}{(\deg D)h(H,\phi^{\circ n}(P))} \qquad\qquad (3·13)\\
&\leq\frac{(e+\epsilon_1)^tC_6h(H,\phi^{\circ n-t}(P))+(e+\epsilon_1)^tc_{X,k}\mathrm{disc}(\phi^{\circ n-t}(P))+C_7}{(\deg D)h(H,\phi^{\circ n}(P))}\\
&\leq\frac{\epsilon}{2}\frac{h(H,\phi^{\circ n}(P))+d^tC_{10}}{h(H,\phi^{\circ n}(P))}+\frac{(e+\epsilon_1)^tc_{X,k}\mathrm{disc}(\phi^{\circ n-t}(P))+C_7}{(\deg D)h(H,\phi^{\circ n}(P))} \qquad (\because (3.10))\\
&\leq\frac{\epsilon}{2}+\frac{(e+\epsilon_1)^tc_{X,k}\mathrm{disc}(\phi^{\circ n-t}(P))+C_{11}}{(\deg D)\cdot d^n\frac{1}{2\delta}\mathrm{disc}(P)} \qquad (C_{11}:=d^t(\deg D)C_{10}+C_7,\ \ (3.12))\\
&\leq\frac{\epsilon}{2}+\frac{2\delta\cdot c_{X,k}}{(\deg D)\cdot d^{n-t}}+\frac{2\delta C_{11}}{(\deg D)\cdot d^n} \qquad (\because (3.9))
\end{aligned}$$

as long as $\phi^{\circ n-t}(P)\notin Z_0$, i.e. as long as $\phi^{\circ n}(P)\notin\phi^{\circ t}(Z_0)$. Note that here we use the fact that the discriminant remains the same in an orbit, as ϕ itself is defined over k. For sufficiently large n, which can be evidently chosen independently of P, the sum of the last two terms is less than $\epsilon/2$. This contradicts the fact that $\phi^{\circ n}(P)$ was (D,S,ϵ)-quasi-integral. Therefore, quasi-integral points must appear in earlier iterations.

Case II: $\mathrm{disc}(P) \leq$ (3·11)

We compute in a similar fashion as above:

$$\begin{aligned}
\epsilon &\leq \frac{\displaystyle\sum_{v\in S}\sum_{\substack{w\in M_{k(P)}\\ w|v}} \lambda_w(D, \phi^{\circ n}(P))}{h(D, \phi^{\circ n}(P))} \\
&\leq \frac{(e+\epsilon_1)^t C_6 h(H, \phi^{\circ n-t}(P)) + (e+\epsilon_1)^t c_{X,k}\mathrm{disc}(\phi^{\circ n-t}(P)) + C_7}{(\deg D)h(H, \phi^{\circ n}(P))} \\
&\leq \frac{\epsilon}{2} + \frac{(e+\epsilon_1)^t c_{X,k}\mathrm{disc}(\phi^{\circ n-t}(P)) + C_{11}}{(\deg D)h(H, \phi^{\circ n}(P))} \\
&\leq \frac{\epsilon}{2} + \frac{(e+\epsilon_1)^t c_{X,k}\max\left(\frac{1}{[k:\mathbb{Q}]}\log|\mathrm{D}_{k/\mathbb{Q}}|, \delta C_8 + \delta(2\delta-2)C_{10}, 1\right) + C_{11}}{(\deg D)h(H, \phi^{\circ n}(P))} \\
&= \frac{\epsilon}{2} + \frac{C_{12}}{h(H, \phi^{\circ n}(P))},
\end{aligned}$$

as long as $\phi^{\circ n}(P) \notin \phi^{\circ t}(Z_0)$, where C_{12} is defined to be $1/\deg D$ times the numerator of the previous equation (which is a constant since t is fixed). In particular,

$$h(\phi^{\circ n}(P)) \leq \frac{2C_{12}}{\epsilon}. \tag{3·14}$$

By the Northcott property, there exists $C_{13} > C_{10}$ such that any point $Q \in \mathbb{P}^M(K)$ with $[K:k] \leq \delta$ and $h(Q) > C_{10}$ satisfies $h(Q) \geq C_{13}$. Therefore, if $h(P) > C_{10}$, (3·7) implies $h(\phi^{\circ n}(P)) > d^n(h(P) - C_{10}) \geq d^n(C_{13} - C_{10})$. This together with (3·14) implies

$$n \leq \log_d \frac{2C_{12}/\epsilon}{C_{13} - C_{10}}.$$

On the other hand, if P is not preperiodic and $h(P) \leq C_{10}$, after at most

$$\#\{Q \in \mathbb{P}^M(K) : [K:k] \leq \delta, h(Q) \leq C_{10}\}$$

iterations, all the remaining iterations have height above C_{10}, so we now obtain a uniform bound for n.

We end this paper with several remarks.

Remark 11. The condition that $e_{\phi,+}(Y) < \alpha_\phi(P)$ is necessary. For example, if $\phi = [Z_0^d : F_1 : \cdots : F_M]$ is a morphism with F_i having coefficients in the ring of integers, then the entire orbit of $P = [1 : a_1 : \cdots : a_M]$ lies in the set of integral points (i.e. in $\mathcal{R}_1$) with respect to the hyperplane $D = (Z_0 = 0)$ if each a_i is integral. In this case, D is ϕ-fixed and $e_{\phi,+}(D) = d = \alpha_\phi(P)$. For general choices of F_i and P, the orbit is conjectured to be also Zariski-dense; for example, [**1**, corollary 2·7] shows the existence of such a P whenever there is a fixed point of ϕ such that the eigenvalues of the tangent map at that point are multiplicatively independent.

Remark 12. We can generalise ϕ to any polarised morphism on a projective variety, as long as D is numerically equivalent to a multiple of H (for example, if the Néron–Severi rank is one). Indeed, (3·6) holds for polarised endomorphisms, the results of Matsuzawa hold for morphisms on projective varieties, and Mahler's inequality (3·5) holds by embedding the variety as a closed subvariety of a projective space. Moreover, if D is numerically equivalent to a positive multiple of H, there exists $d' > 0$ such that $h(D, Q) \geq d' h(H, Q)$ for all Q with sufficiently large height, so by replacing $\deg D$ by d' in (3·3) and (3·13) and in equations thereafter, the argument goes through.

Remark 13. From Sano [22, theorem 1·1], we know

$$C_{14} n^l \alpha_\phi(P)^n < h(\phi^{\circ n}(P)) < C_{15} n^l \alpha_\phi(P)^n$$

for some l, but as C_{14} and C_{15} depend on the canonical height of P with respect to a basis of Néron–Severi group, it seems difficult to use these bounds for our purposes of showing uniformity. Here, we use a more elementary height inequality involving $h(\phi^{\circ t}(P))$ instead.

Remark 14. It is natural to ask whether the exceptional set Z can be removed from the statement of the theorem. That is, does there exist a subvariety $Z \subsetneq \mathbb{P}^M$ such that $Z \cap \mathcal{R}_\epsilon$ contains $\phi^{\circ n}(P)$ for arbitrarily large n with $P \in \mathbb{P}^M(K)$, $[K : k] \leq \delta$, P non-ϕ-preperiodic, and every ϕ-periodic subvariety Y with $Y \cap D \neq \emptyset$ satisfying $e_{\phi,+}(Y) < \alpha_\phi(P)$?

If we ask Z to satisfy the stronger condition that it contains infinitely many points of the orbit of a single point, then the dynamical Mordell–Lang conjecture would imply that there is a periodic subvariety containing infinitely many integral points. Replacing by an iterate, we may assume that we have a fixed subvariety. If ϕ is a map on $\mathbb{P}^2$, then such a fixed curve must be rational as it contains infinitely many integral points, and by Silverman's theorem, the second iterate of the map restricted to this rational curve must be a polynomial. Then the intersection point of this curve with $|D|$ must be a fixed point whose multiplicity is equal to (so not strictly less than) the degree. Since the arithmetic degree is equal to the degree, this is a contradiction to the assumption. This heuristic argument seems to indicate that if such a Z exists for a morphism on $\mathbb{P}^2$, it must contain integral points which are arbitrarily high iterates coming from different initial points.

REFERENCES

[1] E. AMERIK, F. BOGOMOLOV and M. ROVINSKY. Remarks on endomorphisms and rational points. *Compositio. Math.* **147**(6) (2011), 1819–1842.

[2] E. BOMBIERI and W. GUBLER. *Heights in Diophantine Geometry*, *New Mathematical Monographs*, vol. **4** (Cambridge University Press, Cambridge, 2006).

[3] V. DIMITROV, Z. GAO and P. HABEGGER. Uniformity in Mordell-Lang for curves, *Ann. of Math.* (2) **194**(1) (2021), 237–298.

[4] C. FAVRE. Dynamique des applications rationnelles, Ph.D. Thesis. Université Paris-Sud (2000).

[5] Z. GAO, T. GE and L. KÜHNE. The uniform Mordell-Lang conjecture, ArXiv 2105.15085 (2023).

[6] J. GUNTHER and W. HINDES. Integral points of bounded degree on the projective line and in dynamical orbits. *Proc. Amer. Math. Soc.* **145**(12) (2017), 5087–5096.

[7] W. HINDES. The average number of integral points in orbits. *Math. Res. Lett.* **26**(1) (2019), 101–120.

[8] M. HINDRY and J. H. SILVERMAN. *Diophantine Geometry*. Graduate Texts in Math. vol. 201 (Springer-Verlag, New York, 2000), an introduction.

[9] L.-C. HSIA and J. H. SILVERMAN. A quantitative estimate for quasiintegral points in orbits. *Pacific J. Math.* **249**(2) (2011), 321–342.

[10] Q. JI, Q. YAN and G. YU. Holomorphic curves into algebraic varieties intersecting divisors in subgeneral position. *Math. Ann.* **373**(3-4) (2019), 1457–1483.

[11] S. KAWAGUCHI and J. H. SILVERMAN. Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties. *Trans. Amer. Math. Soc.* **368**(7) (2016), 5009–5035.

[12] S. KAWAGUCHI and J. H. SILVERMAN. On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties. *J. Reine Angew. Math.* **713**(2016), 21–48.

[13] G. LE. A Wirsing-type theorem for numerically equivalent divisors. *Kodai Math. J.* **46**(2) (2023), 228–241.

[14] A. LEVIN. The exceptional set in Vojta's conjecture for algebraic points of bounded degree. *Proc. Amer. Math. Soc.* **140**(7) (2012), 2267–2277.

[15] N. LOOPER. The uniform boundedness and dynamical lang conjectures for polynomials, ArXiv 2105.05240 (2001).

[16] N. R. LOOPER. Dynamical uniform boundedness and the abc-conjecture. *Invent. Math.* **225**(1) (2021) 1–44.

[17] K. MAHLER. An inequality for the discriminant of a polynomial. *Michigan Math. J.* **11**(1964), 257–262.

[18] D. W. MASSER. On abc and discriminants. *Proc. Amer. Math. Soc.* **130**(11) (2002), 3141–3150.

[19] Y. MATSUZAWA. On upper bounds of arithmetic degrees. *Amer. J. Math.* **142**(6) (2020), 1797–1820.

[20] Y. MATSUZAWA. Growth of local height functions along orbits of self-morphisms on projective varieties. *Int. Math. Res. Not.* (2023), no. 4, 3533—3575.

[21] J. MELLO. On quantitative estimates for quasiintegral points in orbits of semigroups of rational maps. *New York J. Math.* **25**(2019), 1091–1111.

[22] K. SANO. Growth rate of ample heights and the dynamical Mordell-Lang conjecture. *Int. J. Number Theory* **14**(10) (2018), 2673–2685.

[23] J. H. SILVERMAN. Lower bounds for height functions. *Duke Math. J.* **51**(2) (1984), 395–403.

[24] J. H. SILVERMAN. Integer points, Diophantine approximation, and iteration of rational maps. *Duke Math. J.* **71**(3) (1993), 793–829.

[25] J. H. SILVERMAN. *The Arithmetic of Dynamical Systems*. Graduate Texts in Math. vol. **241** (Springer, New York, 2007).

[26] J. H. SILVERMAN. Dynamical degree, arithmetic entropy, and canonical heights for dominant rational self-maps of projective space. *Ergodic Theory Dynam. Systems* **34**(2) (2014), 647–678.

[27] P. VOJTA. *Diophantine Approximations and Value Distribution Theory*. Lecture Notes in Math. vol. 1239, (Springer-Verlag, Berlin, 1987).

[28] P. VOJTA. A more general abc conjecture. *Int. Math. Res. Not.* (1998), no. 21, 1103–1116.

[29] Y. YASUFUKU. Integral points and relative sizes of coordinates of orbits in $\mathbb{P}^N$. *Math. Z.* **279**(3-4) (2015), 1121–1141.