Bull. Aust. Math. Soc. **111** (2025), 48–55
doi:10.1017/S0004972724000534

ON A CONJECTURE ON SHIFTED PRIMES WITH LARGE PRIME FACTORS, II

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(Received 28 March 2024; accepted 20 May 2024; first published online 13 September 2024)

Abstract

Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding x. Let $P^+(n)$ be the largest prime factor of n, with the convention $P^+(1) = 1$, and $T_c(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p-1) \ge p^c\}$. Motivated by a conjecture of Chen and Chen [‘On the largest prime factor of shifted primes’, *Acta Math. Sin. (Engl. Ser.)* **33** (2017), 377–382], we show that for any c with $8/9 \le c < 1$,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \le 8(1/c - 1),$$

which clearly means that

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \to 0 \quad \text{as } c \to 1.$$

2020 *Mathematics subject classification*: primary 11N05; secondary 11N36.

Keywords and phrases: shifted primes, sieve method, Brun–Titchmarsh inequality, primes in arithmetic progressions, Elliott–Halberstam conjecture.

1. Introduction

The investigation of shifted primes with large prime factors began in a brilliant article of Goldfeld [12]. Historically, this topic had aroused great interest because of its unexpected connection with the first case of Fermat’s last theorem, thanks to the theorems of Fouvry [11] and Adleman and Heath-Brown [1].

For any positive integer n, let $P^+(n)$ be the largest prime factor of n with the convention $P^+(1) = 1$. Let $\mathcal{P}$ be the set of primes and $\pi(x)$ the number of primes not exceeding x. For $0 < c < 1$, let $T_c(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p-1) \ge p^c\}$. As early as 1969, Goldfeld [12] proved

$$\liminf_{x\to\infty} T_{1/2}(x)/\pi(x) \ge 1/2.$$

The author is supported by National Natural Science Foundation of China (Grant No. 12201544), Natural Science Foundation of Jiangsu Province, China (Grant No. BK20210784) and China Postdoctoral Science Foundation (Grant No. 2022M710121).

Goldfeld further remarked that his argument also leads to

$$\liminf_{x\to\infty} T_c(x)/\pi(x) > 0, \tag{1.1}$$

provided that $c < 7/12$. It turns out that exploring large c which satisfy (1.1) is rather difficult and important. For improvements on the values of c, see Motohashi [21], Hooley [15, 16], Deshouillers and Iwaniec [7], and Fouvry [11]. Up to now, the best numerical value of c satisfying (1.1), with a cost of replacing $\pi(x)$ with $\pi(x)/\log x$, is 0.677, obtained by Baker and Harman [3].

In an earlier note [8] on this topic, I showed that

$$\limsup_{x\to\infty} T_c(x)/\pi(x) < 1/2 \tag{1.2}$$

holds for some absolute constant $c < 1$. As a corollary, I disproved a 2017 conjecture of Chen and Chen [6] that

$$\liminf_{x\to\infty} T_c(x)/\pi(x) \geq 1/2$$

for any c with $1/2 \leq c < 1$. The proof in my earlier note is based on the following deep result which is a corollary of the Brun–Titchmarsh inequality.

Proposition 1.1 [24, Lemma 2.2]. *There exist two functions $K_2(\theta) > K_1(\theta) > 0$, defined on the interval $(0, 17/32)$ such that for each fixed real $A > 0$ and all sufficiently large $Q = x^\theta$, the inequalities*

$$K_1(\theta)\frac{\pi(x)}{\varphi(m)} \leq \pi(x; m, 1) \leq K_2(\theta)\frac{\pi(x)}{\varphi(m)}$$

hold for all integers $m \in (Q, 2Q]$ with at most $O(Q(\log Q)^{-A})$ exceptions, where the implied constant depends only on A and θ. Moreover, for any fixed $\varepsilon > 0$, these functions can be chosen to satisfy the following properties:

- *$K_1(\theta)$ is monotonic decreasing and $K_2(\theta)$ is monotonic increasing;*
- *$K_1(1/2) = 1 - \varepsilon$ and $K_2(1/2) = 1 + \varepsilon$.*

The constant c in (1.2) is not specified because of the indeterminate nature of $K_1(\theta)$ in Proposition 1.1. In fact, $K_1(\theta)$ (and hence c) can be explicitly given if one checks carefully the articles of Baker and Harman [2] for $1/2 \leq \theta \leq 13/25$, and Mikawa [19] for $13/25 \leq \theta \leq 17/32$. This gives $K_1(\theta) \geq 0.16$ for $1/2 \leq \theta \leq 13/25$ [2, Theorem 1] and $K_1(\theta) \geq 1/100$ for Mikawa's range [19, (4)]. However, it seems that the constant c in (1.2) obtained in this way will be very close to 1 (see the proofs in [8]).

In [8], I also pointed out that Chen and Chen's conjecture is already in contradiction with the Elliott–Halberstam conjecture (from Pomerance [22], Granville [13], Wang [23] and Wu [24]). In fact,

$$\limsup_{x\to\infty} T_c(x)/\pi(x) = \lim_{x\to\infty} T_c(x)/\pi(x) = \left(1 - \rho\left(\frac{1}{c}\right)\right) \to 0 \quad \text{as } c \to 1, \tag{1.3}$$

https://doi.org/10.1017/S0004972724000534 Published online by Cambridge University Press

under the assumption of the Elliott–Halberstam conjecture, where $\rho(u)$ is the Dickman function, defined as the unique continuous solution of the differential-difference equation

$$\begin{cases} \rho(u) = 1 & \text{for } 0 \le u \le 1, \\ u\rho'(u) = -\rho(u-1) & \text{for } u > 1. \end{cases}$$

However, there are earlier results related to the conjecture of Chen and Chen, and my earlier result (1.2). In fact, as indicated by the proof of a result of Erdős [9, Lemma 4], as early as 1935, one could already conclude from Erdős' proof combined with Lemma 2.2 of Wu (see below) that (1.3) is true in part.

THEOREM 1.2 (Erdős). *Unconditionally,*

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \to 0 \quad \textit{as } c \to 1.$$

Essentially, Theorem 1.2 can be deduced from Erdős' proof by adding Wu's lemma (see Erdős' argument in [9, from page 212, line 6 to page 213, line 4]). Since Erdős' conclusion is not clearly formulated, it is meaningful to restate it explicitly as Theorem 1.2. It is also of interest to pursue Erdős' theorem a little further to reach the following quantitative form.

THEOREM 1.3. *For* $8/9 \le c < 1$*,*

$$\limsup_{x\to\infty} T_c(x)/\pi(x) \le 8(1/c - 1).$$

We note that the restriction on $c \ge 8/9$ in our theorem is natural since otherwise, the upper bound would exceed 1 which is certainly meaningless. Theorem 1.3 can also be compared with the results of Goldfeld [12], Luca *et al.* [18], and Chen and Chen [6] which state that

$$\liminf_{x\to\infty} T_c(x)/\pi(x) \ge 1 - c$$

for $0 < c \le 1/2$. These bounds were recently improved in part by Feng and Wu [10], and Liu, Wu and Xi [17]. From Theorem 1.3, we clearly have two corollaries, one of which is Erdős' theorem (Theorem 1.2) while the other revisits the main result (1.2) of my earlier note in a quantitative form.

COROLLARY 1.4. *For* $c > 16/17$*,*

$$\limsup_{x\to\infty} T_c(x)/\pi(x) < 1/2.$$

2. Proofs

From now on, p will always be a prime. The proof of Theorem 1.3 is based on the following lemma deduced from the sieve method (see, for example, [14, Theorem 5.7, page 172]).

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LEMMA 2.1. *Let g be a natural number and let a_i, b_i $(i = 1, 2, \ldots, g)$ be integers satisfying*

$$E := \prod_{i=1}^{g} a_i \prod_{1 \le r < s \le g} (a_r b_s - a_s b_r) \ne 0.$$

Let $\rho(p)$ denote the number of solutions in n modulo p of

$$\prod_{i=1}^{g} (a_i n + b_i) \equiv 0 \pmod{p},$$

and suppose that

$$\rho(p) < p \quad \textit{for all } p.$$

If the real numbers y and z satisfy $1 < y \le z$, then

$$|\{n : z - y < n \le z, a_i n + b_i \textit{ prime for } i = 1, 2, \ldots, g\}|$$
$$\le 2^g g! \prod_p \Big(1 - \frac{\rho(p) - 1}{p - 1}\Big)\Big(1 - \frac{1}{p}\Big)^{-g+1} \frac{y}{\log^g y}\Big(1 + O\Big(\frac{\log\log 3y + \log\log 3|E|}{\log y}\Big)\Big),$$

where the constant implied by the O-symbol depends at most on g.

We also need the following important relation established by Wu [24, Theorem 2].

LEMMA 2.2. *For $0 < c < 1$, let*

$$T'_c(x) = \#\{p \le x : p \in \mathcal{P}, P^+(p - 1) \ge x^c\}.$$

Then for sufficiently large x,

$$T_c(x) = T'_c(x) + O\Big(\frac{x \log\log x}{(\log x)^2}\Big).$$

We now turn to the proof of Theorem 1.3.

PROOF OF THEOREM 1.3. Let x be a sufficiently large number throughout the proof. Instead of investigating $T_c(x)$, we first deal with $T'_c(x)$. For $1/2 \le c < 1$, it is easy to see that

$$T'_c(x) = \sum_{\substack{x^c \le q < x \\ q \in \mathcal{P}}} \sum_{\substack{p \le x \\ q \mid p-1}} 1. \tag{2.1}$$

On putting $p - 1 = qh$ in the sum (2.1) and then exchanging the order of summation,

$$T'_c(x) = \sum_{\substack{x^c \le q < x \\ q \in \mathcal{P}}} \sum_{\substack{h < x/q \\ qh+1 \in \mathcal{P}}} 1 \le \sum_{\substack{h < x^{1-c} \\ 2 \mid h}} \sum_{\substack{2 < q < x/h \\ q, qh+1 \in \mathcal{P}}} 1. \tag{2.2}$$

For any h with $2 \mid h$ and $h < x^{1-c}$, let $\rho(p)$ denote the number of solutions of

$$n(hn + 1) \equiv 0 \pmod{p}.$$

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Then

$$\rho(p) = \begin{cases} 1 & \text{if } p \mid h, \\ 2 & \text{otherwise.} \end{cases}$$

Now, by Lemma 2.1 with $g = 2, a_1 = 1, b_1 = 0, a_2 = h, b_2 = 1$ and $z = y = x/h$,

$$3|E| = 3h \ll x, \quad 3y = 3x/h \ll x \quad \text{and} \quad y = x/h \geq \sqrt{x},$$

from which it follows that

$$\sum_{\substack{2<q<x/h \\ q,qh+1\in\mathcal{P}}} 1 \leq 16\mathfrak{S} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \frac{x/h}{\log^2(x/h)} \Big(1 + O\Big(\frac{\log\log x}{\log x}\Big)\Big), \tag{2.3}$$

where an empty product for $\prod_{p|h,p>2}$ above denotes 1 as usual and

$$\mathfrak{S} = \prod_{p>2} \Big(1 - \frac{1}{p-1}\Big)\Big(1 - \frac{1}{p}\Big)^{-1} = \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big).$$

Inserting (2.3) into (2.2) gives

$$T_c'(x) \leq (1 + o(1))16\mathfrak{S} \sum_{\substack{h<x^{1-c} \\ 2|h}} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \frac{x/h}{\log^2(x/h)}. \tag{2.4}$$

Note that

$$\prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \leq 2 \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p}\Big) \tag{2.5}$$

since the gaps between odd primes are at least 2, from which we can already give a nontrivial upper bound of $T_c'(x)$ via partial summations. To make our bound more explicit than (2.5), we employ a nice result of Banks and Shparlinski [4, Lemma 2.3] (on taking $a = 1$ therein), which states that for $z \geq 2$,

$$S(z) := \sum_{\substack{h<z \\ 2|h}} \frac{1}{h} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) = \frac{1 + o(1)}{2\mathfrak{S}} \log z. \tag{2.6}$$

For $1 \leq z < 2$, we set $S(z) = 0$. By partial summation,

$$\sum_{\substack{h<x^{1-c} \\ 2|h}} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p}\Big) \frac{1/h}{\log^2(x/h)} = \frac{S(x^{1-c})}{(\log x^c)^2} - \int_1^{x^{1-c}} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2}. \tag{2.7}$$

Note also that for $z \geq 2$,

$$S(z) \leq \sum_{h<z} \frac{1}{h} \prod_{\substack{p|h \\ p>2}} \Big(1 + \frac{1}{p-2}\Big) \leq \sum_{h<z} \frac{1}{h} \prod_{p|h} \Big(1 + \frac{3}{p}\Big) = \sum_{h<z} \frac{1}{h} \sum_{d|h} \frac{3^{\omega(d)}\mu^2(d)}{d},$$

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where $\mu(d)$ is the Möbius function and $\omega(d)$ is the number of distinct prime factors of d. Exchanging the order of summation,

$$S(z) \le \sum_{d<z} \frac{3^{\omega(d)}\mu^2(d)}{d} \sum_{\substack{h<z \\ d|h}} \frac{1}{h} \le 3 \sum_{d<z} \frac{3^{\omega(d)}\mu^2(d)}{d^2} \log z < 3K \log z, \tag{2.8}$$

where

$$K = 3 \sum_{d=1}^{\infty} \frac{3^{\omega(d)}\mu^2(d)}{d^2}.$$

From (2.8),

$$\int_1^{\log x} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2} \ll_K \frac{\log\log x}{(\log x)^2} = o((\log x)^{-1}). \tag{2.9}$$

Now, routine computations yield

$$\frac{S(x^{1-c})}{(\log x^c)^2} = \frac{1+o(1)}{\mathfrak{S}} \frac{(1-c)}{2c^2} (\log x)^{-1} \tag{2.10}$$

and

$$\begin{aligned}
\int_{\log x}^{x^{1-c}} S(z)\, d\Big(\log \frac{x}{z}\Big)^{-2} &= \frac{1+o(1)}{\mathfrak{S}} \int_{\log x}^{x^{1-c}} \frac{\log z}{z} \Big(\log \frac{x}{z}\Big)^{-3} dz \\
&= \frac{1+o(1)}{\mathfrak{S}} \int_{1}^{x^{1-c}} \frac{\log z}{z} \Big(\log \frac{x}{z}\Big)^{-3} dz + o((\log x)^{-1}) \\
&= \frac{1+o(1)}{\mathfrak{S}} \int_{x^c}^{x} \frac{\log x - \log u}{u} (\log u)^{-3}\, du + o((\log x)^{-1}) \\
&= \frac{1+o(1)}{\mathfrak{S}} \Big(\frac{1-c}{2c^2} + \frac{1}{2} - \frac{1}{2c}\Big)(\log x)^{-1},
\end{aligned} \tag{2.11}$$

thanks to the estimate (2.6). Combining (2.9), (2.10) and (2.11), one sees that the right-hand side of (2.7) equals

$$\frac{1+o(1)}{\mathfrak{S}} \Big(\frac{1}{2c} - \frac{1}{2}\Big)(\log x)^{-1}. \tag{2.12}$$

Taking (2.12) into (2.4), we immediately obtain

$$T_c'(x) \le (1+o(1))8\Big(\frac{1}{c} - 1\Big)\frac{x}{\log x}.$$

Therefore, by Lemma 2.2,

$$T_c(x) = T_c'(x) + O\Big(\frac{x \log\log x}{(\log x)^2}\Big) \le (1+o(1))8\Big(\frac{1}{c} - 1\Big)\frac{x}{\log x}.$$

Our theorem now follows from the prime number theorem. □

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3. Remarks

Under the assumption of the Elliott–Halberstam conjecture, it is reasonable to predict that the exact value of c in Corollary 1.4 should be $e^{-1/2} = 0.60653\ldots$ from (1.3) and the recursion formula (see, for example, [20, (7.6)]) for Dickman's function:

$$\rho(v) = u - \int_u^v \frac{\rho(t-1)}{t}\, dt \quad (1 \le u \le v).$$

It therefore seems to be of interest to improve, as far as possible, the numerical value of c in Corollary 1.4. We leave this as a challenge to readers.

Though we provided nontrivial upper bounds on $T_c(x)$ for $8/9 \le c < 1$ in Theorem 1.3, the extension of these bounds to $1/2 \le c < 1$ is an unsolved problem.

Acknowledgments

The author would like to thank the anonymous referees for their very helpful comments. He is really grateful to the one who pointed out the facts involving Erdős' article. Shortly after the manuscript of this note was put on the arXiv, Brad Rodgers informed the author that, joint with Bharadwaj, they independently obtained the same result of Theorem 1.2 with a general form in the probabilistic language (see [5, Theorem 8 and Corollary 9]). (At that time, all of us were unaware of Erdős' article.)

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