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A C^∞ Denjoy counterexample

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Abstract. In this paper we construct an example of a homeomorphism of the circle onto itself which is C^∞, has no periodic points and no dense orbits. Moreover, the homeomorphism will have no more than two points of zero derivative. We alter this example to form a C^∞ map of an interval to itself which has homtervals.

Introduction

Let $\mathbb{T}$ denote the smooth manifold $\mathbb{R}/\mathbb{Z}$, the circle with unit circumference and, for any $\rho \in \mathbb{R}$, let $r_\rho : \mathbb{T} \to \mathbb{T}$ be the map given by

$$r_\rho(\theta + \mathbb{Z}) = (\theta + \rho) + \mathbb{Z}.$$

A theorem of Denjoy [**2**] says that if $f : \mathbb{T} \to \mathbb{T}$ is a C^1 diffeomorphism, f has irrational rotation number ρ and the derivative of f has bounded variation, then f is conjugate to r_ρ, i.e. there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such $f \circ h = h \circ r_\rho$. In fact, Denjoy showed that if f satisfies the hypotheses above then f has a dense orbit and it follows easily from this that f is conjugate to r_ρ. Poincaré showed that a homeomorphism $f : \mathbb{T} \to \mathbb{T}$ has a periodic orbit if and only if it has a rational rotation number. With the theorem of Denjoy this implies that every C^1 diffeomorphism $f : \mathbb{T} \to \mathbb{T}$ whose derivative has bounded variation either has a periodic orbit or every orbit is dense, depending on whether the rotation number is rational or irrational respectively. Denjoy [**2**] (see [**3**], [**5**]) constructed examples of C^1 diffeomorphisms $f : \mathbb{T} \to \mathbb{T}$ with arbitrary irrational rotation number which have no dense orbits. Katok (see [**4**]) has constructed homeomorphisms $f : \mathbb{T} \to \mathbb{T}$ with arbitrary irrational rotation number which have no dense orbits and which are C^∞ diffeomorphisms away from one point of $\mathbb{T}$.

In this paper we show that there exist homeomorphisms $f : \mathbb{T} \to \mathbb{T}$ with arbitrary irrational rotation number which are C^∞ on all of $\mathbb{T}$ and which have no dense orbits. Moreover, the derivative of f will be zero at no more than two points of $\mathbb{T}$. Following the method of Coven & Nitecki [**1**], we show that such a map of $\mathbb{T}$ may be modified to form a C^∞ map of an interval to itself which has homtervals, partially answering a question of Nitecki [**6**].

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Notations and definitions

Let λ denote Haar measure on $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. For any $\rho\in\mathbb{R}$ the map $r_\rho:\mathbb{T}\to\mathbb{T}$ defined by

$$r_\rho(\theta+\mathbb{Z})=(\theta+\rho)+\mathbb{Z}$$

will be called rigid rotation by ρ. Let $\eta:\mathbb{R}\to\mathbb{Z}$ be the natural covering map

$$\eta(\theta)=\theta+\mathbb{Z}.$$

Then any continuous map $f:\mathbb{T}\to\mathbb{T}$ has a unique lift $F:\mathbb{R}\to\mathbb{R}$ such that $F(0)\in[0,1)$ and

$$\eta\circ F=f\circ\eta.$$

We shall always denote maps on $\mathbb{T}$ with small Latin letters and the corresponding lift with the corresponding capital letter. Given $f:\mathbb{T}\to\mathbb{T}$, the lift $F:\mathbb{R}\to\mathbb{R}$ will have the same continuity and differentiability properties as f. The nth iterate of a map $f:\mathbb{T}\to\mathbb{T}$ or $F:\mathbb{R}\to\mathbb{R}$ will be denoted by f^n or F^n respectively, i.e.

$$f^n=f\circ f^{n-1}\quad\text{and}\quad F^n=F\circ F^{n-1}.$$

Definition. For $F:\mathbb{R}\to\mathbb{R}$ a j-times differentiable map we define

$$\|F\|_{C^j}=\sup_{\substack{x\in\mathbb{R}\\ 1\le i\le j}}\left|\frac{d^iF}{dx^i}(x)\right|+\sup_{x\in\mathbb{R}}|F(x)|.$$

Definition. For $f:\mathbb{T}\to\mathbb{T}$ continuous, we say that f is degree one if for all $x\in\mathbb{R}$, $F(x+1)=F(x)+1$.

Definition. A map $f:\mathbb{T}\to\mathbb{T}$ will be called non-decreasing if, for all $x, y\in\mathbb{R}$,

$$x\le y\quad\text{implies}\quad F(x)\le F(y).$$

Some lemmas

The following lemmas will be needed.

LEMMA 1. *If $f:\mathbb{T}\to\mathbb{T}$ is a continuous, non-decreasing, degree one map then*

$$\lim_{n\to\infty}\frac{F^n(x)}{n}$$

exists for every $x\in\mathbb{R}$ and is independent of x.

Remark. This limit is called the rotation number of f and will be denoted rot (f). Recall that we require that F satisfy $F(0)\in[0,1)$. If we were to drop this condition on the lift of f then only the fractional part of the rotation number would be independent of the choice of the lift.

LEMMA 2. *If $f_n:\mathbb{T}\to\mathbb{T}$ are continuous, non-decreasing, degree one maps for $n=0,1,2,\ldots$ and if $f_n\to f_0$ uniformly then*

$$\text{rot}\,(f_n)\to\text{rot}\,(f_0).$$

LEMMA 3. *If* $f, g: \mathbb{T} \to \mathbb{T}$ *are continuous, non-decreasing, degree one maps and if there exists* $\varepsilon > 0$ *such that for all* $x \in \mathbb{R}$

$$G(x) \geq F(x) + \varepsilon$$

then rot $(g) \geq \text{rot}(f)$. *Moreover, if rot* (f) *or* rot (g) *is irrational then rot* $(g) > \text{rot}(f)$.

The proofs of these three lemmas are only slight modifications of results for homeomorphisms of $\mathbb{T}$ which may be found in [**3**] (II 2.3, II 2.7 and III 4.1.1 respectively).

Main theorem

We can now state the main result of this paper.

THEOREM 1. *For any irrational* $\rho \in [0, 1)$ *there exists a homeomorphism* $f: \mathbb{T} \to \mathbb{T}$ *such that* rot $(f) = \rho$, *f is* C^{∞} *and f has no dense orbits.*

More specifically, we prove the following theorem which clearly implies theorem 1.

THEOREM 2. *For any irrational* $\rho \in [0, 1)$ *there exists a homeomorphism* $f: \mathbb{T} \to \mathbb{T}$ *such that* rot $(f) = \rho$, *f is* C^{∞}, *f has no dense orbits and* $F: \mathbb{R} \to \mathbb{R}$ *has at most two points of zero derivative in* $[0, 1)$.

We shall proceed by proving four easy but technical lemmas.

LEMMA 4. *If* $f: \mathbb{T} \to \mathbb{T}$ *is a continuous, non-decreasing, degree one map with irrational rotation number* $\rho \in [0, 1)$ *then the following are equivalent*:

(i) *f has no dense orbits*;

(ii) *there exists a non-empty interval* $I \subset \mathbb{T}$ *and* $\theta \in \mathbb{T}$ *such that* $f^n(\theta) \notin I$ *for all* $n > 0$;

(iii) *there exists an interval* $I \subset \mathbb{T}$ *such that* $\lambda(I) > 0$ *and* $\lambda(f^n(I)) \to 0$ *as* $n \to \infty$.

Proof. As we noted in the introduction, f will be conjugate to rigid rotation by its irrational rotation number ρ if and only if f has a dense orbit. Hence f has one dense orbit if and only if all its orbits are dense. So (i)$\leftrightarrow$(iii) is clear.

One can easily show that there exists a continuous, non-decreasing map $h: \mathbb{T} \to \mathbb{T}$ such that

$$h \circ f = r_\rho \circ h,$$

i.e. f is semi-conjugate to rigid rotation by ρ (see [**3**], II 7.1). Then h is injective if and only if f has dense orbits, while if h is not injective then there is a non-trivial interval $I \subset \mathbb{T}$ such that $h(I)$ is a singleton. It follows easily that $\{f^n(I)\}_{n \geq 0}$ is a disjoint family of intervals. Hence we have (i)$\leftrightarrow$(iii) and the proof of the lemma is complete. □

LEMMA 5. *Suppose* $f: \mathbb{T} \to \mathbb{T}$ *is a* C^{∞}, *non-decreasing, degree one map with irrational rotation number* $\rho \in [0, 1)$. *Suppose there exist* $c, d \in \mathbb{R}$ *such that* $\frac{1}{2} \leq c \leq d \leq \frac{3}{4}$ *and* $dF(x)/dx = 0$ *if and only if* $\eta(x) \in \eta([c, d])$. *Then for an arbitrary positive integer n and* $\varepsilon > 0$ *with* $\frac{1}{8}(d - c) > \varepsilon$, *for any* $\delta > 0$ *there exists a* C^{∞}, *non-decreasing, degree one map* $g: \mathbb{T} \to \mathbb{T}$ *depending on n,* δ *and* ε *which satisfies*:

(i) $\|G - F\|_{C^n} < \delta$;

(ii) $|dG(x)/dx - dF(x)/dx| < \delta\, dF(x)/dx$ *for all x such that*

$$\eta(x) \notin \eta((c - \tfrac{1}{4}(d-c),\ d + \tfrac{1}{4}(d-c)));$$

(iii) $\operatorname{rot}(g) = \rho$;

(iv) $dG(x)/dx = 0$ *if and only if*

$$\eta(x) \in \eta([c, c + \tfrac{1}{2}(d-c) - \tfrac{1}{2}\varepsilon]) \cup \eta([c + \tfrac{1}{2}(d-c) + \tfrac{1}{2}\varepsilon, d]).$$

Proof. To define a map $g : \mathbb{T} \to \mathbb{T}$ we shall first define a map $\tilde{G}$: $[0, 1) \to \mathbb{R}$ such that

$$\tilde{G} \geq 0 \quad \text{and} \quad \int_0^1 \tilde{G}(x)\, dx = 1.$$

Next we extend $\tilde{G}$ to a periodic map $\tilde{G}: \mathbb{R} \to \mathbb{R}$ with period one. For any $a \in \mathbb{R}$, if we define

$$G_a(x) = \int_0^x \tilde{G}(y)\, dy + a,$$

then G_a is the lift of a unique non-decreasing, degree one map $g_a : \mathbb{T} \to \mathbb{T}$. Moreover, if $\tilde{G}: \mathbb{R} \to \mathbb{R}$ is C^∞ then so is g_a.

Fix a positive integer n and real numbers ε, $\delta > 0$. Without loss of generality we may assume $\delta < \frac{1}{2}$. Since

$$\frac{dF}{dx}(x) > 0 \quad \text{for all } x \in [\tfrac{1}{16}, \tfrac{3}{16}]$$

there exists $\zeta > 0$ such that

$$\frac{dF}{dx}(x) > \zeta \quad \text{for all } x \in [\tfrac{1}{16}, \tfrac{3}{16}].$$

Let $\phi: \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function with support in $[-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon]$ such that $\|\phi\|_{C^n} < \frac{1}{4}\delta$, $\phi(x) > 0$ for all $x \in (-\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon)$ and $\phi(x) < \delta\zeta$ for all $x \in \mathbb{R}$. Let

$$\tilde{G}(x) = \frac{dF}{dx}(x) + \phi(x - (c + \tfrac{1}{2}(d-c))) - \phi(x - \tfrac{1}{8})$$

for all $x \in [0, 1)$. Note that $\tilde{G} \geq 0$ and

$$\begin{aligned} \int_0^1 G(x)\, dx &= \int_0^1 \frac{dF}{dx}(x)\, dx + \int_0^1 \phi(x - (c + \tfrac{1}{2}(d-c)))\, dx - \int_0^1 \phi(x - \tfrac{1}{8})\, dx \\ &= 1. \end{aligned}$$

Moreover, since $\phi(x - (c + \frac{1}{2}(d-c)))$ and $\phi(x - \frac{1}{8})$ are both zero in neighbourhoods of zero and one it follows that, if we extend $\tilde{G}$ to a periodic map $\tilde{G}: \mathbb{R} \to \mathbb{R}$ with period one, then this map is C^∞ on $\mathbb{R}$. For each $a \in \mathbb{R}$ define

$$G_a(x) = \int_0^x \tilde{G}(y)\, dy + F(0) + a.$$

For each $a \in \mathbb{R}$, G_a is the lift of a unique C^∞, non-decreasing, degree one map $g_a: \mathbb{T} \to \mathbb{T}$. By lemma 2 we may fix a' such that the map $g = g_{a'}$ has rotation number ρ. Noting that

$$\left| \int_0^x \tilde{G}(y)\, dy + F(0) - F(x) \right| < 2 \int_{-\varepsilon}^{\varepsilon} \phi(x)\, dx < \tfrac{1}{2}\delta,$$

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we see by lemma 3 that $|a'|<\frac{1}{2}\delta$. Hence it is clear that g satisfies (i), (iii) and (iv). Finally, since

$$\sup_{x\in[0,\,1/4)}\left|\tilde{G}(x)-\frac{dF}{dx}(x)\right|\leq \sup_{x\in[-\varepsilon,\varepsilon]}(\phi(x))\leq \zeta\delta,$$

we see that (ii) is also satisfied by g and the proof of the lemma is complete. □

LEMMA 6. *Suppose $g:\mathbb{T}\to\mathbb{T}$ is a C^∞, non-decreasing, degree one map with irrational rotation number $\rho\in[0,1)$. Suppose that for some c, d, $\varepsilon\in\mathbb{R}$ with $\frac{1}{2}\leq c\leq d\leq\frac{3}{4}$ and $\frac{1}{8}(d-c)\geq\varepsilon>0$ we have $dG(x)/dx=0$ if and only if*

$$\eta(x)\in\eta([c,c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon])\cup\eta([c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon,d])$$

and suppose there exists a positive integer r such that

$$g^r(\eta(d))\in\eta((c,c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon)).$$

Then for arbitrary $n\in\mathbb{Z}^+$ and $\delta>0$ there exists a C^∞, non-decreasing, degree one map $h:\mathbb{T}\to\mathbb{T}$ depending on n, δ and ε which satisfies

(i) $\|H-G\|_{C^n}<\delta$;

(ii) $|dH(x)/dx-dG(x)/dx|<\delta\, dG(x)/dx$ *for all $x\in\mathbb{R}$ such that*

$$\eta(x)\notin\eta((c-\tfrac{1}{4}(d-c),d+\tfrac{1}{4}(d-c)));$$

(iii) $\operatorname{rot}(h)=\rho$;

(iv) $dH(x)/dx=0$ *if and only if*

$$\eta(x)\in\eta([c,c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon]);$$

(v) $h^r(\eta([c+\frac{1}{2}(d-c)+\frac{1}{2}\varepsilon,d]))\subset\eta((c,c+\frac{1}{2}(d-c)-\frac{1}{2}\varepsilon))$.

Proof. We define $h:\mathbb{T}\to\mathbb{T}$ by the same procedure as in the proof of lemma 5, i.e. we define the derivative of its lift on $[0,1)$. Fix a positive integer n and a real number $\delta>0$. Without loss of generality we may assume $\delta<\frac{1}{2}$. Let $\zeta>0$ be such that

$$\frac{dG}{dx}(x)>\zeta\quad\text{for all } x\in[0,\tfrac{1}{4}].$$

Let $\psi:\mathbb{R}\to\mathbb{R}$ be a C^∞ bump function with support in $[c+\frac{1}{2}(d-c),d+\frac{1}{2}\varepsilon]$ such that

$$\|\psi\|_{C^n}<\tfrac{1}{4}\delta,\quad \psi(x)>0\quad\text{for all } x\in(c+\tfrac{1}{2}(d-c),d+\tfrac{1}{2}\varepsilon)$$

and

$$\psi(x)<\delta\zeta\quad\text{for all } x\in\mathbb{R}.$$

Then for each α, $0<\alpha<1$, define for all $x\in[0,1)$

$$\tilde{H}_\alpha(x)=\frac{dG}{dx}(x)+\alpha\psi(x)-\alpha\psi(x-(c+\tfrac{1}{2}(d-c))).$$

Since $\tilde{H}_\alpha>0$, $\int_0^1\tilde{H}_\alpha(x)\,dx=1$ and both $\psi(x)$ and $\psi(x-(c+\frac{1}{2}(d-c)))$ are C^∞ flat at zero and one we may proceed as in the proof of lemma 5. First extend $\tilde{H}_\alpha$ to a C^∞ periodic map $\tilde{H}_\alpha:\mathbb{R}\to\mathbb{R}$ with period one. Then for each α, $0<\alpha<1$, we know by lemma 2 that there exists $\Delta(\alpha)\in\mathbb{R}$ such that the map $h_\alpha:\mathbb{T}\to\mathbb{T}$ whose lift is given by

$$H_\alpha(x)=\int_0^x\tilde{H}_\alpha(y)\,dy+G(0)+\Delta(\alpha)$$

has rotation number ρ. As in lemma 5, h_α will satisfy (i)–(iv) for each α, $0<\alpha<1$. Moreover, since

$$g^r(\eta(d)) = g^r(\eta([c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d]))$$

is a single point and $h^r \to g^r$ as $\alpha \to 0$ we may choose α', $0<\alpha'<1$, so small that

$$h^r_{\alpha'}(\eta([c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d])) \subset \eta((c, c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon)).$$

Hence $h_{\alpha'}$ is the required map and the proof of the lemma is complete. □

Remark. If in lemma 6 we replace the condition

$$g^r(\eta(d)) \in \eta((c, c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon))$$

with

$$g^r(\eta(c)) \in \eta(c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d))$$

and the conclusions (iv) and (v) with

(iv′) $dH(x)/dx = 0$ if and only if $\eta(x) \in \eta([c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d])$,

(v′) $h^r(\eta([c, c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon])) \subset \eta((c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d))$,

then we obtain another lemma which we will call lemma 6′. The proof of lemma 6′ is, of course, almost the same as the proof of lemma 6.

LEMMA 7. *Suppose $g: \mathbb{T} \to \mathbb{T}$ is a C^∞, non-decreasing, degree one map with irrational rotation number $\rho \in [0, 1)$. Suppose for some c, d, $\varepsilon \in \mathbb{R}$ with $\tfrac{1}{2} \le c \le d \le \tfrac{3}{4}$ and $\tfrac{1}{8}(d-c) > \varepsilon > 0$ we have that $dG(x)/dx = 0$ if and only if*

$$\eta(x) \in \eta([c, c+\tfrac{1}{2}(d-c)-\tfrac{1}{2}\varepsilon]) \cup \eta([c+\tfrac{1}{2}(d-c)+\tfrac{1}{2}\varepsilon, d]).$$

Finally, suppose that for all integers $i>0$,

$$g^i(\eta(d)) \notin \eta((c, d)).$$

Then there exists a homeomorphism $h: \mathbb{T} \to \mathbb{T}$ which is C^∞, has rotation number ρ and for all $i>0$,

$$h^i(\eta(d)) \notin \eta((c, d)).$$

Moreover, $dH/dx = 0$ if and only if $\eta(x) = \eta(c)$ or $\eta(x) = \eta(d)$.

Proof. Let $\alpha = G(d) - G(c)$. Define

$$H_1 : [c, d] \to [G(c), G(d)]$$

$$H_1(x) = G(c) + \frac{\alpha \displaystyle\int_c^x \exp(-1/(y-c)^2 - 1/(y-d)^2)\,dy}{\displaystyle\int_c^d \exp(-1/(y-c)^2 - 1/(y-d)^2)\,dy}$$

for all $x \in (c, d)$ and

$$H_1(c) = G(c) \quad \text{and} \quad H_1(d) = G(d).$$

Then

$$\frac{d^n H_1}{dx^n}(c) = 0 = \frac{d^n G}{dx^n}(c), \quad \frac{d^n H_1}{dx^n}(d) = 0 = \frac{d^n G}{dx^n}(d) \quad \text{for all } n>0$$

and

$$\frac{dH_1}{dx}(x) > 0 \quad \text{for all } x \in (c, d).$$

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Define $H:\mathbb{R}\to\mathbb{R}$ by

$$H(x)=\begin{cases} G(x) & \text{if } \eta(x)\notin\eta((c,d)),\\ H_1(x-[x])+[x] & \text{if } \eta(x)\in\eta((c,d)),\end{cases}$$

where $[\cdot]$ denotes the greatest integer function. Then $H:\mathbb{R}\to\mathbb{R}$ is a C^∞ map with $dH(x)/dx\ge 0$ for all $x\in\mathbb{R}$ and $dH(x)/dx=0$ if and only if $\eta(x)=\eta(c)$ or $\eta(x)=\eta(d)$. Also,

$$H(x+1)=H(x)+1$$

for all $x\in\mathbb{R}$, so H is the lift of a unique homeomorphism $h:\mathbb{T}\to\mathbb{T}$ which is C^∞ Since G and H differ only at points $x\in\mathbb{R}$ such that $\eta(x)\in\eta((c,d))$ and since

$$g^i(\eta(d))\notin\eta((c,d))\quad\text{for all } i>0$$

we see that

$$G^i(d)=H^i(d)$$

and hence

$$g^i(\eta(d))=h^i(\eta(d))\quad\text{for all } i>0.$$

Hence

$$\operatorname{rot}(h)=\operatorname{rot}(g)=\rho$$

and

$$h^i(\eta(d))\notin\eta((c,d))\quad\text{for all } i>0.$$

So h is the required map and the proof of the lemma is complete. □

Remark. Again we note that if in lemma 7 we replace the condition

$$g^i(\eta(d))\notin\eta((c,d))$$

with

$$g^i(\eta(c))\notin\eta((c,d))\quad\text{for all } i>0$$

and the conclusion

$$h^i(\eta(d))\notin\eta((c,d))$$

with

$$h^i(\eta(c))\notin\eta(c,d))\quad\text{for all } i>0$$

then we obtain another lemma whose proof is essentially the same as the proof above. We shall call this version lemma 7′.

Proof of theorem 2. Fix an irrational $\rho\in[0,1)$ to serve as rotation number. Let $f_{1,0}:\mathbb{T}\to\mathbb{T}$ be a C^∞, non-decreasing, degree one map such that

$$F_{1,0}(0)=0\quad\text{and}\quad\frac{dF_{1,0}}{dx}(x)=0$$

if and only if $\eta(x)\in\eta([\frac{1}{2},\frac{3}{4}])$. Then for each $\Delta\in[0,1)$ let $f_{1,\Delta}:\mathbb{T}\to\mathbb{T}$ be the unique map whose lift is $F_{1,0}(x)+\Delta$. Lemma 2 implies that there exists a $\Delta'\in[0,1)$ such that

$$\operatorname{rot}(f_{1,\Delta'})=\rho.$$

Let $f_1 = f_{1,\Delta'}$ and let

$$a = F_1^{-1}\left(\tfrac{5}{8}+\tfrac{1}{64}\right), \quad b = F_1^{-1}\left(\tfrac{3}{4}-\tfrac{1}{64}\right).$$

Suppose we have constructed maps

$$f_1, \cdots, f_n : \mathbb{T} \to \mathbb{T},$$

sequences

$$\tfrac{1}{2} = c_1 = c_2 < c_3 = c_4 < \cdots c_n, \quad \tfrac{3}{4} = d_1 > d_2 = d_3 > d_4 = \cdots d_n$$

and integers

$$1 = m_1 < m_2 < m_3 < \cdots < m_n$$

such that for $i = 1, 2, \ldots, n$ the following conditions are satisfied:

(1) f_i is C^∞, non-decreasing and degree one;

(2) $\mathrm{rot}\,(f_i) = \rho$;

(3) $dF_i(x)/dx = 0$ if and only if $\eta(x) \in \eta([c_i, d_i])$;

(4) $0 < d_i - c_i \leq 1/2^{i+1}$;

(5) $\|F_i - F_{i+1}\|_{C^i} < 1/2^{i+1}$;

(6) $|dF_i(x)/dx - dF_{i+1}(x)/dx| < (1/2^{i+1})\, dF_i(x)/dx$, whenever

$$\eta(x) \notin \eta((c_i - \tfrac{1}{4}(d_i - c_i),\, d_i + \tfrac{1}{4}(d_i - c_i)));$$

(7) $0 < \lambda(f_i^j(\eta([a, b]))) < 1/2^{k-1}$, whenever $m_{k-1} \leq j < m_k$ for $k = 1, 2, \ldots, i$;

(8) $f_i^j(\eta([a, b])) \cap \eta([c_i, d_i]) = \varnothing$, whenever $0 \leq j < m_i$ and

$$f_i^{m_i}(\eta([a, b])) \subset \begin{cases} \eta((c_i + \tfrac{1}{2}(d_i - c_i),\, d_i)) & \text{if } i \text{ is odd,} \\ \eta((c_i,\, c_i + \tfrac{1}{2}(d_i - c_i))) & \text{if } i \text{ is even.} \end{cases}$$

We shall attempt to perturb the map f_n to form $f_{n+1} : \mathbb{T} \to \mathbb{T}$ so that (1)–(8) above are satisfied for $i = 1, 2, \ldots, n+1$. First, we shall assume that n is odd; the construction for n even is very similar and will be mentioned later.

Fix $\varepsilon > 0$ so that

$$f_n^{m_n}(\eta([a, b])) \subset \eta((c_n + \tfrac{1}{2}(d_n - c_n) + \tfrac{1}{2}\varepsilon,\, d_n - \tfrac{1}{2}\varepsilon)).$$

By lemma 5, for each $\delta > 0$ there exists a C^∞, non-decreasing, degree one map $g_\delta : \mathbb{T} \to \mathbb{T}$ such that

$$\|G_\delta - F_n\|_{C^n} < \delta/2^{n+2},$$

$$\left|\frac{dG_\delta}{dx}(x) - \frac{dF_n}{dx}(x)\right| < \frac{1}{2^{n+4}} \frac{dF_n}{dx}(x),$$

whenever $\eta(x) \notin \eta((c_n - \tfrac{1}{4}(d_n - c_n),\, d_n + \tfrac{1}{4}(d_n - c_n)))$,

$$\mathrm{rot}\,(g_\delta) = \rho,$$

and

$$\frac{dG_\delta}{dx}(x) = 0$$

if and only if

$$\eta(x) \in \eta([c_n,\, c_n + \tfrac{1}{2}(d_n - c_n) - \tfrac{1}{2}\varepsilon]) \cup \eta([c_n + \tfrac{1}{2}(d_n - c_n) + \tfrac{1}{2}\varepsilon,\, d_n]).$$

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Since $G^i_\delta \to F^i_n$ as $\delta \to 0$ uniformly for $i = 1, 2, \ldots, m_n$ we may fix δ', $1 > \delta' > 0$, so small that for $g = g_{\delta'}$

$$\lambda(g^j(\eta([a, b]))) < 1/2^{k-1} \quad \text{for } m_{k-1} \le j < m_k,\ k = 1, 2, \ldots, n$$

and

$$g^j(\eta([a, b])) \cap \eta([c_n, d_n]) = \varnothing \quad \text{for } 0 \le j < m_n.$$

Now we must consider the following two cases:
Case A. There exists $r > 0$ such that $g^r(\eta(d_n)) \in \eta((c_n, d_n))$,
Case B. For all $r > 0$, $g^r(\eta(d_n)) \notin \eta((c_n, d_n))$.

If g satisfies case A, then we shall be able to perturb it to form a suitable map f_{n+1}. If g satisfies case B then we shall be able to alter g to form the map required to complete the proof of the theorem.
Case A. We may assume that r is the smallest positive integer such that

$$g^r(\eta(d_n)) \in \eta((c_n, d_n)).$$

Noting that

$$f^i_n(\eta(d_n)) \notin \eta([c_n, d_n]) \quad \text{for all } i > 0$$

and

$$g^i_\delta \to f^i_n \quad \text{as } \delta \to 0$$

uniformly for $0 < i < r$, we may assume that

$$g^r(\eta(d_n)) \in \eta((c_n, c_n + \tfrac{1}{2}\{\tfrac{1}{2}(d_n - c_n) - \tfrac{1}{2}\varepsilon\}))$$

by taking δ' in the definition of g above smaller if necessary.

By lemma 6 there exists for each $\sigma > 0$ a C^∞, non-decreasing, degree one map $h_\sigma : \mathbb{T} \to \mathbb{T}$ such that

$$\|H_\sigma - G\|_{C^{n+1}} < \sigma/2^{n+2},$$

$$\text{rot}\,(h_\sigma) = \rho,$$

$$\left|\frac{dH_\sigma}{dx}(x) - \frac{dG}{dx}(x)\right| < \frac{1}{2^{n+4}}\frac{dG}{dx}(x) \quad \text{for all } x \in \mathbb{R}$$

such that

$$\eta(x) \in \eta((c_n - \tfrac{1}{4}(d_n - c_n), d_n + \tfrac{1}{4}(d_n - c_n))),$$

$$\frac{dH_\sigma}{dx}(x) = 0 \quad \text{if and only if } \eta(x) \in \eta((c_n, c_n + \tfrac{1}{2}(d_n - c_n) - \tfrac{1}{2}\varepsilon))$$

and

$$h^r_\sigma(\eta([c_n + \tfrac{1}{2}(d_n - c_n) + \tfrac{1}{2}\varepsilon])) \subset \eta((c_n, c_n + \tfrac{1}{4}(d_n - c_n) - \tfrac{1}{4}\varepsilon)).$$

Since $h^i_\sigma \to g^i$ as $\sigma \to 0$ uniformly for $i = 1, 2, \ldots, m_n + r$ and since $g^i(\eta([a, b]))$ is a single point for $i > m_n$, we may fix σ', $0 < \sigma' < 1$, so small that if $h = h_{\sigma'}$ then

$$h^i(\eta([a, b])) \cap \eta([c_n, c_n + \tfrac{1}{2}(d_n - c_n) - \tfrac{1}{2}\varepsilon]) = \varnothing \quad \text{for } 0 \le i < m_n + r$$

and

$$\lambda(h^j(\eta([a, b]))) < 1/2^{k-1}$$

whenever $m_{k-1} \le j < m_k$ for $k = 1, 2, \ldots, n$,

$$\lambda(h^j(\eta([a, b]))) < 1/2^n$$

whenever $m_n \le j < m_n + r$.

We define

$$f_{n+1} = h, \quad c_{n+1} = c_n, \quad d_{n+1} = c_n + \tfrac{1}{2}(d_n - c_n) - \tfrac{1}{2}\varepsilon, \quad \text{and} \quad m_{n+1} = m_n + r.$$

With these choices, $f_1, f_2, \ldots, f_{n+1}$ satisfy (1)–(8) above.

Case B. By lemma 7 there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which is C^∞, has rotation number ρ and which satisfies

$$h^i(\eta(d_n)) \notin \eta((c_n, d_n)) \quad \text{for all } i > 0$$

and

$$\frac{dH}{dx}(x) = 0 \quad \text{if and only if } \eta(x) = \eta(c_n) \text{ or } \eta(x) = \eta(d_n).$$

So h satisfies condition (ii) of lemma 4 and hence h has no dense orbits.

Remark. If n is odd then we obtain two cases A' and B' depending on whether

$$\{g^i(\eta(c_n))\}_{i=1}^{\infty} \cap \eta((c_n, d_n))$$

is empty or not, respectively. In case A' we may use lemma 6′ to produce the map f_{n+1}. In case B' we may use lemma 7′ to construct the map h which satisfies the same conditions as the map h in case B above.

To conclude the proof of theorem 2 we note that the above procedure either produces an infinite sequence of maps $f_1, f_2, \ldots$ which satisfy (1)–(8) for all $n \ge 1$, or for some finite $n \ge 1$ we encounter case B (or B'). In the former situation, by conditions 1, 2 and 5, the sequence $f_1, f_2, \ldots$ converges to some map $f : \mathbb{T} \to \mathbb{T}$ which is C^∞, non-decreasing, degree one and has rotation number ρ. Moreover, if

$$\tilde{c} = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n$$

(the limits are equal by condition 4), then by conditions 3 and 6 we have

$$\frac{dF}{dx}(x) = 0 \quad \text{if and only if } \eta(x) = \eta(\tilde{c}).$$

Hence f is a homeomorphism of $\mathbb{T}$ onto itself. Finally, by conditions 7 and 8,

$$\lambda(f_n^i(\eta([a, b]))) \to 0 \quad \text{as } i \to \infty$$

uniformly in n, so

$$\lambda(f^i(\eta([a, b]))) \to 0 \quad \text{as } i \to \infty,$$

i.e. f satisfies condition (iii) of lemma 4. Hence f has no dense orbits and therefore f is the map whose existence is claimed by the theorem. Note that the lift of f has exactly one point of zero derivative in $[0, 1)$.

We have already seen that if case B (or B') is encountered for some finite $n \ge 1$ then a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ which is C^∞, has rotation number ρ and has no dense orbits may be constructed. Recall that the lift of h has exactly two points of zero derivative in $[0, 1)$.

In either case the proof of theorem 2 is complete. □

Application to maps of the interval

Let $I \subset \mathbb{R}$ be an interval and let $F: I \to I$ be a continuous map.

Definition. An interval $J \subset I$ will be called non-degenerate if J is not empty and not a singleton.

Definition. A homterval for $F: I \to I$ is a closed, non-degenerate interval $J \subset I$ which is not in the domain of attraction of any periodic orbit of F such that

$$\{F^n(J): n \geq 0\}$$

is a collection of disjoint, non-degenerate, closed intervals and $F^n|_J$ is a homeomorphism for every $n > 0$.

Coven & Nitecki [1] have shown that the example of Denjoy of a C^1 diffeomorphism $g: \mathbb{T} \to \mathbb{T}$ with no periodic points and no dense orbits can be modified to form a C^1 map of an interval which has homtervals. Following their construction, we can modify a C^∞ map $g: \mathbb{T} \to \mathbb{T}$ which is a homeomorphism with irrational rotation number and no dense orbits given by theorem 1 above to form a C^∞ map of an interval with homtervals, partially answering a question of Nitecki [5]. The construction proceeds as follows.

Fix an irrational $\rho \in [0, 1)$ and let $h: \mathbb{T} \to \mathbb{T}$ be a homeomorphism with rotation number ρ which is C^∞ and has no dense orbits. Let $\tilde{J} \subset \mathbb{T}$ be a closed connected interval with positive length satisfying

$$h^n(\tilde{J}) \cap h^m(\tilde{J}) = \emptyset \quad \text{for all } n, m \in \mathbb{Z}, n \neq m$$

(any interval in the complement of an orbit will do). We may assume that $\tilde{J} = \eta(J)$ for an interval $J = [a, b] \subset (0, 1)$. Let $H: \mathbb{R} \to \mathbb{R}$ be the lift of h with $H(0) \in [0, 1)$, let

$$\beta = H(a) + \tfrac{1}{2}(H(b) - H(a))$$

and let $H_1: \mathbb{R} \to \mathbb{R}$ be the lift of h with $H_1(0) \in [-1, 0)$, so

$$H_1(x) = H(x) - 1 \quad \text{for all } x \in \mathbb{R}.$$

Note that H is strictly increasing, so $H^{-1}(\beta) \in (a, b)$. Define

$$F_1: [\beta - 1, \beta] \to [\beta - 1, \beta]$$

$$F_1(x) = \begin{cases} H(x) & \text{if } H(x) \leq \beta, \\ H_1(x) & \text{otherwise.} \end{cases}$$

Then F_1 is a C^∞ map away from the discontinuity at $H^{-1}(\beta) \in (a, b)$. By modifying F_1 only on the interval J we can form a C^∞ map

$$F: [\beta - 1, \beta] \to [\beta - 1, \beta].$$

Moreover, if an interval J_1 is chosen so that

$$\eta(J_1) = h^n(\tilde{J}) \quad \text{for some } n > 0$$

so that

$$\eta(\beta) \notin h^i(\eta(J_1)) \quad \text{for all } i > 0$$

(which may be guaranteed by taking n sufficiently large) then

$$\{F^i(J_1): i > 0\}$$

is a collection of non-degenerate, closed intervals. Since for each $i \geq 0$ there exists $m, j \in \mathbb{Z}$ such that

$$F^i(J_1) = H^j(J) - m,$$

it follows that J_1 is a homterval for the map F and the construction is complete.

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