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FULL SUBRINGS OF E-RINGS

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A ring R is said to be an E-ring if the map $R \to E(R^+)$, of R into the ring of endomorphisms of its additive group via $a \hookrightarrow a_l$ = left multiplication by a, is an isomorphism. In this note torsion free rings R for which the group R_l, of left multiplication maps by elements of R, is a full subgroup of $E(R^+)^+$ will be considered. These rings are called TE-rings. It will be shown that TE-rings satisfy many properties of E-rings, and that unital TE-rings are E-rings. If R is a TE-ring, then $E(R^+)$ is an E-ring, and $E(R^+)^+/R_l$ is bounded. Some results concerning additive groups of TE-rings will be obtained.

1. INTRODUCTION

A ring R is said to be an E-ring if the map $\lambda\colon R \to E(R^+)$ of R into the ring of endomorphisms of its additive group defined by $\lambda(a) = a_l$ = left multiplication by a, is an isomorphism. The additive group of an E-ring is called an E-group. E-rings and E-groups have received considerable attention, (see [**1**, **2**, Chapter 4, Section 7, **6**, **7**]). In this note torsion free rings R for which $E(R^+)^+/R_l$ is a torsion group will be considered.

Definitions and notation will follow [**2**, **4**, **5**].

NOTATION.

R	a ring, not necessarily associative, or unital
R^+	the additive group of R
$E(R^+)$	the endomorphism ring of R^+
a_l	left multiplication by $a \in R$
a_r	right multiplication by $a \in R$
R_l	the group or ring $\{a_l \mid a \in R\}$
λ	the map $R \to R_l$ via $a \hookrightarrow a_l$
t	the type function

Several conditions on a unital ring R are equivalent to R being an E-ring. Some of these conditions will be recalled.

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Proposition 1. *Let R be a unital ring. The following are equivalent:*

(1) *R is an E-ring,*
(2) *every ring S with $S^+ = R^+$ is associative,*
(3) *every ring S with $S^+ = R^+$ is commutative,*
(4) *$E(R^+)$ is commutative,*
(5) *$\alpha(ab) = a\alpha(b)$ for all $\alpha \in E(R^+)$, and all $a, b \in R$,*
(6) *$\alpha(ab) = \alpha(a)b$ for all $\alpha \in E(R^+)$, and all $a, b \in R$,*
(7) *$\alpha(x) = \alpha(1)x$ for all $\alpha \in E(R)$, and all $x \in R$.*

Proof: The equivalence of (1), (2) and (3) is proved in [**7**, Lemma 8], and that of (1), (4) and (5) in [**1**, Proposition 1.2]. If R satisfies (5) then R is an E-ring by [**1**, Proposition 1.2], and so R is commutative. Therefore for $\alpha \in E(R^+)$, and $a, b \in R$, it follows that $\alpha(ab) = \alpha(ba) = b\alpha(a) = \alpha(a)b$, and so (6) is satisfied. If R satisfies (6) then for all $a, b \in R$, the product $ab = b_r(1 \cdot a) = b_r(1)a = ba$ and so R is commutative. Therefore, for $\alpha \in E(R^+)$, and $a, b \in R$, one has $\alpha(ab) = \alpha(ba) = \alpha(b)a = a\alpha(b)$, and so (5) is satisfied. The equivalence of (1) and (7) follows from [**7**, Lemma 6]. ∎

Definition: An associative torsion free ring R is a *TE-ring* (*BE-ring*) if $E(R^+)^+/R_l$ is a torsion (bounded) group. The additive group of a TE-ring (BE-ring) is called a *TE-group* (*BE-group*).

Example. $2Z$ the ring of even integers is a BE-ring but is not an E-ring. An example of a TE-ring which is not a BE-ring cannot be given, because TE-rings and BE-rings are one and the same as will be shown later, Corollary 3. The ring of even integers satisfies properties (2)–(7) of Proposition 1. This is typical of TE-rings, as will also be shown.

Theorem 2. *Let R be a TE-ring. Then*

(1) *there exists an element $e \in R$, and a positive integer n such that $ea = na$ for all $a \in R$;*
(2) *R is commutative;*
(3) *$n\alpha = [\alpha(e)]_l$ for all $\alpha \in E(R^+)$.*

Proof: (1) Let i be the identity map on R^+. There exists $e \in R$, and a positive integer n such that $ni = e_l$. This clearly implies that $ea = na$ for all $a \in R$.

(2) Let $a \in R$. Since $a_r \in E(R^+)$, there exist $b \in R$, and a positive integer m such that $mxa = bx$ for all $x \in R$. Substituting e for x yields that $mna = be$. Therefore, $mnax = nbx$ for all $x \in R$. Since R^+ is torsion free, this implies that $max = bx$ for all $x \in R$. Therefore $max = mxa$ which, by the torsion freeness of R^+, implies that $ax = xa$.

(3) Let $\alpha \in E(R^+)$. There exist $a \in R$, and a positive integer m such that

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$m\alpha = a_l$. Therefore $m\alpha(e) = na$, and so $m\alpha(e)x = nax = mn\alpha(x)$ for all $x \in R$. Since R^+ is torsion free, this implies that $n\alpha(x) = \alpha(e)x$.

From now on e will always denote the distinguished element of a TE-ring R satisfying $ex = nx$ for all $x \in R$. ∎

An immediate consequence of Theorem 2 (3) is:

COROLLARY 3. *Every TE-ring is a BE-ring.*

COROLLARY 4. *Let R be a TE-ring. Then the map $\lambda\colon R \to E(R^+)$ defined by $\lambda(a) = a_l$ for all $a \in R$, is a ring monomorphism.*

PROOF: Clearly $\lambda(a+b) = \lambda(a) + \lambda(b)$, and the associativity of R insures that $\lambda(ab) = \lambda(a)\lambda(b)$ for all $a, b \in R$. Let $a \in \ker\lambda$. Since R is TE there exists a positive integer n such that $na = a_l(e) = 0$, and so $a = 0$. ∎

COROLLARY 5. *Let R be a unital TE-ring. Then R is an E-ring.*

PROOF: By Corollary 4, it suffices to show that $\lambda\colon R \to E(R^+)$ is onto. Let $\alpha \in E(R^+)$. Then $n\alpha(x) = \alpha(e)x$ for all $x \in R$, and so $n\alpha(1) = \alpha(e)$. Therefore $n\alpha(x) = n\alpha(1)x$ for all $x \in R$, which implies that $\alpha = [\alpha(1)]_l = \lambda[\alpha(1)]$. ∎

LEMMA 6. *Let R be a TE-ring. Then $E(R^+)$ is commutative.*

PROOF: Let $\alpha, \beta \in R$. Then $n^2\alpha\beta = (n\alpha)(n\beta) = [\alpha(e)]_l[\beta(e)]_l = [\beta(e)]_l[\alpha(e)]_l = n^2\beta\alpha$. Since $E(R^+)^+$ is torsion tree, $\alpha\beta = \beta\alpha$. ∎

THEOREM 7. *Let R be a TE-ring, and let S be a ring with $S^+ = R^+$. Then S is commutative and associative.*

PROOF: Let $*$ denote multiplication in S. For every $a \in S$ the map $R^+ \to R^+$ via $x \hookrightarrow a * x$ belongs to $E(R^+)$. Theorem 2 yields that $na * x = (a * e)x$ for all $x \in S$. The map $R^+ \to R^+$ via $x \hookrightarrow x * e$ belongs to $E(R^+)$ so, again by Theorem 2, $na * e = (e * e)a$ and so $n^2a * x = (e * e)ax$ for all $a, x \in S$. Hence for $a, b \in S$, one has $n^2a * b = (e * e)ab = (e * e)ba = n^2b * a$. Since S^+ is torsion free it follows that S is commutative. Let $a, b, c \in S$. Direct computation shows that $n^3[(a * b) * c] = n^3[a * (b * c)] = [e * (e * e)]abc$, and so S is associative. ∎

LEMMA 8. *Let R be a TE-ring, $\alpha \in E(R^+)$, and let $a, b \in R$. Then*

(1) $\alpha(ab) = \alpha(a)b$, *and*
(2) $\alpha(ab) = a\alpha(b)$.

PROOF: (1) $\alpha(ab) = \alpha \circ b_r(a) =$ (by Lemma 6) $b_r \circ \alpha(a) = \alpha(a)b$.
(2) $\alpha(ab) = \alpha(ba) =$ (by (1)) $\alpha(b)a = a\alpha(b)$. ∎

Rings satisfying property (1) of Lemma 8 were studied in [**3**], and were called E-associative rings.

THEOREM 9. *Let R be a TE-ring. Then $E(R^+)$ is an E-ring.*

PROOF: Let $*$ be a ring multiplication on $E(R^+)$. By Proposition 1 it suffices to show that $\left(E(R^+)^+, *\right)$ is commutative. It follows from Theorem 2 that $nE(R^+) \subseteq R_l$. Define multiplication in R_l by $a_l \circ b_l = n(a_l * b_l)$ for all $a, b \in R$. It is readily seen that $\circ$ is a ring multiplication on R_l. Since $R_l \simeq R^+$ it follows from Theorem 7 that $(R_l, \circ)$ is commutative. Let $\alpha, \beta \in E(R^+)^+$. Then $n\alpha = [\alpha(e)]_l$, and $n\beta = [\beta(e)]_l$ by Theorem 2. Therefore $n^3\alpha * \beta = n(n\alpha) * (n\beta) = [\alpha(e)]_l \circ [\beta(e)]_l = [\beta(e)]_l \circ [\alpha(e)]_l = n^3\beta * \alpha$. Since $E(R^+)^+$ is torsion free it follows that $\alpha * \beta = \beta * \alpha$. ∎

COROLLARY 10. *Let R be a torsion free ring. Then R is a TE-ring if and only if there exists an E-ring S, and an embedding of R into S with bounded index.*

PROOF: If R is a TE-ring then $\lambda: R \to E(R^+)$ is an embedding of R into an E-ring by Corollary 4 and Theorem 9. It follows from Corollary 3 that $\lambda(R) = R_l$ has bounded index in $E(R)$.

Conversely suppose that R is a subring of an E-ring S, and that $nS \subseteq R$, n a positive integer. Let 1 be the unity of S. Then $e = n \cdot 1 \in R$. For $\alpha \in E(R^+)$ define $\widehat{\alpha}: S^+ \to S^+$ by $\widehat{\alpha}(x) = \alpha(nx)$. It is readily seen that $\widehat{\alpha} \in E(S^+)$. Therefore, by Proposition 1, $\widehat{\alpha}(x) = \widehat{\alpha}(1)x = \alpha(e)x$ for all $x \in S$. Therefore $n\alpha(x) = \alpha(nx) = \alpha(e)x$ for all $x \in R$, that is, $n\alpha = [\alpha(e)]_l$ and so R is a TE-ring. ∎

LEMMA 11. *Let R be a TE-ring, and let S be a unital subring of R such that S^+/R^+ is bounded. Then S is an E-ring.*

PROOF: The ring $Q \otimes E(S^+) \simeq Q \otimes E(R^+)$ is commutative by Lemma 6. Therefore $E(S^+)$ is commutative, and so S is an E-ring by Proposition 1. ∎

LEMMA 12. *Let R be a TE-ring. Then every group direct sum $R^+ = H \oplus K$ is a ring direct sum.*

PROOF: Let $h \in H$, and let $x \in R$. The natural projection of R^+ onto H along K, π_H, belongs to $E(R^+)$. It follows from Lemma 8 that $hx = \pi_H(h)x = \pi_H(hx) \in H$, that is, $HR \subseteq H$, and similarly $RH \subseteq H$. Therefore H is an ideal in R. The same argument, using the projection of R^+ onto K, yields that K is an ideal in R. Since $HK \subseteq H \cap K = \{0\}$, and similarly $KH = \{0\}$, it follows that $R = H \oplus K$ is a ring direct sum.

The result parallel to Lemma 12 for E-rings, was proved in [7, Corollary 2]. ∎

COROLLARY 13. *Let G be a TE-group. Then G cannot be an infinite direct sum of non-trivial groups.*

PROOF: Let $G = \bigoplus_{i \in I} G_i$, and let R be a TE-ring with $R^+ = G$. There exists a finite subset $\{1, \ldots, k\} \subseteq I$ such that $e = e_1 + \cdots + e_k$, with $e_i \in G_i$ for $i = 1, \cdots, k$.

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Let $x \in G$. Then $nx = xe = xe_1 + \cdots + xe_k$. Since $xe_i \in G_i$ for each i by Lemma 12, it follows that $nx \in G_1 \oplus \cdots \oplus G_k$. The fact that $G_1 \oplus \cdots \oplus G_k$ is a pure subgroup of G, and that G is torsion free, yields that $x \in G_1 \oplus \cdots \oplus G_k$, that is, $G = G_1 \oplus \cdots \oplus G_k$. ∎

LEMMA 14. *Let G be a completely decomposable torsion free group. The following are equivalent:*

(1) *G is an E-group,*

(2) *G is a TE-group,*

(3) *$G = \bigoplus_{i=1}^{k} G_i$, with k a positive integer, G_i a rank one torsion free group, $t(G_i)$ idempotent, and $t(G_i) \not\leqslant t(G_j)$ for all $1 \leqslant i, j \leqslant k$, $i \neq j$.*

PROOF: Clearly (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Suppose that G is a completely decomposable TE-group. It follows from Corollary 13 that G is a finite direct sum of rank one torsion free groups, $\bigoplus_{i=1}^{k} G_i$. Let $A = \bigoplus_{i=1}^{k} E(G_i)$. Since $E(R^+) = A \oplus \bigoplus_{i \neq j} \operatorname{Hom}(G_i, G_j)$, and $r(G) = r(E(G))^+ = k$, it follows that $\operatorname{Hom}(G_i, G_j) = 0$ for all $i \neq j$. If $t(G_i) \leqslant t(G_j)$ for $i \neq j$, then $\operatorname{Hom}(G_i, G_j) \neq 0$, a contradiction. Let R be a TE-ring with $R^+ = G$, and let $a \in G_j$, $a \neq 0$, for some $1 \leqslant j \leqslant k$. Let $e = e_1 + \cdots + e_k$ with $e_i \in G_i$ for all $1 \leqslant i \leqslant k$. Then $t(G_j) = t(na) = t(ae_j) \geqslant 2t(G_j)$, and so $t(G_j)$ is idempotent for all $1 \leqslant j \leqslant k$. ∎

(3) $\Rightarrow$ (1): [**7**, Theorem 2].

QUESTION 1. Is every TE-group an E-group? It follows from Corollary 3, and Theorem 9, that every TE-group is quasi-isomorphic to an E-group.

QUESTION 2. Let R be a subring of a torsion-free E-ring S, with S^+/R^+ a torsion group. Is R a TE-ring?

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