The Value of H_0 from Gaussian Processes

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Abstract. A new non-parametric method based on Gaussian Processes (GP) was proposed recently to measure the Hubble constant H_0. The freedom in this approach comes in the chosen covariance function, which determines how smooth the process is and how nearby points are correlated. We perform coverage tests with a thousand mock samples within the ΛCDM model in order to determine what covariance function provides the least biased results. The function Matérn(5/2) is the best with sligthly higher errors than other covariance functions, although much more stable when compared to standard parametric analyses.

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1. Introduction

The difference between the value of the Hubble constant H_0 determined by *Planck* (Planck Collaboration 2014) and by local measurements (Riess *et al.* 2011) shows a 2.3σ tension. In order to understand what could generate such a discrepancy, many attempts in the literature were done searching for new physics or systematic errors (e.g. Marra *et al.* 2013, Spergel *et al.* 2015, Efstathiou 2014, Wyman *et al.* 2014, Holanda *et al.* 2014, Clarkson *et al.* (2014)).

Recently, we proposed a new method to determine H_0 by applying *Gaussian Processes* (GP), which is a non-parametric procedure, to reconstruct $H(z)$ data and extrapolating to redshift zero (Busti *et al.* 2014). We selected 19 $H(z)$ measurements (Simon *et al.* 2005, Stern *et al.* 2010, Moresco *et al.* 2012) based on cosmic chronometers and obtained $H_0 = 64.9 \pm 4.2$ km s^{-1} Mpc^{-1}, which is compatible with *Planck* but shows a tension with local measurements.

Here, we use mock samples in order to test our method. Basically, we are interested to see which covariance function adopted in the GP analysis provides the best match with the fiducial cosmological model. As we shall see, the Matérn(5/2) performs better, with the standard squared exponential covariance function underestimating the errors. In the next section GP will be briefly described, showing our results in Sec. 3. We draw our conclusions and discuss future improvements in Sec. 4.

2. Gaussian Processes (GP)

A gaussian process allows one to reconstruct a function from data without assuming a parametrisation for it. While a gaussian distribution is a distribution over random variables, a gaussian process is a distribution over functions. We use GaPP (Gaussian Processes in Python)† (Seikel *et al.* 2012)) in order to reconstruct the Hubble parameter as a function of the redshift.

† http://www.acgc.uct.ac.za/~seikel/GAPP/index.html

Table 1. H_0 constraints from 19 $H(z)$ measurements.

Method	$H_0 \pm 1\sigma$ (km s^{-1} Mpc^{-1})	Coverage 1σ	Coverage 2σ
Sq. Exp.	$64.9 \pm 4.2(5.9)$	0.527	0.905
Matérn(9/2)	$65.9 \pm 4.5(5.6)$	0.594	0.939
Matérn(7/2)	$66.4 \pm 4.7(5.7)$	0.610	0.946
Matérn(5/2)	$67.4 \pm 5.2(5.5)$	0.665	0.959
ΛCDM	68.9 ± 2.8	0.676	0.938
XCDM	69.0 ± 6.7	0.685	0.939

Basically, the reconstruction is given by a mean function with gaussian error bands, where the function values at different points z and $\tilde{z}$ are connected through a covariance function $k(z, \tilde{z})$. This covariance function depends on a set of hyperparameters. For example, the general purpose squared exponential (Sq. Exp.) covariance function is given by

$$k(z, \tilde{z}) = \sigma_f^2 \exp \left\{ -\frac{(z - \tilde{z})^2}{2l^2} \right\}. \qquad (2.1)$$

In the above equation we have two hyperparameters, the first σ_f is related to typical changes in the function value while the second l is related to the distance one needs to move in input space before the function value changes significantly. We follow the steps of Seikel *et al.* (2012) and determine the maximum likelihood value for σ_f and l in order to obtain the value of the function. In this way, we are able to reconstruct the Hubble parameter as a function of the redshift from $H(z)$ measurements. Many choices of covariance function are possible, and we consider a variety below.

3. Results

The freedom in the GP approach comes in the covariance function. While in traditional parametric analyses we choose a model to characterise what is our prior belief about the function in which we are interested, with GP we ascribe in the covariance function our priors about the expected function properties (e.g. smoothness, correlation scales etc.).

We consider the Sq. Exp. covariance function and three examples from the Matérn family:

$$k(z, \tilde{z}) = \sigma_f^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{\sqrt{2\nu(z - \tilde{z})^2}}{l} \right)^{\nu} K_\nu \left(\frac{\sqrt{2\nu(z - \tilde{z})^2}}{l} \right), \qquad (3.1)$$

where K_ν is a modified Bessel function and we choose $\nu = 5/2$, $7/2$ and $9/2$ (see Seikel & Clarkson 2013 for more discussions). Writing $\nu = p + 1/2$, each Matérn function is p times differentiable as are functions drawn from it, and the squared exponential is recovered for $\nu \rightarrow \infty$. Increasing ν increases the width of the covariance function near the peak implying stronger correlations from nearby points for a fixed correlation length ℓ. For comparison purposes, we also consider two standard parametric models: a flat ΛCDM model and a flat XCDM model.

The results are shown in Table 1 together with the constraints from the 19 $H(z)$ data. The coverage test of each covariance function and parametric model was performed by creating 1000 mock catalogues of 19 data points with the same redshifts and error-bars of the measured points in a fiducial ΛCDM model. For each model realisation a value of H_0

was derived. The third and fourth columns of Table 1 show the frequency the true value for H_0 was recovered inside the 1σ and 2σ regions. So, for example, the Matérn(9/2) covariance function captures the true value at $1(2)\sigma$ about $60\%(94\%)$ of the time – alternatively, the $1(2)\sigma$ region should be interpreted as a $60\%(94\%)$ confidence interval. This provides a way to re-normalise the $n\sigma$ intervals for a given covariance function and a prior model assumption, which we show between parentheses for $1\sigma(68\%)$ errors in Table 1. Therefore, this is an attempt to quantify a possible systematic error from the covariance functions *assuming the true model is ΛCDM*. We also considered some different fiducial models with a time-varying dark energy equation of state, 64 data points in the redshift range $0.1 < z < 1.8$, with coverages showing the same pattern as depicted in Table 1. It is important to note this is a model-dependent comparison which relies on the knowledge of the true model in advance, which is never the case, and changes with the quality of the data. The coverage can change with a different underlying model as well – but note that the errors are actually much more stable than switching from ΛCDM to XCDM.

4. Conclusions

We have applied GP to reconstruct $H(z)$ data and from it extrapolate to redshift zero to obtain H_0 (Busti *et al.* 2014). Based on a set of 1000 mock samples, we have tested the method assuming a fiducial flat ΛCDM model by considering four different covariance functions and applying a coverage test. We have shown Matérn(5/2) represents better the errors, with errors slightly higher than the other covariance functions. A heuristic method to recalibrate the errors for different covariance functions was also provided within the ΛCDM model.

Possible improvements can be achieved by marginalizing over the hyperparameters and comparing the results using Bayesian model comparison tools, which will allow a direct assessment of performance with no need to rely on a fiducial model.

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