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A CHARACTERISATION OF CENTRAL ELEMENTS IN C^*-ALGEBRAS

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Abstract

Wu ['An order characterization of commutativity for C^*-algebras', *Proc. Amer. Math. Soc.* **129** (2001), 983–987] proved that if the exponential function on the set of all positive elements of a C^*-algebra is monotone in the usual partial order, then the algebra in question is necessarily commutative. In this note, we present a local version of that result and obtain a characterisation of central elements in C^*-algebras in terms of the order.

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1. Introduction

Let $\mathcal{A}$ be a (unital) C^*-algebra and denote by $\mathcal{A}_s$ the space of all of its self-adjoint elements. An element $x \in \mathcal{A}_s$ is called positive, $x \geq 0$, if its spectrum $\sigma(x)$ lies in the nonnegative part of the real line. The set of all positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}_+$. The usual partial order $\leq$ on $\mathcal{A}_s$ is then defined in the following way: for any $x, y \in \mathcal{A}_s$ we write $x \leq y$ if and only if $y - x \in \mathcal{A}_+$.

There are some classical results in the literature which characterise the commutativity of C^*-algebras in terms of certain properties of the order. For example, a result of Sherman [7] says that a C^*-algebra $\mathcal{A}$ is commutative if and only if $\mathcal{A}_s$ is a lattice (compare with [1]). Another famous result, due to Ogasawara [4], says that squaring is monotone on $\mathcal{A}_+$ if and only if $\mathcal{A}$ is commutative. The slightly more general result [5, Proposition 1.3.9] shows that if the power function $t \mapsto t^\beta$, where $\beta > 1$ is monotone with respect to the usual order on $\mathcal{A}_+$ (meaning that $x, y \in \mathcal{A}_+$, $x \leq y$ implies $x^\beta \leq y^\beta$), then the algebra $\mathcal{A}$ is necessarily commutative. Wu [8] presented a similar statement saying that the same conclusion holds if the power function is replaced by the exponential function.

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In this note we present a local version of Wu's result. Namely, we show that the 'points of monotonicity' of the exponential function on $\mathcal{A}_s$ necessarily belong to the centre of $\mathcal{A}$. This implies Wu's result as an immediate consequence.

2. The result

THEOREM 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $x \in \mathcal{A}_s$. The following assertions are equivalent:*

(i) $e^x \le e^y$ *for every $y \in \mathcal{A}_s$ with $x \le y$;*
(ii) $\int_0^1 e^{tx} z e^{(1-t)x}\, dt \in \mathcal{A}_+$ *for all $z \in \mathcal{A}_+$; and*
(ii) *x is a central element of $\mathcal{A}$.*

For the proof of the theorem, we need the following auxiliary lemma.

LEMMA 2.2. *Let H be a complex Hilbert space and denote by $B(H)$ the algebra of all bounded linear operators on H. Let $T \in B(H)$ be self-adjoint. Assume that $0 = \min \sigma(T)$ and $r = \max \sigma(T)$. For every $\epsilon > 0$, we can choose orthogonal unit vectors $\xi, \eta \in H$ such that, for any $A \in B(H)$ with the properties $\|A\| \le \sqrt{2}$, $A\xi = A\eta$ and $\|A\xi\| = \|A\eta\| = 1$, if*

$$\int_0^1 \exp(tT) A^* A \exp((1-t)T)\, dt \ge 0,$$

then

$$\left(\frac{e^r - 1}{r}\right)^2 \le (1 + 2\epsilon)(e^r + 2\epsilon).$$

PROOF. It is easy to see that, for any pair $f, g : [0, 1] \to B(H)$ of continuous functions, the transformation

$$X \mapsto \int_0^1 f(t) X g(t)\, dt$$

is a bounded linear map on $B(H)$ and its norm is majorised by the product of the supremum norms of f and g. It follows that the above integral depends continuously on the functions f and g, meaning that if $f_n, g_n : [0, 1] \to B(H)$ are sequences of continuous functions uniformly converging to f and g, respectively, then the corresponding sequence

$$X \mapsto \int_0^1 f_n(t) X g_n(t)\, dt$$

of bounded linear maps on $B(H)$ converges to the map

$$X \mapsto \int_0^1 f(t) X g(t)\, dt$$

in the operator norm.

It is also easy to see that if (T_k) is a sequence in $B(H)$ which converges in norm to T, then the sequence $t \mapsto \exp(tT_k)$ of operator valued functions converges to $t \mapsto \exp(tT)$

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uniformly in $t \in [0, 1]$. It follows that, given $T \in B(H)$, for every real number $\epsilon > 0$ there is a real number $\delta > 0$ such that

$$\sup_{\|X\|\le 1} \left\| \int_0^1 \exp(tT)X\exp((1-t)T)\,dt - \int_0^1 \exp(tT')X\exp((1-t)T')\,dt \right\| \le \epsilon \quad (2.1)$$

holds whenever $T' \in B(H)$ with $\|T - T'\| \le \delta$. Obviously, we may assume that $2\delta < r$. Consider a continuous function $h : [0, r] \to [0, r]$ which is zero on the interval $[0, \delta]$, it equals r on $[r - \delta, r]$ and its distance to the identity function on $[0, r]$ in the supremum norm is not greater than δ. Then $\|T - h(T)\| \le \delta$ and hence we obtain from (2.1) that

$$\Big| \int_0^1 \langle \exp(tT)A^*A\exp((1-t)T)\zeta, \zeta\rangle\,dt - \int_0^1 \langle \exp(th(T))A^*A\exp((1-t)h(T))\zeta, \zeta\rangle\,dt \Big| \le \epsilon\|A\|^2\|\zeta\|^2 \quad (2.2)$$

holds for every operator $A \in B(H)$ and vector $\zeta \in H$. Observe that, by elementary change of variables, for any self-adjoint operator $S \in B(H)$,

$$\int_0^1 \exp(tS)A^*A\exp((1-t)S)\,dt = \int_0^1 \exp((1-t)S)A^*A\exp(tS)\,dt, \quad A \in B(H)$$

implying that the values of these integrals are self-adjoint operators. Therefore, if

$$\int_0^1 \exp(tT)A^*A\exp((1-t)T)\,dt \ge 0, \quad (2.3)$$

then it follows from (2.2) that

$$\begin{aligned} 0 &\le \int_0^1 \langle \exp(tT)A^*A\exp((1-t)T)\zeta, \zeta\rangle\,dt \\ &\le \int_0^1 \langle \exp(th(T))A^*A\exp((1-t)h(T))\zeta, \zeta\rangle\,dt + \epsilon\|A\|^2\|\zeta\|^2 \\ &= \int_0^1 \langle A\exp((1-t)h(T))\zeta, A\exp(th(T))\zeta\rangle\,dt + \epsilon\|A\|^2\|\zeta\|^2. \end{aligned}$$

Denote by E the spectral measure of T on the Borel subsets of $[0, r]$. Pick a unit vector ξ from the range of $E([0, \delta])$ and another one η from the range of $E([r - \delta, r])$. Clearly, ξ is orthogonal to η. Let s be an arbitrary real number and set $\zeta = s\xi + \eta$. We compute

$$A\exp((1-t)h(T))\zeta = A(s\xi + e^{(1-t)r}\eta), \quad A\exp(th(T))\zeta = A(s\xi + e^{tr}\eta).$$

Hence, for any $A \in B(H)$ satisfying (2.3), $\|A\| \le \sqrt{2}$, $A\xi = A\eta$ and $\|A\xi\| = \|A\eta\| = 1$, we obtain

$$0 \le \int_0^1 (s + e^{(1-t)r})(s + e^{tr})\,dt + \epsilon 2(s^2 + 1)$$

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for every real number s. This implies that

$$0 \leq s^2 + e^r + 2s\frac{e^r - 1}{r} + \epsilon 2(s^2 + 1)$$

for every real number s. Examining the discriminant of the corresponding quadratic equation, yields

$$4\Big(\frac{e^r - 1}{r}\Big)^2 - 4(1 + 2\epsilon)(e^r + 2\epsilon) \leq 0,$$

which gives the statement of the lemma. □

We are now in a position to prove the theorem.

Proof of Theorem 2.1. According to the bottom line on [6, page 148], the (Fréchet-) derivative of the exponential function $T \mapsto \exp T$ on $B(H)$ at the point T is the linear map

$$X \mapsto \int_0^1 \exp(tT)X\exp((1-t)T)\,dt.$$

This implies that the function $x \mapsto e^x$ on the C^*-algebra $\mathcal{A}$ is differentiable at x and its derivative is the linear map

$$z \mapsto \int_0^1 \exp(tx)z\exp((1-t)x)\,dt.$$

Now, assuming (i), we clearly obtain (ii).

Suppose (ii) holds. Select an irreducible representation π of $\mathcal{A}$ on a Hilbert space H. Then

$$\int_0^1 \exp(t\pi(x))\pi(z)^*\pi(z)\exp((1-t)\pi(x))\,dt \geq 0$$

holds for all $z \in \mathcal{A}$. Since $\pi(1) = I$, adding a real constant times the identity to x, if necessary, we may assume that the operator $\pi(x)$ is positive, zero belongs to its spectrum and the largest element of the spectrum is r. By Lemma 2.2, for every $\epsilon > 0$, we can choose orthogonal unit vectors $\xi, \eta \in H$ such that, for any $A \in B(H)$ with the properties $\|A\| \leq \sqrt{2}$, $A\xi = A\eta$ and $\|A\xi\| = \|A\eta\| = 1$, the positivity of the operator

$$\int_0^1 \exp(t\pi(x))A^*A\exp((1-t)\pi(x))\,dt$$

implies that

$$\Big(\frac{e^r - 1}{r}\Big)^2 \leq (1 + 2\epsilon)(e^r + 2\epsilon).$$

Pick a unit vector $v \in H$ and define the operator $A \in B(H)$ by $A\zeta = \langle \zeta, \xi + \eta \rangle v$ for all $\zeta \in H$. Clearly, $\|A\| = \sqrt{2}$, $A\xi = A\eta$ and $\|A\xi\| = \|A\eta\| = 1$. Since π is an irreducible representation, by a sharper version of the Kadison transitivity theorem (see [2, Exercise 5.7.41.(ii) on page 379]), there is an element $z \in \mathcal{A}$ such that $\|\pi(z)\| \leq \sqrt{2}$ and $\pi(z)\xi = A\xi, \pi(z)\eta = A\eta$. It then follows that

$$\left(\frac{e^r - 1}{r}\right)^2 \leq (1 + 2\epsilon)(e^r + 2\epsilon).$$

But here $\epsilon > 0$ is arbitrary, so, consequently,

$$\left(\frac{e^r - 1}{r}\right)^2 \leq e^r.$$

It is easy to check that for a nonnegative real number r this holds only if $r = 0$. Therefore, $\pi(x) = 0$. Since we may have added a constant multiple of the identity to x, this means, for the original element x, that $\pi(x) = \lambda I$ holds for some real number λ. We know that this is true for all irreducible representations π of $\mathcal{A}$ and claim that x is central. Indeed, if $a \in \mathcal{A}$ is an element and $xa - ax \neq 0$, then, by [3, Corollary 10.2.4], we have an irreducible representation π such that $0 \neq \pi(xa - ax) = \pi(x)\pi(a) - \pi(a)\pi(x)$, which is clearly a contradiction.

Finally, to see the implication (iii)⇒(i), let $x \in \mathcal{A}_s$ be central and select an arbitrary element $y \in \mathcal{A}_s$ such that $y \geq x$. Since $x, y - x$ commute and $y - x \geq 0$,

$$e^x = e^{x/2} 1 e^{x/2} \leq e^{x/2} e^{y-x} e^{x/2} = e^{x/2+(y-x)+x/2} = e^y.$$

The proof of the theorem is complete. □

As an immediate corollary we obtain the following statement which is formally stronger than Wu's original theorem.

Corollary 2.3. *Let $\mathcal{A}$ be a C^*-algebra such that the exponential function is monotone on a nongenerate interval I of the real line, meaning that I is of positive length and, for any $x, y \in \mathcal{A}_s$ with $\sigma(x), \sigma(y) \subset I$ and $x \leq y$, we have $e^x \leq e^y$. Then $\mathcal{A}$ is commutative.*

Proof. Let I' be a nongenerate compact interval in the interior of I. Select $x \in \mathcal{A}_s$ such that $\sigma(x) \subset I'$. For any element $z \in \mathcal{A}_+$, the inclusion $\sigma(x + tz) \subset I$ holds for small enough $t > 0$. It follows that the directional derivative of the exponential function on $\mathcal{A}_s$ at x along z, that is, the limit $\lim_{t \to 0+}(e^{x+tz} - e^x)/t$, belongs to $\mathcal{A}_+$. As mentioned in the proof of Theorem 2.1, the (Fréchet-) derivative of the exponential function at x is the linear transformation

$$z \to \int_0^1 e^{tx} z e^{(1-t)x} \, dt$$

on $\mathcal{A}$. It follows that

$$\lim_{t \to 0+} \frac{e^{x+tz} - e^x}{t} = \int_0^1 e^{tx} z e^{(1-t)x} \, dt$$

belongs to $\mathcal{A}_+$ for every $z \in \mathcal{A}_+$. By the implication (ii) ⇒ (iii) in Theorem 2.1, x is central in $\mathcal{A}$. We then easily obtain the desired conclusion. □

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